Deep-Inelastic Neutrino Scattering in a Resonance Model*

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Deep-inelastic neutrino scattering is investigated in the framework of a direct-channel resonance model. Using empirical properties of the baryon spectrum and making a universality hypothesis for the transition form factors, we deduce that in the Bjorken limit the structure functions W_1, νW_2, and νW_3 become functions of the scale variable (ratio of center-of-mass energy to momentum transfer) alone. By imposing generalized Goldberger-Treiman-Nambu relations on the axial-vector transition matrix elements and fixing the coupling constants from low-energy experiments, we predict all structure functions without free parameters. The predictions of the theory are compared with neutrino scattering data obtained at CERN. Good agreement is found between theory and experiment in the deep-inelastic region.

I. INTRODUCTION

Not long ago we called attention to the fact that most results of deep-inelastic electron-nucleon scattering experiments can be satisfactorily explained on the basis of the assumption that the differential cross section is dominated by the contribution of nucleon resonances.[1,2] Similar conclusions were reached by Landshoff and Polkinghorne – who studied an explicit model – and by Bloom and Gilman on the basis of an analysis of the experimental data.[3]

In the framework of the resonance model, the qualitative explanation of the well-known scaling behavior of the structure functions arises from the "conspiracy" of two seemingly unrelated phenomena. First, with increasing resonance mass, the level degeneracy increases rapidly. Second, the transition form factors themselves exhibit a peculiar "shrinking" behavior, viz., the rms radius of the transition form factor decreases with increasing resonance mass, roughly as the Compton wavelength of the final state. As a result, the differential cross section approaches a multiple of the point cross section as both the invariant momentum transfer squared and the energy transfer tend to infinity in such a way that their ratio remains finite.

While the "shrinking" of the transition form factors can be easily motivated on the basis of various nonrelativistic models (in fact, the present authors conjectured such a behavior on the basis of nonrelativistic analogies), at present we know of no completely satisfactory explanation for it. Nevertheless, close examination of the existing data on resonance production and the investigations of Elitzur[4] lend strong support to this conjectured behavior, so we are going to accept it. The purpose of the present work is to generalize the resonance model to include neutrino-induced reactions. While such an extension seems to be a straightforward matter in principle, the situation becomes quite complicated due to the presence of the nonconserved axial-vector current and great care has to be exercised in choosing the appropriate form factors.

The effort is worthwhile, however. In particular – *assuming that the leptonic part of the differential cross section is known* – the vector–axial-vector interference term yields essentially the same type of information about the hadronic part as a polarized-beam–polarized-target experiment does in electron scattering.

The present work is based on the same physical assumptions as our previous investigations of inelastic electron scattering. For the reader's convenience, we briefly summarize them here; for details, he is referred to Ref. 2.

(i) *Resonance spectrum.* The baryon resonances exhibit a structure characteristic of a symmetric quark model with orbital excitations, i.e., *on the average,*

$$M_n{}^2 \approx m^2(1+n) \quad (n=0, 1, 2, \ldots),$$

where m is the nucleon mass and M_n stands for the mass of the nth excitation. Also, $n=0, 2, \ldots$ and $n=1,3,\ldots$ correspond to $I=\frac{1}{2}$ and $I=\frac{3}{2}$ nonstrange resonances, respectively. The total widths satisfy Suranyi's formula

$$M_n\Gamma_n \approx 0.13nm^2 \quad (n \gg 1).$$

(ii) The *level degeneracy* increases with n. A convenient way of taking this fact into account is to define (as in nuclear physics) average transition matrix elements and a strength function. The latter is assumed to behave as $fn^{-1/2}$ for high excitations, where f is a phenomenological parameter. We assume that, on the average, exchange degeneracy between parity doublets holds.

(iii) The appropriately chosen *average transition form factors* have a "dipole" form, viz.,

$$F(q^2)=F(0)(1+r_n{}^2q^2)^{-2},$$

where $r_n{}^2\approx(m^2/m_\rho{}^2)n^{-1}$ (for $n\gg1$) in the case of vector form factors. A similar assumption will be made for the axial-vector form factors, with the difference, however, that we replace m_ρ by m_{A1}.

The plan of the paper is the following. In Sec. II we briefly review the pertinent kinematic relations and *summarize the results of the calculation.* The matrix elements of the currents, together with the appropriate generalization of the Goldberger-Treiman relations are obtained in Sec. III. In the same section we discuss the methods and sources from which the various coupling constants, entering the transition form factors, are determined. We outline the derivation of the structure functions from the matrix elements of the currents in Sec. IV. This derivation proceeds in exactly the same way as in the case of electroproduction, and the reader should be able to fill in the gaps using the technique described in Ref. 2. Section V contains the discussion of our results.

Even at the cost of some repetition, we intend to make Sec. II reasonably self-contained. The reader not interested in the (sometimes cumbersome) details of the calculation may read Secs. I, II, and V only.

Throughout the paper we use a metric with $g_{00}=1$, $g_{in}=-\delta_{in}$. The representation of the Dirac matrices used is one with $\gamma_i^\dagger=-\gamma_i$, $\gamma_0^\dagger=\gamma_0$, γ_0 diagonal.

II. KINEMATICS AND SUMMARY OF RESULTS

We consider the differential cross section of the process schematically depicted in Fig. 1. In a standard fashion, the differential cross section is expressed through the absorptive part of the forward scattering amplitude for weak-current–nucleon scattering. The latter is characterized by five structure functions (invariant amplitudes) chosen as follows[5]:

$$W_{\mu\nu}=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1+\frac{1}{m^2}\left(p_\mu-q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu-q_\nu\frac{p\cdot q}{q^2}\right)W_2-\frac{i}{2m^2}\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta W_3+\frac{q_\mu q_\nu}{m^2}W_4+\frac{p_\mu q_\nu+p_\nu q_\mu}{m^2}W_5\,. \tag{2.1}$$

T invariance is assumed to hold. The nucleon polarization has been summed over. Here p_α and q_α stand for the four-momenta of the nucleon and W, respectively. The invariant amplitudes depend on two independent kinematical invariants, which we choose conventionally as $\nu=(p\cdot q)/m$ and q^2, the invariant energy transfer and momentum transfer squared at the lepton vertex, respectively.

