*Supported in part by the National Science Foundation.
†Present address: Department of Physics, Louisiana State University, Baton Rouge, La. 70803.

[1]M. Gell-Mann, Physics 1, 63 (1964).
[2]S. L. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[3]S. Gasiorowicz and D. Geffen, Rev. Mod. Phys. 41, 531 (1969). This work contains many useful references to work on chiral symmetry.
[4]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[5]S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).
[6]R. Dashen, Phys. Rev. 183, 1245 (1969).
[7]L-F. Li and H. Pagels, Phys. Rev. Letters 26, 1204 (1971).
[8]P. Carruthers and Richard W. Haymaker, Phys. Rev. Letters 27, 455 (1971).
[9]M. Lévy, Nuovo Cimento 52, 23 (1967).
[10]S. Okubo and V. Mathur, Phys. Rev. D 1, 2046 (1970).
[11]P. Carruthers and Richard W. Haymaker, following paper, Phys. Rev. D 4, 1815 (1971).
[12]Note that we are dealing with a linear realization and an old-fashioned renormalizable theory.
[13]$\mathfrak{M}$ is related to σ_i and ϕ_i by $\mathfrak{M} = \sum_{j=0}^{8}(\sigma_j + i\phi_j)\lambda_j/\sqrt{2}$, where λ_i are the usual SU_3 matrices.
[14]A discussion of the transformation properties of the various terms in $\mathfrak{L}$ under $U_3\times U_3$ and dilations can be found in Ref. 11.
[15]This is discussed in many places – in particular Refs. 3, 6, and 11.
[16]This result is easily read from Table I of Ref. 11.
[17]The mixing angles are defined by

$$\eta=\phi_8\cos\theta_P - \phi_0\sin\theta_P ,$$
$$\eta' =\phi_8\sin\theta_P +\phi_0\cos\theta_P .$$

We use the same convention for the scalars ϵ and ϵ' for the sake of uniformity.
[18]*Proceedings of a Conference on* $\pi\pi$ *and* $K\pi$ *Interactions at Argonne National Laboratory, 1969,* edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, Ill., 1969).
[19]Particle Data Group, Rev. Mod. Phys. 42, 87 (1970).
[20]P. Carruthers, Phys. Rev. D 3, 959 (1971). This work gives some recent references on the somewhat controversial subject of scalar mesons.
[21]P. Carruthers, Phys. Rev. D 2, 2265 (1970).

PHYSICAL REVIEW D VOLUME 4, NUMBER 6 15 SEPTEMBER 1971

Matching Explicit and Spontaneous Scale-Invariance Breaking in Lagrangian Models*

P. Carruthers and Richard W. Haymaker
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 27 May 1971)

Effective Lagrangian models, constructed from fields transforming as $(3,\overline{3})+(\overline{3},3)$ and $(1,1)$ in $SU_3\times SU_3$, are studied in the tree approximation to learn how solutions having broken scale and chiral symmetries are related to underlying limit solutions exhibiting spontaneous breakdown of these symmetries. The requirement of a smooth transition to the limit of scale invariance leads to restrictive conditions on the structure of the model and its solutions. For a special case of the general model, it is possible to compute explicitly the squared masses of the single-particle excitations in terms of symmetry-breaking parameters. The condition of stability ($m^2 \geq 0$) leads to the allowed domains of Okubo and Mathur, and hence, provides a physical interpretation to their result. It is noted that there are several ways to normalize the ninth axial-vector current by placing its divergence and the trace of the energy-momentum tensor in a representation of $U_1\times U_1$.

I. INTRODUCTION

The study of Lagrangian models has provided considerable insight into the nature of approximate chiral symmetry.[1] Much of this work has been concerned with "nonlinear realizations" since current-algebra soft-meson theorems are automatically satisfied, and the coordinates of heavy or dubious particles can be eliminated. A similar situation obtains in much recent work on models of scale invariance and its breaking.[2-4] For many purposes[5,6] it is useful to study ordinary linear realizations, not only because the particles in question might actually exist, but because the dynamics of the underlying spontaneous-breakdown mechanism is much clearer at this level. Further, the nonlinear realizations can be regarded[7] as certain limits of the usual linear theories.

