PHYSICAL REVIEW D VOLUME 4, NUMBER 2 15 JULY 1971

Projection Operator in the Dual-Resonance Model

K. Kikkawa

Department of Physics, The City College of the City University of New York, New York, New York 10031

(Received 18 March 1971)

It is proved that the projection operator of the nonspurious states has a well-defined property in a certain space if and only if the Regge intercept is noninteger. A discussion is also given on the off-mass-shell extrapolation.

The projection operator which eliminates the spurious states plays an important role in the dual-resonance model.[1,2] Although the formally constructed operator has been employed in calculations of general diagrams, no mathematical discussion for the applicability range has been given. In particular, the question on the existence of inverse operator appearing in the projection operator is a dubious point. In this note we show that the projection operator $P^\dagger$ is well defined in a certain restricted space if the Regge intercept is not an integer. Then, we show that the conjugate of $P^\dagger$, P, is well defined in another space as it should be.

The space H we consider is that spanned by eigenvectors $|n\rangle$ of the generator $L_0(\pi)$:

$$L_0(\pi)|n\rangle = (-\tfrac{1}{2}\pi^2 + n)|n\rangle. \tag{1}$$

Here, the notations and the metric conventions of Ref. 1 are used. The momentum π^2 is so fixed at the mass-shell point of $n=N$ that the eigenvalue of $L_0(\pi)$ ranges from $l_0=-N-\alpha(0)$ to infinity in unit steps. In the following, to express the state vectors, we use the Ida-Soffer orthogonal basis[3] for convenience. The general state vector $|f\rangle$ in H is, therefore, represented by an infinite set of numbers

$$\{x_0, x_1, x_2, \ldots\}, \tag{2}$$

where x_n stands for the projection of $|f\rangle$ onto $|n\rangle$, and the x_n's are arranged in increasing order of the eigenvalue of $L_0(\pi)$. Each projection x_n still has a number of components because of the existence of degeneracy. The norm of $|f\rangle$ in our discussion is defined by

$$\|f\| = \sum_n |x_n\langle n|n\rangle|^2, \tag{3}$$

which is always positive, and $\|f\|=0$ implies $|f\rangle = 0$.

In order to treat the dual amplitude, we must consider two kinds of physical states; the first is

$$|\Psi_n\rangle = \prod_{i=0}^{n} \{DV(k_i)\}\,|0\rangle, \tag{4}$$

which satisfies the gauge-invariance condition

$$[A(\pi) - \alpha(0)]|\Psi_n\rangle = 0, \tag{5}$$

where $A(\pi) = L_0(\pi) - L_-(\pi)$, and the second is

$$|\Psi_n'\rangle = V(k_{n+1})\prod_{i=0}^{n}\{DV(k_i)\}\,|0\rangle, \tag{6}$$

which obeys, however, a different gauge condition, viz.,

$$A(\pi)|\Psi_n'\rangle = 0, \tag{7}$$

where D and V are the propagator and the vertex operator, respectively.

Corresponding to (5) and (7), we introduce a pair of subspaces in H. The first one, signified by H_1, is a subspace which contains only normalizable vectors in the sense of (3). We further assume that $\sum |x_n|^2 |\langle n|n\rangle|^2$ converges so strongly that the generators L_0, $L_\pm$, and $\Omega^\dagger = (-)^R e^{L_-}$ are applicable a finite number of times in H_1. As was noted previously,[3] all the physical states satisfying (5) are contained in H_1.

The second subspace H_2 is defined by

$$H_2 = V(k)H_1, \tag{8}$$

where $V(k)$ is the vertex operator

$$V(k) = \exp\left(\sum k \frac{a^{(n)\dagger}}{\sqrt{n}}\right) \exp\left(\sum k \frac{a^{(n)}}{\sqrt{n}}\right), \tag{9}$$

with $k^2 = -2\alpha(0)$. On H_2 there exists an inverse operator

$$V^{-1}(k) = \exp\left(-\sum k \frac{a^{(n)}}{\sqrt{n}}\right) \exp\left(-\sum k \frac{a^{(n)\dagger}}{\sqrt{n}}\right), \tag{10}$$

which transforms a vector in H_2 back to that in H_1.[4] Therefore, the relation (8) is a one-to-one mapping between H_1 and H_2. The state vectors in H_2 are not normalizable; the physical states obeying (7) belong to H_2.

The projection operators we consider are

$$P(\pi) = 1 - [A^\dagger(-\pi) - \alpha(0)]\{A(\pi)[A^\dagger(-\pi) - \alpha(0)]\}^{-1}A(\pi) \tag{11}$$

and

$$P^\dagger(\pi)=1-A^\dagger(-\pi)\{[A(\pi)-\alpha(0)]A^\dagger(-\pi)\}^{-1}[A(\pi)-\alpha(0)]. \tag{12}$$

Our question is whether the inverse operators in (11) and (12) are well defined.