Neglecting the mass of the charged lepton, the differential cross section becomes

$$\frac{d^2\sigma^{\nu,\bar\nu}}{dq^2d\nu}=\frac{G^2E'}{2\pi mE}\left(\cos^2(\tfrac12\theta)W_2^{\nu,\bar\nu}+2\sin^2(\tfrac12\theta)W_1^{\nu,\bar\nu}\mp\frac{E+E'}{m}\sin^2(\tfrac12\theta)W_3^{\nu,\bar\nu}\right); \tag{2.2}$$

E, E', and θ are the energy of the neutrino, the energy of the charged lepton, and the scattering angle, respectively, measured in the laboratory system. The minus (plus) sign before W_3 refers to neutrino- (antineutrino-) induced reactions. G is the weak-interaction coupling constant.

As described in the Introduction, we approximate the amplitude by a sum over the nucleon-resonance contributions. The contribution of strangeness-changing currents – being proportional to $\sin^2\theta_c$, where θ_c is the Cabibbo angle – is neglected. Thus, only $I=\frac12$, $Y=1$ (nucleon) and $I=\frac32$, $Y=1$ (Δ) resonances contribute. Their contribution to the structure functions is denoted by $W_i^{(N)}$ and $W_i^{(\Delta)}$, respectively. Assuming, as usual, that the weak-interaction current is pure isovector, the ith structure function of a given initial state is expressed through $W_i^{(N)}$ and $W_i^{(\Delta)}$ as summarized in Table I.

After performing the summation over resonances, we obtain the following results.

In the deep-inelastic region, with $Q^2=-q^2$,

$$\nu\to\infty,\quad Q^2\to\infty,\quad \omega=2m\nu/Q^2=O(1),$$

the structure functions entering Eq. (2.2) can be expressed as functions of ω multiplied by certain

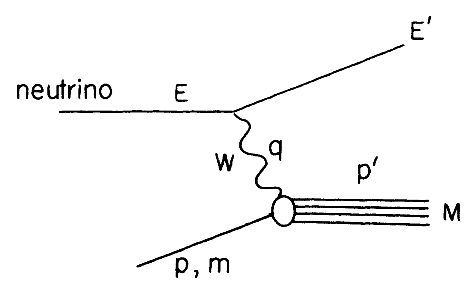

FIG. 1. Kinematics of inelastic neutrino scattering.

TABLE I. Contribution of nucleon and Δ intermediate states to neutrino-induced reactions.

Initial state	Composition of structure functions
νp	$3W_i^{(\Delta)}$
νn	$W_i^{(N)}+W_i^{(\Delta)}$
$\bar{\nu} p$	$W_i^{(N)}+W_i^{(\Delta)}$
$\bar{\nu} n$	$3W_i^{(\Delta)}$

powers of ν ("scaling"). Specifically, we define the following functions:

$$
\begin{aligned}
f_V(\omega)&=\frac{\omega(\omega-1)^3}{(\omega-1+r_V{}^2)^4},\\
f_A(\omega)&=\frac{\omega(\omega-1)^3}{(\omega-1+r_A{}^2)^4},\\
f_{VA}(\omega)&=\frac{\omega(\omega-1)^3}{(\omega-1+r_V{}^2)^2(\omega-1+r_A{}^2)^2},\\
S_V(\omega)&=\tfrac{1}{4}\left(1+\lambda^2\,\frac{(\frac{1}{2}r_V+3)\omega-3}{(\frac{1}{2}r_V+1)\omega-1}\right),\\
S_A(\omega)&=\tfrac{1}{4}\left(1+\lambda^2\,\frac{(\frac{1}{2}r_A+3)\omega-3}{(\frac{1}{2}r_A+1)\omega-1}\right),\\
S_{VA}(\omega)&=\tfrac{1}{4}\left(1+\lambda^2\,\frac{[\frac{1}{2}(r_Ar_V)^{1/2}+3]\,\omega-3}{[\frac{1}{2}(r_Ar_V)^{1/2}+1]\,\omega-1}\right),
\end{aligned}
\qquad (2.3)
$$

where $r_V=m/m_\rho$, $r_A=m/m_A$, m_ρ and m_A being approximately equal to the masses of the ρ and $A1$ mesons, respectively. The parameter λ has the same meaning as in Ref. 2; it accounts for the symmetry breaking in the quark model, in the sense that $\lambda=1$ corresponds to complete symmetry, while experimentally $\lambda\approx 0.93$.

$$W_1^{(N)}=f[(g_3^{(N)})^2f_V(\omega)+(h_3^{(N)})^2f_A(\omega)], \qquad (2.4a)$$

$$\nu W_2^{(N)}=\frac{2}{\omega}W_1^{(N)}, \qquad (2.4b)$$

$$\nu W_3^{(N)}=4g_3^{(N)}h_3^{(N)}ff_{VA}(\omega), \qquad (2.4c)$$

$$W_1^{(\Delta)}=f[(g_3^{(\Delta)})^2f_V(\omega)S_V(\omega)+(h_3^{(\Delta)})^2f_A(\omega)S_A(\omega)], \qquad (2.5a)$$

$$\nu W_2^{(\Delta)}=\frac{2}{\omega}W_1^{(\Delta)}, \qquad (2.5b)$$

$$\nu W_3^{(\Delta)}=4g_3^{(\Delta)}h_3^{(\Delta)}ff_{VA}(\omega)S_{VA}(\omega). \qquad (2.5c)$$

The quantities $g_1^{(N)},\ldots,h_3^{(\Delta)}$ play the role of coupling constants; their values have been determined from independent, low-energy experiments. The details of this procedure are described in Sec. IV. In Table II their numerical values, as used in the evaluation of the structure functions, are summarized.

From Eqs. (2.4a), (2.4b) and (2.5a), (2.5b), we can recover the corresponding electroproduction structure functions if we set all axial-vector coupling constants (h_i) equal to zero and the vector coupling constants ($g_i^{(N)}$) are changed so as to include the isoscalar contribution in the nucleon channel, i.e., $g_1^{(\mathrm{proton})}=1$, $g_1^{(\mathrm{neutron})}=0$, $g_2^{(\mathrm{neutron})}=g_2^{(\mathrm{proton})}=0$, $g_3^{(\mathrm{neutron})}=\mu_n=-1.91$, $g_3^{(\mathrm{proton})}=\mu_p=2.78$.

With increasing ν and Q^2, the structure functions approach the scaling limit as rapidly as in the case of electroproduction. We observe that the Callan-Gross[6] relation $\nu W_2=(2/\omega)W_1$ is satisfied in the scaling limit.

In Figs. 2 and 3 we plotted the curves $W_i^{(N)}$ and $W_i^{(\Delta)}$ from which, using Table I, the structure functions can be calculated for all neutrino-induced reactions.

TABLE II. Numerical values of the coupling constants.