The present paper is concerned with the phenomenon of spontaneous breakdown of scale invariance and the additional stringent requirement that such solutions change smoothly when scale-invariance-breaking operators are turned on. The calculations are carried out in the semiclassical (tree-graph) approximation, with canonical commutation relations assumed. (Modifications due to renormalization will occur in the next approximation to the self-consistent ground state used here.) The theory used is described by an old-fashioned renormalizable polynomial Lagrangian which is a slight generalization of the SU_3 "σ model,"[8] called the "20-meson model" in Ref. 7.

The existence of finite masses in a theory having no fixed dimensional quantities involves a number of tricky questions which we plan to discuss elsewhere. Some authors[9] have, in fact, claimed that the foregoing situation is impossible. Here we only note that the key assumption of Ref. 9, that the dilation current can be cast in the form $x^\nu\Theta_{\mu\nu}$, depends on the possibility[10] of "improving" the canonical energy-momentum tensor by certain terms which do not modify the generators of the Poincaré group. In fact, the extra terms *do* change these generators when there exists a massless scalar particle (dilaton) in the theory.[11] Since the latter particle is *required* in theories exhibiting spontaneous breakdown of scale invariance, the aforementioned assumption is wrong and the attendant rigor is beside the point.

The idea[12] that there exists a limit of spontaneously broken scale invariance is frequently accompanied by a more optimistic assumption of "PCDC" (partial conservation of dilation current). PCDC hopes to imitate the success of PCAC (partial conservation of axial-vector current) by supposing that the trace of the energy-momentum tensor ($\Theta^\mu{}_\mu$) be dominated by the dilaton (and perhaps a few other scalars). It is important to realize that the existence of the dilaton does not necessarily imply PCDC, although it would be hard to imagine the success of the latter in the absence of the former. Although a phenomenological analysis[13] of hadron masses based on PCDC gives reasonable results, several difficult experiments need to be performed to test this idea.[14]

In Sec. II the properties of the model are set forth in canonical detail. Special attention is given to the question of the "ninth" axial baryon current $\mathfrak{F}^5_{0\mu}$, although these ideas are more relevant to an earlier paper by the authors[15] than to the subsequent development. The new idea set forth here is that it may be possible to normalize the ninth current by means of a nonlinear commutation relation relating F^5_0, $\partial^\mu\mathfrak{F}^5_{0\mu}$, and $\Theta^\mu{}_\mu$ by placing the latter two operators in a representation of $U_1\times U_1$. (This result generalizes the proposal of Ref. 7 and provides examples of the theoretical structure presumed in Ref. 14.) Possible experimental manifestations of the current $\mathfrak{F}^5_{0\mu}$ have recently been discussed.[16]

In Sec. III various possibilities for spontaneous breakdown of scale invariance are studied. It is found that the quantity $b=\xi_8/\sqrt{2}\,\xi_0$ ($\xi_0=\langle\sigma_0\rangle$, $\xi_8=\langle\sigma_8\rangle$ being vacuum expectation values of the SU_3 singlet and octet scalar fields σ_0 and σ_8) has only four possible values in this limit. These four values can only exist when a certain constraint is satisfied by the quartic coupling constants of the theory. A further interesting point is that the four allowed values for b are the boundaries of the "allowed domains" for the symmetry-breaking parameters discovered by Okubo and Mathur.[17] The next interesting question is whether the foregoing solutions are stable in the presence of an external perturbation which violates scale invariance. Here only linear fields are allowed as perturbations so that we have operator PCDC. (The case of cubic scale-invariance-breaking operators was considered in Ref. 7.) It is found in the 18-meson model that, if the breaking term $\mathfrak{L}_{\text{S. B.}}$ is $-(\epsilon_0\sigma_0+\epsilon_8\sigma_8)$, then the requirement that the solution join smoothly with a spontaneously broken one in the limit $\epsilon_0\to 0$, $\epsilon_8\to 0$ determines the ratio $a=\epsilon_8/\sqrt{2}\,\epsilon_0$, when the quartic coupling constants are held fixed. However, the limiting values of a lie outside the allowed domains[17] and correspond to some (mass)^2 being negative. This difficulty is avoided by allowing the quartic coupling constants to vary as ϵ_0, ϵ_8 are turned on.

