Dimension of Scale-Invariance-Breaking Interactions*

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Assuming the hadronic energy density has the decomposition $\Theta_{00}=\overline{\Theta}_{00}+\delta+u$ (where $\overline{\Theta}_{00}$ is chiral- and scale-invariant, δ is chiral-invariant but violates scale invariance, and u violates both chiral and scale invariance), we find rigorous bounds on the dimensions of δ and u in the (actual) SU(2)-symmetric limit and in the SU(3)-symmetric limit. If we further assume that δ is a c number, we derive a new sum rule which leads to satisfactory bounds on the dimension of u in the SU(2)-symmetric limit but not in the SU(3)-symmetric limit. Saturation of this sum rule with a scalar meson having a mass of 700 MeV leads to a value of 1.9 for the dimension of u.

In a recent paper Li and Pagels[1] have derived an upper bound on the derivative of a certain combination of K_{l3} form factors. Subsequently their method was strengthened and generalized by several authors.[2,3] In the following we apply the same powerful analysis to derive bounds on the dimensions of scale-invariance-breaking interactions.

We begin the derivation by noting the Lehmann-Källén representation for the propagator of Θ, the trace of the symmetric energy-momentum tensor $\Theta_{\mu\nu}$:

$$\begin{aligned}\Delta(q^2)&=i\int d^4x\, e^{iqx}\langle 0|T[\Theta(x)\Theta(0)]\,|0\rangle\\&=\int_{t_0}^{\infty}dt\,\frac{\rho_\Theta(t)}{t+q^2-i\epsilon}\,,\end{aligned}\tag{1}$$

where the positive spectral function is

$$\rho_\Theta(-k^2)=(2\pi)^3\sum_n\delta^{(4)}(p_n-k)\,|\langle 0|\Theta(0)|n\rangle|^2.$$

In an SU(2)-symmetric theory there are three states of two Cartesian pions which contribute at the threshold of ρ_Θ ($t_0=4m_\pi^2$); in the limit of SU(3) symmetry there are eight threshold states, each containing two pseudoscalar mesons with common octet mass, m_0 ($t_0=4m_0^2$). In each case the intermediate states with lowest mass contribute the following terms to ρ_Θ:

$$\rho_T(t)=\frac{N}{32\pi^2}\,\frac{|H(t)|^2(t-t_0)^{1/2}}{t^{1/2}}\,,\tag{2}$$

where $H(t)$ is the "mechanical" form factor[4] of the pion,

$$(4p_0p_0'V^2)^{1/2}\langle 0|\Theta(0)|\pi^+(p),\pi^-(p')\rangle=H(-(p+p')^2)\,,$$

and

$$N=\begin{cases}3 & \text{for SU(2) symmetry}\\ 8 & \text{for SU(3) symmetry.}\end{cases}$$

Since each intermediate state contributes a positive amount to the spectral function, we have $\rho_\Theta(t)\geq\rho_T(t)$ for $t\geq t_0$; therefore, Eq. (2) implies that, for $t\geq t_0$,

$$|H(t)|\leq\frac{4\pi\sqrt{2}}{\sqrt{N}}\,\frac{t^{1/4}}{(t-t_0)^{1/4}}\,[\rho_\Theta(t)]^{1/2}\,.\tag{3}$$

Thus, the form factor is bounded by the spectral function *in the region above threshold*. The techniques of Refs. 2 and 3 use the analytic properties of $H(t)$ to transform Eq. (3) into a similar inequality *in the region below threshold*. A straightforward application of those methods[5] leads to the result ($t<t_0$)

$$\begin{aligned}|H(t)|\leq(t_0-t)^{(n+1)/2}&\left[1+\left(\frac{t_0}{t_0-t}\right)^{1/2}\right]^{n+1/2}\\&\times\left(\frac{8\pi}{N}\int_{t_0}^{\infty}ds\,\frac{\rho_\Theta(s)}{s^n}\right)^{1/2},\end{aligned}\tag{4}$$