Symbol	Value	Remark
$g_1^{(N)}$	1.0 [a]	Charge of proton
$g_3^{(N)}$	4.7	Isovector magnetic moment of nucleon
$h_1^{(N)}$	$-\frac{1}{4}h_3^{(N)}$ [a]	Eq. (3.13)
$h_3^{(N)}$	-1.2	"Axial-vector renormalization constant"
$g_1^{(\Delta)}$	0.1 [a]	Static limit of Coulomb transition form factor in $\gamma N\to\Delta$
$g_2^{(\Delta)}$	$0.93g_3^{(\Delta)}$	
$g_3^{(\Delta)}$	3.3	$\gamma N\to\Delta$ "static" magnetic moment
$h_1^{(\Delta)}$	1.25 [a]	PCAC
$h_2^{(\Delta)}$	$-h_3^{(\Delta)}$	Quark-model prediction
$h_3^{(\Delta)}$	-0.75	

[a] Does not enter the expression of structure functions in scaling limit.

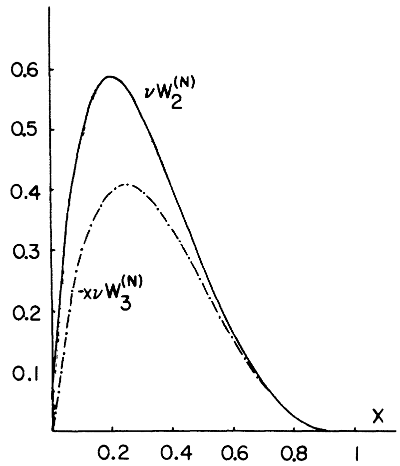

FIG. 2. Contributions of the nucleon trajectory to the structure functions W_i from Eqs. (2.4a), (2.4b), and (2.4c).

In Fig. 4 we plotted the quantity

$$\frac{dn}{dx} = \tfrac{1}{2}(\nu W_2 + \tfrac{2}{3}x\, W_1 - \tfrac{2}{3}x\nu W_3)$$

as a function of $x = \omega^{-1}$, where

$$W_i = \tfrac{1}{2}(W_i^{(N)} + 4W_i^{(\Delta)}) .$$

Thus, dn/dx is proportional to the event rate, averaged over protons and neutrons, as measured in the CERN neutrino experiment. For comparison, the data taken from Ref. 7 are plotted together with the theoretical curve. [The normalization of the experimental data is fixed by one theoretical prediction – the "PCAC (partial conservation of axial-vector current) limit" – plotted together with the data.] Figure 5 shows the distribution of the quantity ν/E (which measures the integrals over the

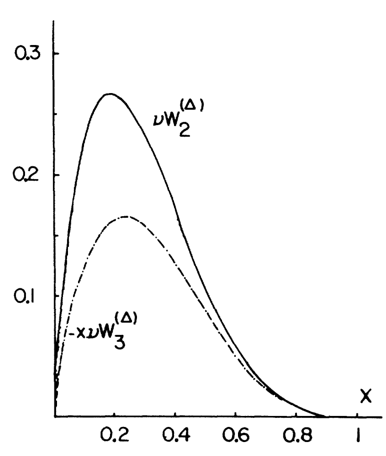

FIG. 3. Contributions of the Δ trajectory to the structure functions W_i from Eqs. (2.5a), (2.5b), and (2.5c).

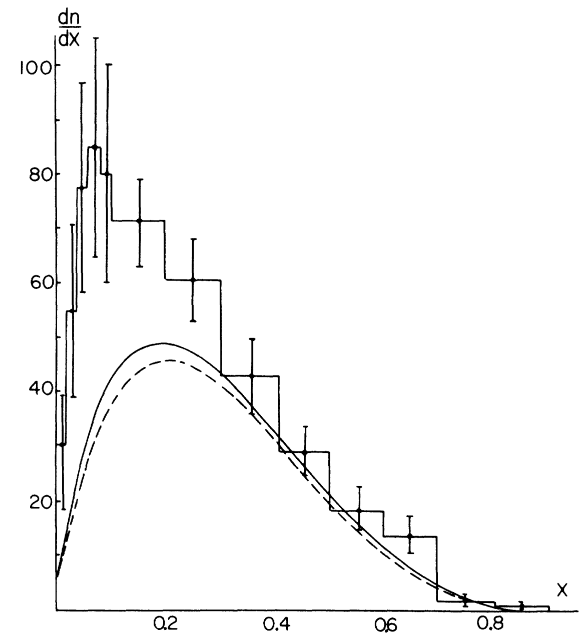

FIG. 4. The event rate averaged over protons and neutrons. The solid curve is obtained by using $h_3^{(N)} = -1.4$ as suggested by the analysis of Nambu and Yoshimura (Ref. 8). The dashed curve is calculated with $h_3^{(N)} = -1.2$, i.e., the static value of the axial-vector renormalization constant. The experimental points are taken from Ref. 7.

structure functions), together with data taken from the same source.[7] In Fig. 6 we also plotted the quantity xW_3/W_2, with the structure functions averaged over protons and neutrons. We wish to emphasize that our theoretical curves are absolute predictions since the only free parameter of the theory (occurring in our approximation to the

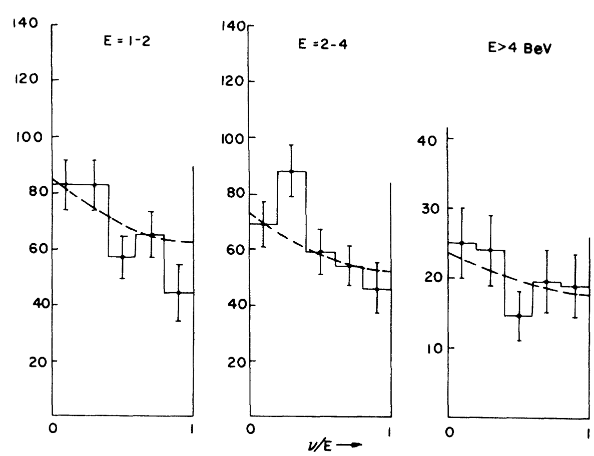

FIG. 5. Distribution of ν/E, averaged over protons and neutrons for various values of neutrino energy. Data from Ref. 7.

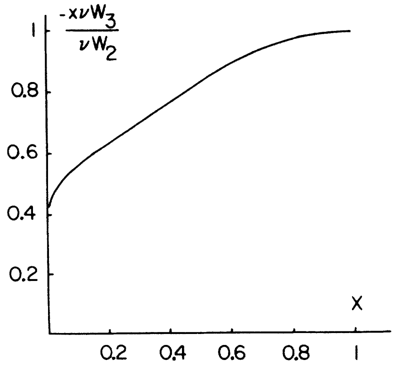

FIG. 6. The ratio of the structure functions $x\nu W_3/\nu W_2$ averaged over protons and neutrons.

strength function) is fixed by our previous fit to the electroproduction data, $f \approx 0.15$. At subasymptotic energies the variable x should be replaced by

$$x' = Q^2/(m^2 + 2m\nu),$$

which is the scale variable given by the resonance model; cf. Ref. 2.