In Sec. IV we compute the squared masses for the 18-meson model in which both scale and $U_3\times U_3$ symmetry are broken by $\mathfrak{L}_{\text{S. B.}}=-(\epsilon_0\sigma_0+\epsilon_8\sigma_8)$. We are able to express these quantities explicitly in terms of the parameters $a=\epsilon_8/\sqrt{2}\,\epsilon_0$ and $b=\xi_8/\sqrt{2}\,\xi_0$. The requirement that several of the m^2 values be positive leads directly to the more formal result of Ref. 17, while the positivity of the remaining m^2 give slightly more stringent conditions on the allowed values of a and b. Therefore, we are led to interpret the physical significance of the "allowed domains" to be simply that of stability of the single-particle excitations. An incidental feature of this model is the η'-π degeneracy, which has been noted elsewhere.[1] This result points to the physical inadequacy of this type of model, as stressed previously.[7] The 20-meson model exhibiting PCDC does not modify the η'-π degeneracy and leads to nothing especially interesting. Finally we study the question of the existence of a perturbation expansion about the chiral-invariant limit (which is, in the present model, the same as the scale-invariant limit) when this limit exhibits spontaneous breakdown. We give an example in which there is a finite radius of convergence in the symmetry-breaking parameters. This feature is true in much more realistic models, as will be reported elsewhere.[18] The radius of convergence of the perturbation series turns out to be much smaller than the value required by experiment. This result suggests an explanation for previously reported[19] failures of perturbation theory about the $SU_3\times SU_3$ limit.

II. A LAGRANGIAN MODEL OF MESON FIELDS

A. Construction of the Lagrangian

In order to analyze in detail the physical questions posed in the Introduction, we construct a fairly simple canonical field theory from the 18 meson fields σ_i, ϕ_i, $i=0, 1, \ldots, 8$, which belong to the representation $(3, \bar{3})+(\bar{3}, 3)$ and two meson fields S and P which are $SU_3\times SU_3$-invariant, but transform into each other under the ninth axial charge F_0^5. The assumed commutators are[20]

$$\begin{aligned}
&[F_i, \sigma_j]=if_{ijk}\sigma_k,\\
&[F_i, \phi_j]=if_{ijk}\phi_k,\\
&[F_i^5, \sigma_j]=-id_{ijk}\phi_k,\\
&[F_i^5, \phi_j]=id_{ijk}\sigma_k,
\end{aligned} \tag{2.1}$$

where i runs from 1 to 8 and (j, k) run from 0 to 8,

$$\begin{aligned}
&[F_0^5, S]=-iP, \quad [F_0^5, \sigma_i]=-i\phi_i,\\
&[F_0^5, P]=iS, \quad [F_0^5, \phi_i]=i\sigma_i.
\end{aligned} \tag{2.2}$$

The charges F_i, F_i^5 can be constructed from the currents given in Eq. (2.6) and the relations (2.1)–(2.2) can be verified explicitly.

Our Lagrangian is a simple extension of the usual SU_3 σ model (without baryons) which was called the "20-meson" model in Ref. 7. We insist that $\mathcal{L}$ be of standard polynomial form with renormalizable interactions, writing

$$\mathcal{L}=\bar{\mathcal{L}}+\mathcal{L}_{\text{S. B.}}\,. \tag{2.3}$$

Here $\bar{\mathcal{L}}$ would have dimension four in the absence of the scale-invariance-breaking term $\mathcal{L}_{\text{S. B.}}$, and has the form

$$\begin{aligned}
\bar{\mathcal{L}}=&\tfrac{1}{2}\operatorname{Tr}\partial_\mu \mathfrak{M}^\dagger \partial^\mu \mathfrak{M}+\tfrac{1}{2}[(\partial_\mu S)^2+(\partial_\mu P)^2]\\
&+f_1(\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M})^2+f_2\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M}\mathfrak{M}^\dagger\mathfrak{M}\\
&+f_3\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M}(S^2+P^2)+f_4(S^2+P^2)^2.
\end{aligned} \tag{2.4}$$

Here the matrix $\mathfrak{M}$ is a convenient way of writing the $(3, \bar{3})$ fields, $\mathfrak{M}=\sum_i(\sigma_i+i\phi_i)\lambda_i/\sqrt{2}$. The scale-invariance-breaking interaction is taken to be

$$\begin{aligned}
\mathcal{L}_{\text{S. B.}}=&g_1(\det\mathfrak{M}+\text{H.c.})+g_2 S\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M}+g_3 S(S^2+P^2)\\
&-\tfrac{1}{2}\mu_0{}^2\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M}-\tfrac{1}{2}m_0{}^2(S^2+P^2)\\
&-\epsilon_0\sigma_0-\epsilon_8\sigma_8-\epsilon_9 S.
\end{aligned} \tag{2.5}$$

The constants g_i have dimension one (in mass units), while the ϵ_i have dimension three. The only violation of $SU_3\times SU_3$ in the Lagrangian is due to the linear terms σ_0 and σ_8.

