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Partial Conservation of Axial-Vector Current in a Nonlinear Conformal and Chiral Lagrangian Model*

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We show that all Lagrangian functions of scalar fields yielding conformally invariant actions may be expressed in covariant derivative notation. We study the nonlinear pion Lagrangian and break both the chiral and conformal symmetry with partial conservation of axial-vector current. If we require smoothness of the $\sigma\pi\pi$ amplitude, we find that the axial-vector current divergence and the pion field have unit scale dimension.

Using the same techniques that Weinberg[1] applied to the study of nonlinear realizations of chiral symmetry, Hankey[2] recently showed that the conformally covariant derivatives derived by Salam and Strathdee[3] are unique except for canonical transformations of fields. These covariant derivatives are functions of the Goldstone fields σ and ϕ_μ as well as of non-Goldstone fields ψ. We shall show that any conformally covariant Lagrangian density of scale dimension four that can be represented by a power-series expansion in scalar fields Θ and χ (where $\chi=e^{-g\sigma}$), and which is second order in derivatives, may be expressed in the covariant derivative formalism. We then show that such an expansion is essentially unique for a chirally symmetric pion Lagrangian density that produces a conformally invariant action. The chiral and conformal symmetry is then broken by the PCAC (partial conservation of axial-vector current) assumption $\partial_\mu A_i^\mu = f_\pi \mu_\pi{}^2 \Phi_i$, where the physical pion field Φ_i is assumed to have scale dimension d. We then calculate the $\pi\pi\sigma$ amplitude. If we require it to satisfy a smoothness criterion, we find that $d=1$.

We shall first review the transformation properties of the Goldstone fields[4] σ and ϕ^μ:

$$[D,\sigma]=i/g,\quad [D,\phi^\lambda]=-i\phi^\lambda,$$
$$[K^\mu,\sigma]=0,\quad [K^\mu,\phi^\lambda]=-ig^{\mu\lambda}/f, \tag{1}$$

where D and K^μ are the generators of scale and special conformal transformations.

If a field ψ satisfies $[D,\psi]=-id\psi$, $[K^\mu,\psi]=0$, it is said to be a conformally covariant field with scale dimension d. For example, $\chi=e^{-g\sigma}$ is such a field with $d=1$.

The following conformally covariant derivatives may be derived using the well-known commutation relations of D and K^μ with the generators of Lorentz translations:

$$D_\mu\sigma=\partial_\mu\sigma+\frac{2f}{g}\phi_\mu,\quad \text{scale dimension 1}$$
$$D_\mu\phi_\nu=\partial_\mu\phi_\nu-2f\phi_\mu\phi_\nu+f\phi^2 g_{\mu\nu},\quad \text{scale dimension 2} \tag{2}$$
$$D_\mu\psi=e^{g\sigma}(\partial_\mu-2df\phi_\mu+2if\phi^\nu\Sigma_{\mu\nu})\psi,\quad \text{scale dimension } d$$

where $\Sigma_{\mu\nu}$ is the spin matrix for the field ψ.

A Lagrangian density formed by combining conformally covariant fields and derivatives in a Lorentz-invariant way, and which has scale dimension four, will produce a conformally invariant action.

Let us write $L(\Theta,\sigma)$ in the covariant derivative formalism and examine the constraints imposed upon L by demanding that there be no conformal symmetry-breaking terms [Θ is a covariant scalar field of scale dimension d and L has scale dimension four and is assumed to be represented by the power-series expansion (choosing $f=g=1$)]:

$$L=\sum_m \exp\{[(m+2)d-2]\sigma\}(\alpha_m\Theta^m\partial_\mu\Theta\partial^\mu\Theta-\beta_m\Theta^{m+1}\partial_\mu\Theta\partial^\mu\sigma+\gamma_m\Theta^{m+2}\partial_\mu\sigma\partial^\mu\sigma). \tag{3}$$

Inserting $\partial_\mu\sigma=D_\mu\sigma-2\phi_\mu$ and $\partial_\mu\Theta=e^{-\sigma}D_\mu\Theta+2d\phi_\mu\Theta$ into Eq. (3), we find terms bilinear in covariant derivatives which are therefore automatically conformally covariant, plus the additional terms

$$\Theta^m\exp\{[(m+2)d-2]\sigma\}[(4\alpha_m d+2\beta_m)\Theta e^{-\sigma}\phi^\mu D_\mu\Theta-2(\beta_m d+2\gamma_m)\Theta^2\phi^\mu D_\mu\sigma+4(d^2\alpha_m+d\beta_m+\gamma_m)\Theta^2\phi^2].$$