where n is an arbitrary real number. From the definition of $H(t)$ we know that $H(0)=\frac{1}{2}t_0$; therefore, at $t=0$, Eq. (4) becomes, for $n=1$ and $n=0$,

$$t_0\leq 16\left(\frac{\pi}{N}\int_{t_0}^{\infty}ds\,\frac{\rho_\Theta(s)}{s}\right)^{1/2},\tag{5a}$$

$$t_0^{3/2}\leq 8\left(\frac{\pi}{N}\int_{t_0}^{\infty}ds\,\rho_\Theta(s)\right)^{1/2}.\tag{5b}$$

These inequalities will be useful if we can evaluate the integrals of $\rho_\Theta(t)$. To do this we exploit the fact[6] that Θ is the divergence of the current associated with the generator of scale transformations (D):

$$\begin{aligned}&\partial_\mu D_\mu(x)=\Theta(x)\,,\\&D(x)=-i\int d^3x\,D_4(x)\,.\end{aligned}\tag{6}$$

We work in the context of a model of scale-invariance and chiral-symmetry breaking, which has been advocated by Gell-Mann and others.[7,8] The energy density is assumed to be the sum of three terms:

$$\Theta_{00}=\overline{\Theta}_{00}+\delta+u\,.$$

$\overline{\Theta}_{00}$ is invariant under scale transformations; that is, it transforms with dimension equal to four. δ and u are Lorentz scalars which break scale invariance; hence they transform with dimensions (d' and d, respectively) not equal to four:

$$\begin{aligned} i[D(0),\delta(0)]&=d'\delta(0)\,,\\ i[D(0),u(0)]&=du(0)\,. \end{aligned} \tag{7}$$

The $\overline{\Theta}_{00}$ and δ terms are invariant under chiral $SU(3)\otimes SU(3)$. Chiral-symmetry breaking is provided by u, which transforms[9] as the $(3,3^*)\oplus(3^*,3)$ representation of $SU(3)\otimes SU(3)$:

$$u=\epsilon_0(S^{(0)}+a\sqrt{2}\,S^{(8)})\,, \tag{8}$$

where[10] ($i=1,\ldots,8;\ j,k=0,1,\ldots,8$)

$$\begin{aligned} [F^{(i)}(x_0),S^{(j)}(x)]&=if_{ijk}S^{(k)}(x)\,,\\ [F^{(i)}(x_0),P^{(j)}(x)]&=if_{ijk}P^{(k)}(x)\,,\\ [F_5^{(i)}(x_0),S^{(j)}(x)]&=id_{ijk}P^{(k)}(x)\,,\\ [F_5^{(i)}(x_0),P^{(j)}(x)]&=-id_{ijk}S^{(k)}(x)\,. \end{aligned}$$

A fit to the pseudoscalar mass spectrum suggests that the physical value of a is about -0.89; in the SU(3) limit, $a=0$. For future reference we note that the above model satisfies the "virial" theorem[7]:

$$\Theta=(4-d')\delta+(4-d)u\,. \tag{9}$$

This equation and the covariance condition, $\langle 0|\Theta_{\mu\nu}|0\rangle=0$, give a useful relation between the vacuum expectation values of δ and u:

$$(4-d')\langle 0|\delta|0\rangle=-(4-d)\langle 0|u|0\rangle\,. \tag{10}$$

We are now in a position to determine the right-hand side of Eq. (5a) in terms of d' and d. We begin by combining Eqs. (1) and (6) and then using the standard Ward-identity techniques:

$$\begin{aligned} \int_{t_0}^{\infty}dt\,\frac{\rho_\Theta(t)}{t}&=-i\int d^4x\,\langle 0|T\{\partial_\mu D_\mu(x)\Theta(0)\}|0\rangle\\ &=-i\langle 0|[D(0),\Theta(0)]|0\rangle\,. \end{aligned} \tag{11}$$