B. The Currents and Their Divergences

The currents of the theory are explicitly

$$\begin{aligned}
&\mathfrak{F}_{0\mu}=0,\\
&\mathfrak{F}_{i\mu}=f_{ijk}(\sigma_j\partial_\mu\sigma_k+\phi_j\partial_\mu\phi_k),\\
&\mathfrak{F}_{0\mu}^5=P\overleftrightarrow{\partial}_\mu S+\phi_i\overleftrightarrow{\partial}_\mu\sigma_i,\\
&\mathfrak{F}_{i\mu}^5=d_{ijk}\phi_j\overleftrightarrow{\partial}_\mu\sigma_k.
\end{aligned} \tag{2.6}$$

The current divergences are given in our model by

$$\begin{aligned}
&\partial^\mu\mathfrak{F}_{i\mu}=i[F_i, \mathcal{L}_{\text{S. B.}}],\\
&\partial^\mu\mathfrak{F}_{i\mu}^5=i[F_i^5, \mathcal{L}_{\text{S. B.}}],
\end{aligned} \tag{2.7}$$

where $i=0, \ldots, 8$. (This makes use of the fact that $\bar{\mathcal{L}}$ has $U_3\times U_3$ symmetry as well as scale symmetry.) In order to categorize more completely the behavior of the various interactions under commutation with the charges, we note the following relations:

$$\begin{aligned}
&[F_0^5, \sigma_i+i\phi_i]=-(\sigma_i+i\phi_i),\\
&[F_0^5, S+iP]=-(S+iP),
\end{aligned} \tag{2.8}$$

or their finite equivalents

$$\begin{aligned}
&e^{iF_0^5\lambda}\mathfrak{M}e^{-iF_0^5\lambda}=e^{i\lambda}\mathfrak{M},\\
&e^{iF_0^5\lambda}(S+iP)e^{-iF_0^5\lambda}=e^{i\lambda}(S+iP).
\end{aligned} \tag{2.9}$$

These relations show that $\mathfrak{M}^\dagger\mathfrak{M}$ and S^2+P^2 are invariant under F_0^5, and that the only violations of $U_3\times U_3$ are due to $\mathcal{L}_{\text{S. B.}}$. The mass terms are $U_3\times U_3$-invariant. Of the remaining terms in $\mathcal{L}_{\text{S. B.}}$, only σ_0 and σ_8 violate $SU_3\times SU_3$, so that the octet currents obey

$$\begin{aligned}
&\partial^\mu\mathfrak{F}_{i\mu}=\epsilon_8 f_{i8j}\sigma_j,\\
&\partial^\mu\mathfrak{F}_{i\mu}^5=-\epsilon_0 d_{i0j}\phi_j-\epsilon_8 d_{i8j}\sigma_j.
\end{aligned} \tag{2.10}$$

The isospin and hypercharge vector currents are conserved, and the octet axial-vector currents obey "operator" PCAC.

The ninth current is nonconserved by the cubic and linear terms occurring in $\mathcal{L}_{\text{S. B.}}$. These operators transform differently but all belong, along with their pseudoscalar counterparts, to simple representations of $U_3\times U_3$ characterized by the index κ,

$$\begin{aligned}
&i[F_0^5, u_9]=\kappa v_9,\\
&i[F_0^5, v_9]=-\kappa u_9.
\end{aligned} \tag{2.11}$$

Table I gives the value of κ for the pairs (u_9, v_9) arising from the constituents of $\mathcal{L}_{\text{S. B.}}$. For brevity in writing we shall frequently denote the operators $\det\mathfrak{M}+\text{H.c.}$ and $(\det\mathfrak{M}-\text{H.c.})/i$ by δ and δ_5, respectively. Noting that (σ_i, ϕ_i) transform into each