Noting that the following interaction is conformally covariant with scale dimension four:

$$L_I^m=\Theta^{m+2}\exp\{[(m+2)d-2]\sigma\}D_\mu\phi^\mu=\Theta^{m+2}\exp\{[(m+2)d-2]\sigma\}(\partial_\mu\phi^\mu+2\phi^2)$$
$$=\partial_\mu(\Theta^{m+2}\exp\{[(m+2)d-2]\sigma\}\phi^\mu)-\phi^\mu\partial_\mu(\Theta^{m+2}\exp\{[(m+2)d-2]\sigma\})+2\Theta^{m+2}\exp\{[(m+2)d-2]\sigma\}\phi^2$$
$$=-\Theta^m\exp\{[(m+2)d-2]\sigma\}\{(m+2)\Theta e^{-\sigma}\phi_\mu D^\mu\Theta+[(m+2)d-2]\Theta^2\phi_\mu D^\mu\sigma+2\Theta^2\phi^2\},$$

we see that the preceding set of terms will equal $\sum_m -2(d^2\alpha_m+d\beta_m+\gamma_m)L_I^m$ if and only if for each m

$$d[(m+2)d-2]\alpha_m+[(m+2)d-1]\beta_m+(m+2)\gamma_m=0. \tag{4}$$

Equation (4) is the same result[5] that one obtains by using the Callan-Coleman-Jackiw[6] condition for L to produce a conformally invariant action, namely, the condition

$$\frac{\delta L}{\delta\partial_\mu\Theta}d\Theta+\frac{\delta L}{\delta\partial_\mu\chi}\chi=\partial_\nu\sigma^{\mu\nu}\qquad(\chi=e^{-\sigma}).$$

It is interesting to note that the presence of both or either of the Goldstone fields σ and ϕ_μ in a Lagrangian density is not required for conformal invariance of the action, in marked contrast to the necessity for the Goldstone pion field's presence in any chirally symmetric nonlinear Lagrangian involving isospin-bearing fields. For example, from Eqs. (3) and (4), we note that both the Lagrangian $\frac{1}{2}\partial_\mu\Theta\partial^\mu\Theta$ of the free massless scalar field with $d=1$ and the Lagrangian $\frac{1}{2}e^{-2\sigma}\partial_\mu\sigma\partial^\mu\sigma$ produce conformally invariant actions. An example of a Lagrangian interaction containing only Θ (with $d=1$) and ϕ_μ which yields a conformally invariant action is

$$\tfrac{1}{2}\Theta^2D_\mu\phi^\mu=\Theta^2\phi^2-\Theta\partial_\mu\Theta\phi^\mu,$$

where we have ignored a divergence term.

We now wish to use the covariant derivative formalism in our study of the scale dimension of the axial-vector current divergence in a nonlinear π-σ Lagrangian model. In this model, we break the chiral and conformal symmetry by the PCAC hypothesis $\partial_\mu A_i^\mu=f_\pi\mu_\pi^2\Phi_i$, where Φ_i is the physical pion field. The requirement that the off-mass-shell $\pi\pi\sigma$ amplitude vary smoothly will determine the scale dimension of the pion field and of the axial-vector current divergence. We shall now leave g and f arbitrary as in Eq. (1).

The commutators $[Q_i^5,\phi_\mu]$ and $[Q_i^5,\sigma]$ are zero, since ϕ_μ and σ are isoscalar fields. Assuming that Φ_i transforms as a conformally covariant field of scale dimension d, we define a covariant field of dimension zero: $\Phi_i^0=e^{dg\sigma}\Phi_i$. Then, from Eq. (2), we note that the following is a conformally and chirally covariant derivative having scale dimension zero:

$$\Delta_\mu\Phi_i^0=e^{g\sigma}\mathfrak{D}_\mu\Phi_i^0$$
$$=f_\pi\left\{\frac{D_\mu\Phi_i^0}{[f^2(\lambda)+\lambda]^{1/2}}-\left(\frac{2f'(\lambda)+v(\lambda)}{f^2(\lambda)+\lambda}\right)(\vec\Phi^0\cdot D_\mu\vec\Phi^0)\Phi_i^0\right\}, \tag{5}$$

where $\lambda=(\Phi^0)^2$ and $\mathfrak{D}_\mu$ is the chirally covariant deriviative operator.[1] Canonical transformations of the form $\Phi_i^0\to F(\lambda)\Phi_i^0$ cannot affect the off-shell behavior of the $\pi\pi\sigma$ amplitude, so that, without loss of generality in this study, we may assume that

$$[Q_i^5,\Phi_j^0]=i\delta_{ij}f(\lambda)=i\delta_{ij}(f_\pi^2-\lambda)^{1/2}. \tag{6}$$

Then the unique π-σ Lagrangian density, second order in field derivatives, that produces a chirally and conformally invariant action, is

$$L_0=\tfrac{1}{2}e^{-4g\sigma}\Delta_\mu\Phi_i^0(\Phi,\sigma)\Delta^\mu\Phi_i^0(\Phi,\sigma)+\tfrac{1}{2}e^{-2g\sigma}\partial_\mu\sigma\partial^\mu\sigma. \tag{7}$$

The absence of the field ϕ_μ in L_0 indicates that the assumption of invariance of the action under special conformal transformations is unnecessary in the derivation of Eq. (7).