The right-hand side can be evaluated[11] with the aid of Eqs. (9), (7), and (10):

$$\int_{t_0}^{\infty}dt\,\frac{\rho_\Theta(t)}{t}=-(d-d')(4-d)\langle 0|u(0)|0\rangle\,. \tag{12}$$

The vacuum expectation value of u can be determined in the following way.[12] Set $\xi_{0,8}=\langle 0|S^{0,8}(0)|0\rangle$, $b=(\frac{1}{2})^{1/2}\xi_8/\xi_0$, and $\gamma=-\frac{2}{3}(\epsilon_0\xi_0)$. Then, Eq. (8) gives

$$\langle 0|u(0)|0\rangle=-\tfrac{3}{2}\gamma(1+2ab)\,. \tag{13}$$

On the other hand, defining the spectral weights $\rho_{ij}(t,A)$ by

$$\begin{aligned} \rho_{ij}(-k^2,A)\equiv(2\pi)^3\sum_n\delta^{(4)}(p_n-k)\langle 0|\partial_\mu A_\mu^{(i)}(0)|n\rangle\\ \times\langle n|\partial_\nu A_\nu^{(j)}(0)|0\rangle\,, \end{aligned}$$

where $A_\mu^{(i)}(x)$ ($i=1,\ldots,8$) is an octet of axial-vector current densities,

$$F_5^{(i)}(x_0)=-i\int d^3x\,A_4^{(i)}(x)\,,$$

and setting

$$\tau_{ij}=\int_0^\infty dt\,\frac{\rho_{ij}(t,A)}{t}\,, \tag{14}$$

we know that

$$\tau_{33}=\gamma\,(1+a)(1+b)\geq 0\,, \tag{15a}$$

$$\tau_{88}=\gamma\,(1-a-b+3ab)\geq 0\,. \tag{15b}$$

Therefore,

$$-\langle 0|u(0)|0\rangle=\tfrac{3}{4}(\tau_{33}+\tau_{88})\geq 0\,. \tag{16}$$

Thus, Eqs. (12) and (16) lead to the well-known inequality[11]

$$(d-d')(4-d)>0\,. \tag{17}$$

Moreover, we note that

$$-\langle 0|u(0)|0\rangle=\frac{3}{2}\,\frac{\tau_{33}(1+2ab)}{(1+a)(1+b)}\,. \tag{18}$$

Since the vacuum is believed to be approximately SU(3)-invariant, we expect[12] $b\approx 0$. Approximating τ_{33} by the pion-pole contribution, we find the relation

$$-\langle 0|u(0)|0\rangle\approx\tfrac{3}{4}\,\frac{m_\pi^2f_\pi^2}{1+a}\,. \tag{19}$$

Equations (12) and (19) imply

$$\int_{t_0}^{\infty}dt\,\frac{\rho_\Theta(t)}{t}\approx(d-d')(4-d)\left(\frac{3f_\pi^2m_\pi^2}{4(1+a)}\right). \tag{20}$$

Finally, Eqs. (5a) and (20) lead to the desired inequality, which strengthens the bound in Eq. (17):

$$(d-d')(4-d)\gtrsim\frac{N(1+a)}{12\pi}\left(\frac{m_\pi}{f_\pi}\right)^2\,. \tag{21}$$

In actual [SU(2)-symmetric] theory, $m_\pi/f_\pi\approx 1.05$, and Eq. (21) becomes

$$(d-d')(4-d)\geq 0.01\quad[SU(2)]. \tag{22}$$

In the SU(3) limit $m_\pi/f_\pi\approx m_0/f_\pi\approx 3.2$, and Eq. (21) means

$$(d-d')(4-d)\geq 2.2\quad[SU(3)]\,. \tag{23}$$

Notice that, if d and d' are restricted to integers in the interval $0\leq d,\ d'\leq 4$, then Eq. (23) implies $d'=0$; this suggests that δ may be a c number (see the related discussion below).