TABLE I. The operator pairs (u_9, v_9) belonging to the representation $U_1 \times U_1$ characterized by the label κ [see Eq. (2.11)] are listed for the model Lagrangian.

u_9	v_9	κ
$\det\mathfrak{M} + \mathrm{H.c.}$	$(\det\mathfrak{M} - \mathrm{H.c.})/i$	3
$S\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M}$	$P\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M}$	1
$S(S^2+P^2)$	$P(S^2+P^2)$	1
S	P	1

other in the same way as (S, P) under F_0^5 [Eq. (2.2)], we find the following:

$$\partial^\mu \mathfrak{F}_{0\mu}^5 = 3g_1\delta_5 + g_2 P\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M} + g_3 P(S^2+P^2) - \epsilon_0\sigma_0 - \epsilon_8\sigma_8 - \epsilon_9 P. \qquad (2.12)$$

Equation (2.12) is extremely complicated and would seem to preclude any simple approximations to matrix elements of $\mathfrak{F}_{0\mu}^5$. However, as stressed by Dashen,[19] the criterion for PCAC is not that operator PCAC hold, but rather that there must exist a Goldstone limit in which the current is conserved. (This means that the limit $\mathfrak{L} \to \bar{\mathfrak{L}}$ must be characterized by the spontaneous breakdown of scale invariance for PCDC to be possible. Even given this circumstance it can happen that the symmetry breaking is too severe for PCDC to be a reasonable approximation.)

C. Connection Between Scale-Invariance-Breaking Operators and F_0^5

Since the nonconservation of $\mathfrak{F}_{0\mu}^5$ is caused by the scale-invariance-breaking interactions, it is useful to study the scaling aspects of the model. These are usefully formulated in terms of the improved energy-momentum tensor of Callan, Coleman, and Jackiw,[10] which is for our model

$$\Theta_{\mu\nu} = \tfrac{1}{2}\,\mathrm{Tr}\,\partial_\mu\mathfrak{M}^\dagger\,\partial_\nu\mathfrak{M} + (\partial_\mu S\partial_\nu S + \partial_\mu P\partial_\nu P) - g_{\mu\nu}(\bar{\mathfrak{L}} + \mathfrak{L}_{\mathrm{S.B.}}) - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\tfrac{1}{2}\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M} + S^2 + P^2). \qquad (2.13)$$

The canonical dimension d is given by commutation

$$i[D(t), O_d(x,t)] = (d + x\cdot\partial)O_d(x), \qquad (2.14)$$

where $D(t) = \int d^3x\, x^\mu \Theta_{\mu 0}$. The scale-invariance breaking is then measured[21] by the trace $\Theta^\mu{}_\mu$,

$$\Theta^\mu{}_\mu = \sum_i (-d_i + 4)w_i, \qquad (2.15)$$

where the operators w_i have dimension d_i. For the model of Eqs. (2.3)–(2.5) we find, denoting the operator $\det\mathfrak{M} + \mathrm{H.c.}$ by $-\delta$,

$$\Theta^\mu{}_\mu = -g_1\delta - g_2 S\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M} - g_3 S(S^2+P^2) + \mu_0{}^2\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M} + m_0{}^2(S^2+P^2) + 3(\epsilon_0\sigma_0 + \epsilon_8\sigma_8 + \epsilon_9 S). \qquad (2.16)$$

We now ask whether $\Theta^\mu{}_\mu$, which is also the divergence of the dilation current, has any simple relation to $\partial^\mu\mathfrak{F}_{0\mu}^5$. To study this question, we ask how $\Theta^\mu{}_\mu$ is transformed by F_0^5:

$$i[F_0^5, \Theta^\mu{}_\mu] = -[3g_1\delta_5 + g_2 P\,\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M} + g_3 P(S^2+P^2)] + 3(\epsilon_0\phi_0 + \epsilon_8\phi_8 + \epsilon_9 P). \qquad (2.17)$$

We note that the right-hand side bears some resemblance to (2.12), except that the linear terms have acquired a factor of 3 because they have dimension unity. If we compute $i[F_0^5, \partial^\mu\mathfrak{F}_{0\mu}^5]$, we see that a structure similar to (2.16) is generated, except for the mass terms. In order to obtain a closed algebraic structure, we set $\mu_0{}^2 = m_0{}^2 = 0$ for the rest of this section.