By using the well-known expression[5] for the total cross sections,

$$\sigma^{(\nu,\bar{\nu})} \sim \frac{G^2 m E}{\pi} \int_0^1 dx\left(\tfrac{1}{2}\nu W_2^{(\nu,\bar{\nu})} + \tfrac{1}{3} x W_1^{(\nu,\bar{\nu})} \mp \tfrac{1}{3} x\nu W_3^{(\nu,\bar{\nu})}\right) \quad (E \to \infty),$$

one can estimate the ratios of total cross sections for various neutrino-induced reactions. We find

$$\frac{\sigma(\bar{\nu}p)}{\sigma(\nu p)} = 0.53, \quad \frac{\sigma(\nu n)}{\sigma(\nu p)} = 1.1, \quad \frac{\sigma(\bar{\nu}n)}{\sigma(\nu p)} = 0.5.$$

It should be emphasized that these are the ratios of the *nondiffractive parts* of the total cross sections, since, presumably, the diffractive part cannot be represented by a sum over resonances.

From the point of view of a resonance model, the quantities to be compared directly with experiments are those combinations of the cross sections which do not contain an $I = 0$ part in the t channel (and hence are free of diffractive contributions). For these we find

$$\left.\begin{aligned} &\sigma(\nu p) - \sigma(\bar{\nu}p) \sim 0.14\, G^2 mE/\pi, \\ &\sigma(\nu n) - \sigma(\bar{\nu}n) \sim 0.19\, G^2 mE/\pi \end{aligned}\right\} \quad (E \to \infty).$$

We also mention in passing an interesting relation we obtained from the generalized Goldberger-Treiman-Nambu (GTN) relations discussed in Sec. III. We find that the coupling constants of the "axial-Coulomb" and "axial-magnetic" transition matrix elements of the $I = \frac{1}{2}$ channel (h_1 and h_3, respectively), satisfy the equation

$$2h_1 + h_3 = 0.$$

This relation can be tested either in neutrino production of nucleon resonances, or – via a Nambu-Yoshimura[8] analysis – in reactions of the type $\gamma N \to \pi N^*$, where N^* is a (preferably low-lying) nucleon resonance.

III. MATRIX ELEMENTS OF CURRENTS

We parametrize the matrix elements of currents in essentially the same way as in Ref. 2. As far as the vector current is concerned, the parametrization of the axial-vector current is slightly more complicated, since the latter is not conserved. We, therefore, split the axial-vector current $A_\mu(x)$ into a solenoidal and an irrotational part:

$$\begin{aligned} &A_\mu(x) = A_\mu^T(x) + \partial_\mu \phi(x), \\ &\partial_\mu A_\mu^T = 0. \end{aligned} \tag{3.1}$$

The matrix elements of A_μ^T can be parametrized in the same fashion as those of the conserved vector current, $V_\mu(x)$, apart from a factor γ_5. Therefore, we obtain the following for a *normal* transition to a resonance of spin j and helicity λ:

$$\begin{aligned} \langle p' j \lambda' | V_\mu(0) + A_\mu^T(0) | p \tfrac{1}{2} \lambda \rangle = \bar{\psi}_{\lambda'}(p')^{\mu_1 \cdots \mu_{j-1/2}} q_{\mu_2} \cdots q_{\mu_{j-1/2}} \Big\{ & q_{\mu_1}[q_\mu(p \cdot q) - p_\mu q^2](v_1 + \gamma_5 a_1) + \frac{2}{m} \epsilon_{\mu_1 \alpha\beta\gamma} p^\alpha q^\beta S_\mu^\gamma (v_2 + \gamma_5 a_2) \\ & + i\frac{M}{m} q_{\mu_1} \gamma_\rho S_\mu^\rho \gamma_5 [v_2 + v_3 + \gamma_5(a_2 + a_3)] \Big\} u_\lambda(p). \end{aligned} \tag{3.2}$$

Here $p\lambda$, $p'\lambda'$ stand for the four-momenta and helicities of the nucleon and resonance (of mass M), respectively, and $q_\mu = p'_\mu - p_\mu$. The tensor $S_{\mu\gamma} = \epsilon_{\gamma\nu\lambda\mu} p^\nu q^\lambda$, while ψ and u stand for the Rarita-Schwinger and Dirac amplitudes of the resonance and nucleon, respectively. The parametrization of the longitudinal part of $A_\mu(x)$ is trivial:

$$\langle p' j \lambda' | \partial_\mu \phi(0) | p \tfrac{1}{2} \lambda \rangle = q_\mu \bar{\psi}_{\lambda'}^{\mu_1 \cdots}(p') q_{\mu_1} \cdots \gamma_5 u_\lambda(p) a_4. \tag{3.3}$$

The matrix elements of an abnormal transition differ from those of a normal transition by an over-all factor γ_5, the functions a_i, v_i are real because of T invariance. (These form factors correspond essentially to the helicity amplitudes in the reference frame where $q_0=0$, $|q|=\sqrt{Q^2}$.)

The dipole approximation for the form factors is very well established for the nucleon and the $\Delta(1236)$. In order to implement the universality assumption for the production form factors of all nucleons and Δ resonances, it is necessary to extract certain kinematical factors from the matrix element. The way in which these kinematical factors are chosen defined what we mean by universal form factors. Although our choice is not unique, a large amount of arbitrariness is removed when we require that the correct threshold behavior for the production of the lowest excitations be obeyed. The residual arbitrariness amounts merely to a redefinition of the strength function. Having extracted the kinematic factors, the remaining "form factors," (denoted by $V_i^{(N)}, \ldots, A_i^{(\Delta)}$, respectively), will be approximated by dipoles, according to assumption (iii) in Sec. I. We thus write