It is important to understand that the derivation of Eq. (23) depends on the assumption that the dimensions of δ and u do not change or become indefinite in the SU(3) limit; that is, we assume that there is no anomalous dimension[13] in the theory. Recently, some authors[14] have considered the possibility that δ is a c number; in that case d' is zero, and Eqs. (22) and (23) serve to restrict d to the following ranges:

$$0.0025 \leq d \leq 3.998 \quad [\mathrm{SU}(2)], \tag{24a}$$

$$0.66 \leq d \leq 3.34 \quad [\mathrm{SU}(3)]. \tag{24b}$$

The evaluation of the right-hand side of Eq. (5b) requires that we assume from the start that δ is a c number ($d'=0$); it is also necessary to use the hypothesis of asymptotic SU(3)×SU(3). We begin by substituting Eq. (9) into the spectral representation for the vacuum expectation value of $[\dot{\Theta},\Theta]$:

$$\int_{t_0}^{\infty} dt\,\rho_\Theta(t) = i\int d^3x\,\langle 0|[\dot{\Theta}(0,\vec{x}),\Theta(0)]|0\rangle$$

$$= i(4-d)^2\int d^3x\,\langle 0|[\dot{u}(0,\vec{x}),u(0)]|0\rangle$$

$$= (4-d)^2\int_0^{\infty} dt\,\rho_u(t), \tag{25}$$

where

$$\rho_u(-k^2) = (2\pi)^3\sum_n \delta^{(4)}(p_n-k)\,|\langle 0|u(0)|n\rangle|^2.$$

Now, asymptotic SU(3)⊗SU(3) relates the integrated spectral weights of the scalar and pseudoscalar densities, $S^{(i)}$ and $P^{(i)}$:

$$\int_0^{\infty}\rho_{ij}(t,S)dt = (\text{constant})\delta_{ij} = \int_0^{\infty}\rho_{ij}(t,P)dt \quad (i,j=0,1,\ldots,8), \tag{26}$$

where

$$\rho_{ij}\left(-k^2,\begin{Bmatrix}S\\P\end{Bmatrix}\right) = (2\pi)^3\sum_n \delta^{(4)}(p_n-k) \times\left\langle 0\left|\begin{Bmatrix}S^{(i)}(0)\\P^{(i)}(0)\end{Bmatrix}\right|n\right\rangle\left\langle n\left|\begin{Bmatrix}S^{(j)}(0)\\P^{(j)}(0)\end{Bmatrix}\right|0\right\rangle.$$

This last result, together with Eq. (8), implies

$$\int_{t_0}^{\infty} dt\,\rho_u(t) = \epsilon_0^2\left[\int_0^{\infty}dt\,\rho_{00}(t,S) + 2a^2\int_0^{\infty}dt\,\rho_{88}(t,S)\right]$$

$$= \epsilon_0^2(1+2a^2)\int_0^{\infty}dt\,\rho_{33}(t,P). \tag{27}$$

We approximate the integral on the right-hand side by saturating $\rho_{33}(t,P)$ with the single-pion state,

$$\int_0^{\infty}dt\,\rho_{33}(t,P) \approx \frac{3m_\pi^4 f_\pi^2}{4\epsilon_0^2(1+a)^2}. \tag{28}$$

Combining Eqs. (25), (27), and (28), we find

$$\int_{t_0}^{\infty} dt\,\rho_\Theta(t) \approx (4-d)^2\left(\frac{3m_\pi^4 f_\pi^2(1+2a^2)}{4(1+a)^2}\right). \tag{29}$$

The desired bound on d is derived by substituting the above expression into Eq. (5b):

$$|4-d| \geq \left(\frac{4N}{3\pi}\right)^{1/2}\left(\frac{m_\pi}{f_\pi}\right)\frac{|1+a|}{(1+2a^2)^{1/2}}. \tag{30}$$

Numerically, this means

$$|4-d| \geq 0.07 \quad [\mathrm{SU}(2)], \tag{31a}$$

$$|4-d| \geq 5.7 \quad [\mathrm{SU}(3)]. \tag{31b}$$

As before, Eq. (31b) is based on the assumption that the dimensions of δ and u do not change or become indefinite in the SU(3) limit.