Before entering into the discussion, we wish to observe the different roles of the two operators in the amplitude. The series of operator products always appear in the following form:

$$\cdots P^\dagger DPVP^\dagger DPV\cdots, \tag{13}$$

where we suppose that all the operators should be operated to their right. Then, since all the physical states Ψ_n are contained in H_1, the intermediate states between V and D where $P^\dagger$ appears can be restricted to H_1, while the intermediate states between D and V where P appears cannot, because the physical states obeying (7) belong to H_2. Owing to the definition, if H_1 is inserted at the $P^\dagger$ position, only H_2 appears at the P position. In the following we prove that $P^\dagger$ and P are well-defined operators on H_1 and H_2, respectively.

$P^\dagger$ on H_1

The relevant term in $P^\dagger$ is

$$\sum_{\overline{n},m} A^\dagger|\overline{n}\rangle\langle\overline{n}|X^{-1}|m\rangle\langle m|[A_1-\alpha(0)], \tag{14}$$

where $X=[A-\alpha(0)]A^\dagger$. The intermediate states $|m\rangle$ and $|\overline{n}\rangle$ span the spaces $H_1'\equiv[A-\alpha(0)]H_1\subset H_1$ and $H_1''\equiv AH_1\subset H_1$, respectively. For any vector $f'\in H_1'$,

$$[A^\dagger-\alpha(0)]|f'\rangle\neq 0, \tag{15}$$

because, if Eq. (15) is zero, the conjugate of (15) leads us to $\langle f'|[A-\alpha(0)]H_1=0$, which means $|f'\rangle$ orthogonal to any element of H_1'. Similarly one can show that if $f''\in H_1''$,

$$A^\dagger|f''\rangle\neq 0. \tag{16}$$

The existence of X^{-1} from H_1' to H_1'' implies a one-to-one mapping of X from H_1'' onto H_1'. In other words, if there is no nonzero vector $|f\rangle\in H_1''$ such that

$$X|f\rangle=0, \tag{17}$$

then X^{-1} exists on its range. Conversely, if there is no state $g\in H_1'$ such that

$$\langle g|X=0 \text{ or } X^\dagger|g\rangle=0, \tag{18}$$

then all the elements in H_1' are uniquely mapped into H_1'', and X^{-1} exists on H_1'.

To prove the first part, assume that there is a nonzero vector $|f\rangle\in H_1''$ satisfying (17). Multiplying $\Omega^\dagger$ on (17), and using $\Omega^\dagger A=L_0\Omega^\dagger$, one gets

$$[L_0-\alpha(0)]\Omega^\dagger A^\dagger|f\rangle=0. \tag{19}$$

Since $\Omega^\dagger A^\dagger|f\rangle\neq 0$ because of (16) and $(\Omega^\dagger)^2=1$, $\Omega^\dagger A^\dagger|f\rangle$ must be an eigenstate of $[L_0-\alpha(0)]$ with zero eigenvalue [$n=N$ in (1)]:

$$\Omega^\dagger A^\dagger|f\rangle=c|N\rangle$$

or (20)

$$A^\dagger|f\rangle=c\Omega^\dagger|N\rangle,$$

with c being a constant. Equation (20) can be explicitly solved:

$$\begin{aligned}(L_0-L_+)\{x_0,x_1,x_2,\ldots\}\\ &=\{l_0x_0,(l_0+1)x_1-x_0,\ldots,-\alpha(0)x_N-x_{N-1},\ldots\}\\ &=c\sum_{j=1}^{N}(-)^{N-j}\frac{1}{j!}(L_-)^j|N\rangle,\end{aligned} \tag{21}$$

where $l_0=-N-\alpha(0)$. One can easily confirm that the solution to (21) is unnormalizable if $\alpha(0)$ is noninteger.[5] Therefore, $f\notin H_1''\subset H_1$, which is a contradiction.

Next, suppose (18) is true for a nonzero vector $|g\rangle\in H_1'$. Multiplying $\Omega^\dagger$ on (18), one gets

$$\Omega^\dagger A[A^\dagger-\alpha(0)]|g\rangle=L_0\Omega^\dagger[A^\dagger-\alpha(0)]|g\rangle. \tag{22}$$

Since L_0 has no vanishing eigenvalue in H_1 if $\alpha(0)$ is not an integer, and since $\Omega^\dagger$ has an inverse, one can operate $\Omega^\dagger L_0^{-1}$ on (22) to get $[A^\dagger-\alpha(0)]|g_1\rangle=0$ which contradicts the relation (15).

If the intercept is an integer, say $\alpha(0)=1$, X^{-1} does not exist because

$$|f\rangle=\Omega^\dagger|N\rangle\in H_1'' \tag{23}$$

is a nontrivial solution to (17).

As a result, $P^\dagger$ is well defined on H_1 if and only if $\alpha(0)$ is not an integer.