$$\begin{aligned}
&V_1^{(N)} = (Mm)^{1/2}[Q^2+(M+m)^2][Q^2+(M-m)^2]\,v_1^{(N)}/4mM\,,\\
&A_1^{(N)} = (Mm)^{1/2}Q^2[Q^2+(M-m)^2]\,a_1^{(N)}/4m(M-m)\,,\\
&\left.\begin{aligned}&V_i^{(N)} = (Mm)^{1/2}[Q^2+(M+m)^2]\,v_i^{(N)}/2\,,\\ &A_i^{(N)} = (Mm)^{1/2}[Q^2+(M-m)^2]\,a_i^{(N)}/2\,;\end{aligned}\right\}\ i=2,3
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
&V_1^{(\Delta)} = \left(\frac{M}{2m}\right)^{1/2}\frac{[Q^2+(M+m)^2][Q^2+(M-m)]^2}{4M^2}\left(\frac{2l+4}{2l+3}\right)^{1/2} v_1^{(\Delta)}\,,\\
&A_1^{(\Delta)} = \left(\frac{M}{2m}\right)^{1/2}\frac{[Q^2+(M+m)^2][a^2+(M-m)^2]Q^2}{4M(M-m)}\left(\frac{2l+4}{2l+3}\right)^{1/2} a_1^{(\Delta)}\,,\\
&\left.\begin{aligned}&V_i^{(\Delta)} = -(2Mm)^{1/2}\frac{[Q^2+(M+m)^2][Q^2+(M-m)^2]}{4M}\left(\frac{2l+4}{2l+3}\right)^{1/2} v_i^{(\Delta)}\,,\\
&A_i^{(\Delta)} = -(2Mm)^{1/2}\frac{[Q^2+(M+m)^2][Q^2+M-m)^2]}{4M}\left(\frac{2l+4}{2l+3}\right)^{1/2} a_i^{(\Delta)}\,.\end{aligned}\right\}\ i=2,3
\end{aligned}\tag{3.5}$$

The symbol l stands for the orbital angular momentum of the resonance in the quark model. Thus, e.g., $l=0$ for the nucleon and $\Delta(1236)$, etc. For the functions V_i, A_i, we assume the universal forms[9]:

$$V_i^{(N,\Delta)}(Q^2,M^2) = \frac{g_i^{(N,\Delta)}}{[1+r_V^{\,2}(Q^2/M^2)]^2}\,, \tag{3.6}$$

$$A_i^{(N,\Delta)}(Q^2,M^2) = \frac{h_i^{(N,\Delta)}}{[1+r_A^{\,2}(Q^2/M^2)]^2}\,. \tag{3.7}$$

The kinematic factors extracted have been chosen so as to take care of threshold and pseudothreshold singularities at the first resonances. (We believe that for higher resonances – because of the increasing level degeneracy – the effect of the threshold factors is largely washed out.) It is easy to check that for $M\to m$, the vector form factors $V_{1,3}^{(N)}$ go over to the isovector Sachs form factors $G_{E,M}$, whereas $V_2^{(N)}$ has to be put equal to zero. Similarly, $A_3^{(N)}|_{M=m}=G_A(q^2)$, the conventional axial-vector form factor of the nucleon[10]; $A_1^{(N)}|_{M=m}=A_1^{(N)}|_{M=m}=0$.

The factors $(2l+4)^{1/2}(2l+3)^{-1/2}$ have been inserted in order to uniformize the summation over resonances; the role they play will become clear in Sec. IV.

We require that the matrix elements of the axial-vector currents satisfy appropriately generalized GTN relations. The procedure we follow can be easily explained on the matrix elements of $A_\mu(0)$ between nucleons. This is written conventionally as

$$\langle p'|A_\mu(0)|p\rangle = \bar{u}(p')[\gamma_\mu F_1(q^2)+q_\mu F_2(q^2)]\gamma_5 u(p)\,. \tag{3.8}$$

Following Nambu's reasoning, we first require that $A_\mu(x)$ be divergenceless. This gives

$$F_2(q^2) = -\frac{2m}{q^2}F_1(q^2)\,,$$

and there appears a pole at $q^2=0$ signifying the presence of a Goldstone boson. Next, "spoil" this relation by giving a finite mass, μ, to the pion,

$$F_2(q^2) = -\frac{2m}{q^2-\mu^2}F_1(q^2)\,, \tag{3.9}$$

so that

$$q^\mu\langle p'|A_\mu(0)|p\rangle = -\frac{2m\mu^2}{q^2-\mu^2}F_1(q^2)\bar{u}(p')\gamma_5 u(p)\,.$$

Now split the axial-vector current, with a *massive* pion, into solenoidal and irrotational parts; i.e., write

$$\langle p'|A_\mu(0)|p\rangle = \bar{u}(p')[\gamma_\mu - (2m/q^2)q_\mu]\gamma_5 u(p)F_1(q^2) - 2m q_\mu \frac{\mu^2}{q^2(q^2-\mu^2)}\bar{u}(p')\gamma_5 u(p)F_1(q^2)\,. \tag{3.10}$$

Clearly, the "Goldstone pole" q^{-2} is canceled as long as $\mu^2 \neq 0$, whereas it reappears if we let $\mu^2 \to 0$ with q^2 fixed. The form (3.10) can now be generalized for matrix elements between arbitrary states.

Replace the divergenceless and irrotational parts by our parametrization [Eqs. (3.2) through (3.7)] of the matrix element. Looking at the decomposition (3.2), (3.3) of the matrix element, we realize that as long as $M \neq m$, the longitudinal part of the axial-vector current entering the GTN relation can come from a_1 only. Therefore, we inserted the "Goldstone pole" into the defining relation of the form factors A_1 in Eqs. (3.4) and (3.5). The generalized GTN relation now reads

$$\lim_{Q^2\to 0} Q^2\left(a_1\frac{\mu^2(p\cdot q)}{Q^2+\mu^2}\right) - a_4 = 0 \quad (M\neq m)\,. \tag{3.11}$$

If $M=m$ (the matrix element between nucleons), the function of a_1 vanishes as a consequence of C and T invariance and $a_2=0$ because of angular momentum selection rules. However, a pole at $Q^2=0$ appears in a_3, as can be seen from Eqs. (3.5) and (3.7). Therefore, after some simple algebra, the GTN relation for the matrix element between nucleons is obtained in the form

$$\lim_{Q^2\to 0} Q^2\left(\frac{\mu^2 m^2}{Q^2+\mu^2}\right)a_3 - a_4 = 0 \quad (M=m)\,. \tag{3.12}$$

Equation (3.12) is just the classical GTN relation.

Two remarks are in order here. First, the somewhat puzzling fact that the "Goldstone pole" moves from one form factor to the other as we go to the nucleon, signifies that the decomposition of the matrix element we use is not convenient from the point of view of writing down the generalized GTN relations. However, since the decomposition used here is very convenient from the point of view of calculating cross sections, we continue to use it. Second, one observes that the longitudinal matrix element, proportional to a_4, does not enter the expression of the cross section [see Eqs. (4.2)], since we neglected the mass of the charged lepton – a good approximation for high ν and Q^2. (This means among other things that we do not have to worry about off-mass-shell corrections to the pion-pole term in a_4.) Nevertheless, the generalized GTN relations play an important role, since they demand the presence of a "Goldstone pole" in a_1 – which in turn influences the behavior of the latter in the scaling limit as well.