Indeed, from Eq. (3), we see that the only π-σ Lagrangian density having scale dimension four, which is second order in field derivatives and is also chirally symmetric, must be

$$\alpha_0e^{-2g\sigma}\mathfrak{D}_\mu\Phi_i^0\mathfrak{D}^\mu\Phi_i^0+\gamma_{-2}e^{-2g\sigma}\partial_\mu\sigma\partial^\mu\sigma.$$

Equation (4) imposes no additional constraints. A glance at Eq. (2) shows that there are no interaction of pions with ϕ_μ's, which are less than third order in field derivatives that yield conformally and chirally invariant actions. Hence, the form of L_0 given in Eq. (7) is unique to within canonical transformations of the fields.

Our PCAC assumption $\partial_\mu A_i^\mu=i[Q_i^5,L]=f_\pi\mu_\pi^2\Phi_i$ implies that $\partial_\mu A_i^\mu$ transforms as a conformally covariant operator with the scale dimension d of the physical pion field. This assumption also requires the chiral and conformal symmetry-breaking term to be

$$L'=f_\pi\mu_\pi^2e^{-dg\sigma}f(\lambda), \tag{8}$$

where $f(\lambda)$ is specified in Eq. (6) and $L=L_0+L'$.

Separating out the $\pi\pi\sigma$ interaction terms of L using Eqs. (7) and (8), we obtain

$$L_{\pi\pi\sigma}=g[(d-1)\sigma\partial_\mu\vec\Phi\cdot\partial^\mu\vec\Phi+d\partial^\mu\sigma(\vec\Phi\cdot\partial_\mu\vec\Phi)-\tfrac{1}{2}\mu_\pi^2d\sigma\Phi^2]. \tag{9}$$

Equation (9) allows us to evaluate the $\pi\pi\sigma$ scattering amplitude in tree-graph approximation:

$$M(\sigma(k)\to\pi(q)+\pi(p)) = M(k^2,p^2,q^2) = M(k^2,q^2,p^2)$$
$$= g[(d-1)(k^2-p^2-q^2) + d(\mu_\pi{}^2-k^2)]. \quad (10)$$

For on-shell pions we obtain Ellis's result[7]: $M(k^2,\mu_\pi{}^2,\mu_\pi{}^2) = g[(2-d)\mu_\pi{}^2 - k^2]$. For off-shell pions, our amplitude differs from Ellis's because our assumption of PCAC relates our pion fields to his by the canonical transformation $e^{-dg\sigma}$. Our PCAC assumption also requires that our amplitude M satisfy

$$M(\mu_\pi{}^2,\mu_\pi{}^2,\mu_\pi{}^2)M(\mu_\pi{}^2,0,\mu_\pi{}^2)=0, \quad (11)$$

since the left-hand side of this equation is just the coupling of the σ pole in the $\pi\pi$ scattering amplitude at the Adler point.[8] Indeed, from Eq. (10) we see that

$$M(\mu_\pi{}^2,\mu_\pi{}^2,\mu_\pi{}^2) = g\mu_\pi{}^2(1-d)$$

and

$$M(\mu_\pi{}^2,\mu_\pi{}^2,0)=0.$$

If we demand that $M(k^2,p^2,q^2)$ be as smoothly varying as possible, i.e.,

$$\frac{\partial M}{\partial p^2} = \frac{\partial M}{\partial q^2} = g(1-d) = 0,$$

then we find that $d=1$.[9] This value of d also produces a second-order zero in the coupling of the σ pole in the π-π amplitude at the Adler point.

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Analytic Properties of Three-Particle Amplitudes in the Scattering-Angle Variable*

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The elastic amplitude describing a scattering of a free particle on a two-particle bound state is here considered as a function of $\cos\theta$, where θ is the scattering angle. The two-body interactions were taken to be s-wave and separable. Our results are: (1) The on-shell amplitude is analytic inside a Lehmann ellipse for all real energies, except for a finite interval on the negative E axis, and (2) for negative energy, the off-shell amplitude is analytic in the whole $\cos\theta$ plane, apart from possible real left and right cuts, and in the Lehmann ellipse. Its asymptotic behavior in $\cos\theta$ is determined by the leading Regge trajectory.

I. INTRODUCTION

The study of multiparticle scattering amplitudes is essential not only for the calculation of cross sections, but also for the establishment of complete S-matrix theory and dispersion relations.

In nonrelativistic potential scattering, dispersion relations in the total energy for fixed directions of the individual momenta have been proved for three-particle amplitudes.[1]

Analytic properties in the scattering-angle variable for the three-particle amplitude were studied by Immirzi,[2] and by Hartle and Sugar.[3] Immirzi, using the invariance of the three-particle Green's function under rotation, has established a Lehmann ellipse in the $\cos\theta$ plane, in which the off-shell elastic amplitude describing a scattering of a particle off a two-particle bound state is analytic for all values of the total energy E. Hartle and Sugar have generalized Immirzi's result, and have also estab-