P on H_2

Let us discuss the properties of P in H_2:

$$P(\pi)H_2=P(\pi)V(k)H_1. \tag{24}$$

Using the relation

$$A(\pi)V(k)=V(k)[A(\pi+k)-\alpha(0)], \tag{25}$$

we rewrite (24) as follows:

$$P(\pi)H_2=V(k)P^\dagger(\pi+k)H_1. \tag{26}$$

Since we know that $P^\dagger$ is well defined on H_1 and $V(k)$ is a one-to-one mapping between H_1 and H_2, Eq. (26) proves that P is a well-defined operator on H_2.

In conclusion, we have proved that $P^\dagger$ and P exist on H_1 and H_2, respectively, if and only if $\alpha(0)$ is not an integer. The generalization of the discussion to an amplitude containing three Reggeon vertices is straightforward. In the Virasoro case[6] where $\alpha(0)=1$, however, the construction of the projection operator requires special care to avoid the singularity of X^{-1}.

Finally, we would like to remark on the extrapolation of momentum π^2 in $P(\pi)$ to the off-mass-shell region. As one can see from the above discussion, a certain singularity occurs in X^{-1} when the eigenvalue of $L_0(\pi)$ approaches zero, where $\frac{1}{2}\pi^2 = n$ $(n = 0, 1, \ldots)$. These singularities, however, can be avoided if the loop momentum integration is performed along a deformed path away off the real axis. In particular, if the Wick rotation is allowable, the zero of L_0 never occurs.

ACKNOWLEDGMENTS

The author thanks N. P. Chang for various discussions, and also thanks R. Keller for a careful reading of the manuscript.

[1]M. Ida, H. Matsumoto, and S. Yazaki, Progr. Theoret. Phys. (Kyoto) 44, 456 (1970).

[2]C. B. Thorn, Phys. Rev. D 1, 1693 (1970); R. C. Brower and J. H. Weis, Lett. Nuovo Cimento 3, 285 (1970).

[3]M. Ida and J. Soffer, CERN Report No. 1241 (unpublished).

[4]Applicability of operators must be carefully examined on each space. $V^{-1}(k)$ is not well defined on H_1, because it sends $|f\rangle \in H_1$ to either the null vector or infinity, depending on whether $k^2 = -2a(0) > 0$ or < 0. On the other hand, the conjugate of the twist operator $\Omega^+ = (-)^R e^{L_{-1}(\pi)}$ should not be allowed to operate on H_2 by a similar argument.

[5]Note that the Ida-Soffer bases have the normalization $\langle n|n\rangle \propto n!\,\Gamma(2a(0)+n)/\Gamma(2a(0))$.

[6]M. A. Virasoro, Phys. Rev. D 1, 2933 (1970).

PHYSICAL REVIEW D VOLUME 4, NUMBER 2 15 JULY 1971

Remarks on the Breaking of Dilation Invariance*

S. K. Bose and W. D. McGlinn
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556
(Received 26 March 1971)

Some formal implications of broken scale invariance are derived.

I. INTRODUCTION

In a previous paper[1] we studied some implications of a conserved dilation current, $J^\mu = x^\nu \Theta^\mu{}_\nu$, where $\Theta^\mu{}_\nu$ is the local energy-momentum tensor. We presented an argument that spontaneously broken scale invariance does not admit massive single-particle states, which indicates that the real world does not have a "nearly" conserved dilation current.

We would like now to discuss some formal implications of broken scale invariance, $\partial^\mu J_\mu \neq 0$. In this paper we assume the existence of a mass gap; that is, that all particles of the theory have a mass greater than zero.

We prove two results (II-1 and II-2) that are essentially Coleman's theorem for dilation symmetry, and another result (II-3) that shows that a usually assumed commutation relation is, in a particular sense, formally true. The techniques used to prove results II-1 and II-2 are trivial extensions of the techniques employed by Schroer and Stichel[2] to prove Coleman's theorem for a nongeometric symmetry.

II. BROKEN SCALE INVARIANCE

Let us consider a field theory with local fields $\Phi_i(x)$, and an energy-momentum tensor $\Theta_{\mu\nu}(x)$, which is local with respect to $\Phi_i(x)$. Let $\Theta_{\mu\nu}(x)$ be symmetric and divergenceless but not traceless. We also assume the restricted spectral condition, i.e., that there exists a smallest positive rest mass in the theory. We choose a space-smearing function $f_R(x)$ as

$$f(x) = f(|x|) = \begin{cases} 1, & |\vec{x}| < R \\ 0, & |\vec{x}| > R + \epsilon \end{cases} \tag{1}$$

and let $O(f_R)$ denote

$$O(f_R) = \int f_R(x) O(x) d^3x \tag{2}$$

for a general local operator (-valued distribution) $O(x)$. Then it is known that the four-momentum operator P_μ is related to $\Theta_{\mu\nu}(x)$ as follows[1]:

$$\langle \Phi|P_\mu|\Psi\rangle = -\lim_{R\to\infty} \langle \Phi|\Theta_{0\mu}(f_R)|\Psi\rangle \tag{3}$$

when the states $|\Phi\rangle$ and $|\Psi\rangle$ belong to a certain dense set of states (quasilocal states). It also