The generalized GTN relations (3.11) and (3.12), combined with a continuity argument, give rise to an interesting relation between the axial-vector coupling constants. Indeed, using Eqs. (3.4) through (3.7), we find from (3.11) for $I=\frac{1}{2}$

$$\lim_{\substack{Q^2\to 0\\ M\neq m}} Q^2 a_4 = -4mh_1 \quad (M-m \ll m)\,,$$

$$\lim_{\substack{Q^2\to 0\\ M=m}} Q^2 a_4 = 2mh_3\,.$$

Apart from threshold singularities, the matrix element of the longitudinal part of the axial-vector current has to be a smooth function of the external mass in the neighborhood of $M=m$. This gives the relation for the $I=\frac{1}{2}$ channel:

$$2h_1 + h_3 = 0\,. \tag{3.13}$$

(There is no physical state with $M=m$ in the $I=\frac{3}{2}$ channel.)

Next we come to the determination of the coupling constants g_i and h_i entering Eqs. (3.6) and (3.7). The symmetric quark model gives immediately $g_2^{(N)} = h_2^{(N)} = 0$, and using the universality hypothesis (iii) in Sec. I we extrapolate this for all $I=\frac{1}{2}$ resonances. The nonvanishing coupling constants are determined as follows. [Assumption (iii) is used everywhere.]

(a) *Nucleon resonances.* $g_1^{(N)}=1$ (charge form factor), $g_3^{(N)}=\mu_1$ (isovector magnetic moment of nucleon). $h_3^{(N)}$ is the static axial-vector coupling constant of the nucleon, determined, e.g., from the Adler-Weisberger sum rule. The constant $h_1^{(N)}$ could be directly determined from neutrino production of individual $I=\frac{1}{2}$ resonances; lacking sufficiently accurate data, we use Eq. (3.13) to fix its value.

(b) Δ *resonances.* We write $g_2^{(\Delta)}=\lambda g_3^{(\Delta)}$, where $\lambda=1$ from the symmetric quark model. Analysis of photoproduction data[11,2] suggests $\lambda\approx 0.93$; we use the latter value. The value of $g_3^{(\Delta)}$ is extracted from Δ photoproduction data.[11] (The symmetric quark model predicts a value too low by a few percent; the difference *is* significant.) The constant $g_1^{(\Delta)}$ was determined by Albright and Liu.[12] The symmetric quark model gives $h_2^{(\Delta)} = -h_3^{(\Delta)}$; using this, Albright and Liu's analysis[12] of neutrino production of $\Delta(1236)$ determines $h_2^{(\Delta)}$. The constant $h_1^{(\Delta)}$ can be estimated from the width of $\Delta(1236)$ via PCAC.

The numerical values of the constants used in the calculations are listed in Table II.

IV. CALCULATION OF THE STRUCTURE FUNCTIONS

The calculation of the structure functions from the matrix elements of currents proceeds in two steps.

Step 1. Calculate the structure functions for individual resonances. This step is the same as in the case of electroproduction. The only essential observation to be made is the following. In calculating the matrix element of an *axial*-vector current for a *normal* transition, we have an extra factor of γ_5 with respect to the vector matrix element. Therefore, the former matrix element can be calculated in the same way as a *vector* matrix element for an *abnormal* transition (cf. Sec. III) and – *mutatis mutandis* – the same rule applies for the calculation of the axial-vector matrix element for an abnormal transition.

Denote the structure functions for individual resonance production by w_i $(i=1,\ldots,5)$. The contribution of a single resonance to the absorptive part of the forward current-hadron scattering amplitude is then given by

$$w_{\mu\nu}=\tfrac{1}{2}\sum_{\lambda,\lambda'}|\langle p'j\lambda'|V_\mu^{\pm}(0)+A_\mu^{\pm}(0)|p\tfrac{1}{2}\lambda\rangle|^2, \quad (4.1)$$

where $V_\mu^{\pm}(0)+A_\mu^{\pm}(0)$ are the appropriate isospin components of the weak currents (e.g., the $I_3=-1$ component for a $p\bar{\nu}$ or $n\nu$ reaction). Upon inserting Eqs. (3.2) and (3.3) into (4.1) and performing the summation over the helicities λ and λ', we can project out the structure functions w_i, defined by Eq. (2.1). Below we summarize the results.

Normal transitions.

$$\begin{aligned}
w_1^{(n)}&=M^2k^2\left[\beta\left(v_3^{\,2}+\frac{2j+3}{2j-1}v_2^{\,2}\right)+\delta\left(a_3^{\,2}+\frac{2j+3}{2j-1}a_2^{\,2}\right)\right],\\
w_2^{(n)}&=\frac{Q^2}{M^2k^2}w_1^{(n)}+\frac{Q^4}{M^2}(\beta v_1^{\,2}+\delta a_1^{\,2}),\\
w_3^{(n)}&=\frac{4m^2M}{\epsilon+m}k^2\beta\left(v_3a_3-\frac{2j+3}{2j-1}v_2a_2\right), \qquad (4.2)\\
w_4^{(n)}&=\delta\frac{m^6}{M^2}\left(a_4^{\,2}+2\frac{p\cdot q}{m^2}a_1a_4\right),\\
w_5^{(n)}&=\delta\frac{m^4Q^2}{M}a_1a_4.
\end{aligned}$$

Abnormal transitions. For $i\neq 3$, $w_i^{(a)}$ is obtained from $w_i^{(n)}$ by the substitution $\beta\longleftrightarrow\delta$; for $i=3$,

$$w_3^{(a)}=w_3^{(n)}. \qquad (4.3)$$

We used the following notation[13]:

$$\begin{aligned}
k^2&=\frac{1}{4M^2}[Q^2+(M-m)^2][Q^2+(M+m)^2],\\
\epsilon+m&=\frac{1}{2M}[Q^2+(m+M)^2],\\
\beta&=\frac{M^3(\epsilon+m)k^{2j-1}}{m^2 2^{j+3/2}}\,\frac{(2j+1)!!}{(2j)!!}, \qquad (4.4)\\
\delta&=\beta k^2/(\epsilon+m)^2.
\end{aligned}$$