If δ is a c number, the rather general condition, Eq. (17), restricts d to the range $0<d<4$. This is not satisfied by the result (31b), which was also derived for theories with a c number δ. Therefore, if the strong interactions are described by the above model with a c-number scale-invariance-breaking term, then there must be a breakdown in our derivation of Eq. (31b). The possible sources of such a breakdown include the following:

(1) u may not have a definite dimension in the SU(3) limit; in this case we can retain only the weak results, Eqs. (24a) and (31a). Notice that the inconsistency cannot be caused simply by a *changed* (but definite) dimension of u in the SU(3) limit, since that *changed* dimension must still satisfy the contradictory conditions (31b) and (17).

(2) Our assumption of asymptotic SU(3)⊗SU(3), Eq. (26), may be wrong; in that event d is restricted only by Eqs. (24a) and (24b).

On the other hand, if we accept the detailed arguments leading to Eq. (31b), then the clash between Eqs. (31b) and (17) forces us to conclude that δ cannot be a c number; however, d' and d must still satisfy Eqs. (22) and (23).

Finally, we remark that Kleinert and Weisz[11] estimated $(d-d')(4-d)$ by saturating $\rho_\Theta(t)$ with a single σ-meson state. For δ a c number, their result may be expressed by the relation

$$F_\sigma^2 m_\sigma^2 \approx d(4-d)\left(\frac{3f_\pi^2 m_\pi^2}{4(1+a)}\right), \tag{32}$$

where the universal coupling constant F_σ is defined by

$$(2p_0V)^{1/2}\langle 0|\Theta(0)|\sigma(p)\rangle = F_\sigma m_\sigma^2.$$

If we approximate the integral on the left-hand side of Eq. (29) by the σ-meson contribution, we obtain the additional relation

$$F_\sigma^2 m_\sigma^4 \approx (4-d)^2\left(\frac{3f_\pi^2 m_\pi^4(1+2a^2)}{4(1+a)^2}\right). \tag{33}$$

Equations (32) and (33) may be used to determine

d and F_σ in terms of the σ-meson mass:

$$d=\frac{4(1+2a^2)}{1+2a^2+(1+a)(m_\sigma/m_\pi)^2}\,,\qquad F_\sigma=\tfrac{1}{2}f_\pi d\left(\frac{3}{1+2a^2}\right)^{1/2}. \tag{34}$$

Taking the σ meson to be the $\epsilon(700)$ gives[15]

$$d=1.9\,,\qquad |F_\sigma|=0.98m_\pi\,. \tag{35}$$

If we use the relation[16]

$$2m_\pi F_\sigma g_{\sigma\pi\pi}\approx m_\sigma^{\,2} \tag{36}$$

to calculate the coupling of the σ to the two-pion system, we find (with $m_\sigma=m_\epsilon$) $g_{\sigma\pi\pi}^{\;2}/4\pi\approx 13$, which is in rough agreement with the experimental value, $g_{\sigma\pi\pi}^{\;2}/4\pi\approx 10$. We emphasize that these results are based on the assumptions that asymptotic $SU(3)\otimes SU(3)$ is valid and that δ is a c number, in addition to σ-meson dominance. The qualitative success of this approach suggests that the error in the derivation of Eq. (31b) is associated with possibility (1), discussed in the last paragraph.

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[10]The generators of $SU(3)\otimes SU(3)$ obey the usual commutation relations:

$$[F^{(i)}(x^0),\,F^{(j)}(x^0)]=if_{ijk}F^{(k)}(x^0),$$
$$[F_5^{(i)}(x^0),\,F^{(j)}(x^0)]=if_{ijk}F_5^{(k)}(x^0),$$
$$[F_5^{(i)}(x^0),\,F_5^{(j)}(x^0)]=if_{ijk}F^{(k)}(x^0),$$

where $F^{(1,2,3)}$ are the components of isospin.

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