Step 2. Sum over the resonances. In order to sum the structure functions over the spectrum, we would have to know the precise form of the latter, together with the expressions of the level density. As we have repeatedly emphasized, however, present-day inelastic scattering experiments are not refined enough to detect such details. Moreover, in the region where resonances strongly overlap, the detection of such details is impossible *in principle* and only certain average quantities have a physical significance. Nevertheless, in order to guide one's intuition, it is advisable to keep in mind some specific excitation scheme compatible with, and suggested by, the observed baryon spectrum. Imagine, therefore, a quark model with "orbital" (including the possibility of radial) excitations. If the total orbital angular momentum of the excitation is l, the allowed values of the total angular momentum are $j=l\pm\frac{1}{2}$ for nucleons, $j=l-\frac{3}{2},\ldots,l+\frac{3}{2}$ for Δ's. Moreover, the observed spectrum suggests the existence of a higher degeneracy, the total mass depending on a "principal quantum number," N, and l running through the integers from 0 to N. (The degeneracy may be still higher, due to the existence of other, so far undetected quantum numbers.) We write the empirical mass formula as follows:

$$M^2=m^2(N+1),$$

where $N=2n$ corresponds to $I=\frac{1}{2}$, $N=2n+1$ to $I=\frac{3}{2}$ $(n=0,1,2,\ldots)$.

In order to uniformize the summation over the resonances, we prefer to sum over the orbital excitations rather than the total angular momentum. We can write uniformly

$$j=l+\tfrac{1}{2}\quad(I=\tfrac{1}{2}),$$

$$j=l+\tfrac{3}{2}\quad(I=\tfrac{3}{2}),$$

if we allow l to run through some "nonsense" (negative integer) values as well. In terms of l rather than j, the j-dependent factors in the expressions (4.2) of the structure functions become

$$\frac{(2j+1)!!}{(2j)!!}=\begin{cases}\dfrac{(2l+2)!!}{(2l+1)!!} & (I=\frac{1}{2})\\[2ex] \dfrac{(2l+2)!!}{(2l+1)!!}\times\dfrac{2l+4}{2l+3} & (I=\frac{3}{2})\end{cases}$$

and (4.5)

$$\frac{2j+3}{2j-1}=\frac{l+3}{l+1}.$$

(The last factor does not occur for $I=\frac{1}{2}$.) In order to obtain the usual expression for the strength functions of the Δ states, we choose to absorb the factors $(2l+4)^{1/2}(2l+3)^{-1/2}$ in the defining relations of the $I=\frac{3}{2}$ transition form factors, and also the appropriate powers of k^2; see Eq. (3.5). We are now ready to put our results together.

Let us characterize each resonance by I, N, l, α, where α stands for the collection of all the other quantum numbers necessary to obtain a complete set of state labels. Denote for a moment the corresponding structure functions in (4.2) by $w_i(INl\alpha)$. W_i is then given by

$$W_i^{(I)} = \sum_{N,l,\alpha} \rho(N,l,\alpha)\frac{M_N{}^2\Gamma_N{}^2}{(s-M_N{}^2)^2+M_N{}^2\Gamma_N{}^2} \times [w_i^{(n)}(INl\alpha)+w_i^{(a)}(INl\alpha)], \qquad (4.6)$$

where $s=(p+q)^2$, $\rho(N,l,\alpha)$ is the density of states, and we used the assumption of average exchange degeneracy. Upon inserting here Eqs. (4.2) through (4.5) and using the relations (3.4) to (3.7), we observe that a common factor,

$$F(N)=\sum_{l,\alpha} \tfrac{1}{2}\rho(N,l,\alpha)\frac{k^{2l}}{2^{l+1}}\frac{(2l+2)!!}{(2l+1)!!}\frac{M^2}{m^2},$$

can be extracted from all the terms. In this factor we recognize the usual expression of a strength function and [see assumption (ii) in Sec. I] we approximate it by

$$F_N \sim f/\sqrt{N}\,.$$

The remaining factors which still contain l are slowly varying and are approximated by their semiclassical expressions by substituting

$$l \sim kR \sim kr/M,$$

where $r=r_V$ or r_A depending on whether the expression in question occurs in a vector or an axial-vector matrix element.

Finally, the summation over N is performed in the narrow-resonance approximation after replacing the Breit-Wigner factor by a δ function and integrating rather than summing over N. (Details of this summation procedure are described in our earlier works.[2]) The final result may be somewhat simplified if we observe that the factor

$$\left(\frac{kr_V+3M}{kr_V+M}\right)^{1/2} \times \left(\frac{kr_A+3M}{kr_A+M}\right)^{1/2}$$

occurring in the expression of w_3, can be well approximated by the expression

$$\frac{k(r_Vr_A)^{1/2}+3M}{k(r_Vr_A)^{1/2}+M}\,.$$

As a result, we obtain the expressions of the structure functions as listed in Sec. II [Eqs. (2.3) through (2.5)].

V. DISCUSSION

Our purpose in this paper was to investigate whether the basic ideas of a resonance model apply as well to neutrino scattering as they do to electroproduction. The satisfactory agreement between the theoretical curves presented in Sec. II and the experimental data suggests rather strongly that this is indeed the case. There is one rather startling prediction of the present model to which we wish to call attention at this point. We have emphasized in our previous paper that the "spin-independent" structure functions (W_1 and W_2) are rather insensitive to the details of various models and that it is the "spin-dependent" ones (W_3, W_4) which merit serious attention if one really wants to distinguish between models. In our calculations we found a substantial contribution from W_3 to the neutrino cross section. Moreover, the ratio xW_3/W_2 varies rather strongly with x. In our opinion, this effect may be detectable even with the present setup at CERN, and certainly by forthcoming experiments in Gargamelle and at the National Accelerator Laboratory.

One interesting question would be to decide whether scaling continues to hold down to very small values of x (large values of ω) or diffractive (Pomeranchuk) contributions begin to dominate there. The success of the resonance model both in electroproduction and neutrino scattering at values of x close to 1 – which is the "truly" deep-inelastic region in the present experiments – suggests that the scattering is mainly nondiffractive. It seems to the present authors that neither the data from SLAC nor the ones from CERN are accurate enough at present to give a definite answer to this question. Nevertheless, *some* information may be gained by considering certain moments of the structure functions.

Consider, in particular, the quantity

$$\langle x^{-1}\rangle_\xi = \int_\xi^1 \frac{dx}{x}\,\nu W_2(x) \Big/ \int_\xi^1 dx\,\nu W_2(x)\,, \qquad (5.1)$$

where ξ is some cutoff value. Evidently, the integrand in the numerator of (5.1) strongly emphasizes small values of x and hence can be considered as a rough measure of the diffraction scattering present. Using our theoretical expressions from Sec. II, we find

$$\langle x^{-1}\rangle_{\xi=0} = 5.7$$

with the structure functions averaged over neutrons and protons. This is to be contrasted with the values found by Myatt and Perkins[7] from the CERN data:

$$\langle x^{-1}\rangle_{\xi=0.08} \cong 4.7, \quad \langle x^{-1}\rangle_{\xi=0} \cong 10\,.$$

These results and the comparison of the theoretical event rate with the data suggest that the resonance model is at least approximately valid for about

TABLE III. Evaluation of sum rules.

	Sum rule	Quark-model commutator	Resonance model	Experiment
Adler sum rule (vector part)	$\int_0^1 \frac{dx}{x}[\nu W_2^V(\bar{\nu}p) - \nu W_2^V(\nu p)]$	1	0.9	
Adler sum rule (axial-vector part)	$\int_0^1 \frac{dx}{x}[\nu W_2^A(\bar{\nu}p) - \nu W_2^A(\nu p)]$	1	0.1	
Adler sum rule (total)	$\int_0^1 \frac{dx}{x}[\nu W_2(\bar{\nu}p) - \nu W_2(\nu p)]$	2	1	$\lesssim(2\pm1)$
Gross–Llewellyn Smith	$\int_0^1 dx[\nu W_3(\nu p) + \nu W_3(\nu n)]$	-6	-2.9	3 ± 2

$0.2 \leqslant x \leqslant 1$ and for the smallest momentum transfers some other mechanism (probably diffraction) takes over.[14] (This conclusion is not a very firm one due to the substantial errors of the experimental numbers quoted.) Qualitatively, this situation is quite similar to what we found in the case of electroproduction; cf. Ref. 2.

It is instructive to evaluate various sum rules in the framework of the resonance model. The sum rules express the predictions of various versions of the quark model; they serve essentially to compare the predictions of the quark and resonance models. We evaluated the Adler sum rules[15] and the Gross-Llewellyn Smith sum rule[16] in the scaling limit. The results are summarized in Table III. The Adler sum rule at $Q^2=0$ depends essentially on the validity of the quark *charge* algebra; we find that it is satisfied to within ≈10% in the resonance model (it involves the *longitudinal* part of the matrix element).

Unfortunately, the numbers quoted in column 4 of Table III are subject to considerable variations depending on the values of the low-energy parameters used. In particular, we find that the values of the integrals in the "vector" Adler sum rule change from ~0.4 to ~0.9 if the $I=\frac{3}{2}$ transition moment is changed from $\mu_\Delta=1.2\mu_p$ (experiment) to $\mu_\Delta=\frac{8}{9}\mu_p$ (quark model), respectively, whereas this variation has an almost negligible effect on the average event rate. (The integrals are found to be rather less sensitive to changes of the asymptotic behavior of the structure functions near $x=0$.)

Qualitatively, we conclude that the vector part of the Adler sum rule has much more chance to be satisfied in the resonance model, whereas it seems impossible to satisfy the axial-vector part without entirely destroying the agreement with the event distribution and the low-energy data. The experimental number quoted has such a large error that it makes any comparison illusory at present.

The Gross-Llewellyn Smith sum rule is almost certainly not satisfied in the resonance model (the integral involved is quite insensitive to small variations of the low-energy parameters and/or the asymptotic behavior). The experimental number quoted seems to favor the resonance model, although the error is again too large to draw definite conclusions.

Finally, we allow ourselves a few speculative remarks about possible relations between the parton and resonance models. Clearly, if partons (quarks?) really existed as some fundamental constituents of hadrons, then the present-day "naive" parton model would turn out to be but a "Weizsäcker-Williams approximation" to a future, more complete theory. On the other hand, as long as the energy of *internal* excitations of the quark system is sufficiently low with respect to some characteristic energy scale (like the quark mass), the sum over intermediate states in the expressions of the various reaction amplitudes can probably be well approximated by a sum over the low-lying excited states of the system. In this spirit, the parton and resonance models need not be contradictory at all; rather, they would correspond to complementary, "single-particle" and "collective" descriptions of the same phenomena. Nevertheless, as the previous discussion indicates, if this qualitative picture is to be taken seriously at all, the quark-parton or the resonance model (or probably both) may have to undergo serious changes (e.g., by taking into account correlation effects in the quark-parton model) before they become entirely compatible with each other.

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Determination of Asymptotic Parameters in the Statistical Bootstrap Model*

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The statistical bootstrap model predicts that the density of hadron states $\rho(m)$ approaches $cm^a e^{bm}$ asymptotically. We consider the consequences of extending the bootstrap condition in the model from asymptotic down to finite masses. This allows us to determine the parameters a, b, and c for various assignments of the hadron volume and low-mass input spectrum, and for the extreme cases of excluding all exotic particles or including all of them. In all cases, $a=-3$ for ρ_{tot} (summed over all internal quantum numbers). The parameter b varies somewhat from case to case but is always of order m_π^{-1}; thus we predict the maximum temperature $T_0 = b^{-1} \approx m_\pi$ in rough agreement with Hagedorn's empirical determination. The inclusion of exotic states has little effect on ρ_{tot} but does redistribute the partial level densities according to a simple rule. The predicted level densities (excluding exotic states) are compared with present data.

I. INTRODUCTION

The statistical model of Hagedorn[1] is a bootstrap: Hadrons are assumed to be made of hadrons, with the density of hadron states, $\rho_{\text{out}}(m)$, approaching the constituent level density $\rho_{\text{in}}(m)$ asymptotically to within a power of the mass,

$$\rho_{\text{out}}(m) \underset{m\to\infty}{\sim} c'\rho_{\text{in}}(m) m^{\alpha}\,. \tag{1}$$

We may call this the "*weak asymptotic bootstrap condition*."[2] It leads to the form

$$\rho(m) \underset{m\to\infty}{\sim} cm^a e^{bm} \tag{2}$$

for the hadron level density $dn/dm \equiv \rho(m)$, with $a \le -\frac{5}{2}$ in ρ_{in}.

Recently the statistical bootstrap discussion of the hadron spectrum has been reformulated[3] purely in terms of phase space. One instructive feature of this approach, in which temperature is never introduced, is to emphasize that the level density of Eq. (2) is independent of any assumptions concerning what happens in hadron collisions. One can, following Hagedorn, go on to assume that local thermal equilibrium is achieved during collisions, but this extra assumption is not needed to derive the particle spectrum.

In fact, the phase-space formulation of the model depends on only two assumptions:

(i) The *statistical assumption* that the level density $\rho_{\text{out}}(m)$ of hadrons is given by the phase space