The right-hand side of (2.17) is proportional to $\partial^\mu\mathfrak{F}_{0\mu}^5$ only for three choices of the constants g_i, ϵ_i.

Case 1. $g_1 = g_2 = g_3 = 0$.

$$i[F_0^5, \tfrac{1}{3}\Theta^\mu{}_\mu] = -\partial^\mu\mathfrak{F}_{0\mu}^5, \qquad i[F_0^5, \partial^\mu\mathfrak{F}_{0\mu}^5] = \tfrac{1}{3}\Theta^\mu{}_\mu. \qquad (2.18)$$

This fits the pattern (2.11) with

$(u_9, v_9) = (\tfrac{1}{3}\Theta, -\partial^\mu\mathfrak{F}_{0\mu}^5)$ and $\kappa = 1$.

Case 2. $\epsilon_0 = \epsilon_8 = \epsilon_9 = g_1 = 0$.

$$i[F_0^5, \Theta^\mu{}_\mu] = -\partial^\mu\mathfrak{F}_{0\mu}^5, \qquad i[F_0^5, \partial^\mu\mathfrak{F}_{0\mu}^5] = \Theta^\mu{}_\mu. \qquad (2.19)$$

Here the operators (u_9, v_9) are $(\Theta^\mu{}_\mu, -\partial^\mu\mathfrak{F}_{0\mu}^5)$ and $\kappa = 1$.

Case 3. $\epsilon_1 = \epsilon_2 = \epsilon_3 = g_2 = g_3 = 0$.

$$i[\tfrac{1}{3}F_0^5, \Theta^\mu{}_\mu] = -\tfrac{1}{3}\partial^\mu\mathfrak{F}_{0\mu}^5, \qquad i[\tfrac{1}{3}F_0^5, \tfrac{1}{3}\partial^\mu\mathfrak{F}_{0\mu}^5] = \Theta^\mu{}_\mu. \qquad (2.20)$$

If we introduce $\mathfrak{F}_{0\mu}^{5\,\prime} = \tfrac{1}{3}\mathfrak{F}_{0\mu}^5$, we get

$(u_9, v_9) = (\Theta^\mu{}_\mu, -\partial^\mu\mathfrak{F}_{0\mu}^{5\,\prime})$.

In each of the foregoing cases, we have been able to put $\Theta^\mu{}_\mu$ and $\partial^\mu\mathfrak{F}_{0\mu}^5$ into a representation of $U_1 \times U_1$ whose form does not depend on a specific canonical description of the ninth axial-vector current. Among the above models only case 1 (absence of cubic and quadratic scale-invariance-breaking terms) possesses *operator* PCAC and PCDC. Cases 2 and 3 have operator PCAC, but not operator PCDC. The advantage of case 1 is somewhat dubious, but we shall give closest attention to this case since it is the simplest to analyze.

It is entirely possible that none of the simple patterns (2.18)–(2.20) holds. That would not necessarily preclude the utility of the idea expressed by such relations. It could easily be that case 3 is close to reality, with the linear terms comprising a small correction.

D. Spontaneous-Breakdown Mechanism in the Tree Approximation

Currently available techniques do not provide a reliable solution to the system described by the Lagrangian of Eqs. (2.3)–(2.5). However the nature of the dynamics characteristic of spontaneous breakdown of symmetry leads naturally to a suggestive approximation scheme justified by many successful antecedents in many-body theory. This approximation, which exhibits many of the interesting features expected of the actual solution, is usually called the tree approximation (since closed loops are banished). It could equally well be called the Weiss or Hartree approximation, since its idea is physically similar to that of the "mean-field" theories which are associated with these names. In these theories certain dynamical coordinates have average values determined self-consistently by interactions with the whole system. Although the actual values may fluctuate about the average values, interesting results follow when the fluctuations are ignored in lowest order.

The linear and cubic terms in $\mathfrak{L}_{\mathrm{S.B.}}$ tend to induce instabilities in a "normal" vacuum characterized by $\langle S\rangle_0 = \langle\sigma_0\rangle_0 = \langle\sigma_8\rangle_0 = 0$. Because of this we expect that, in general, the physical vacuum will be characterized by nonvanishing values for the constants

$$\begin{aligned} &\xi_0 \equiv \langle\sigma_0\rangle_0 ,\\ &\xi_8 \equiv \langle\sigma_8\rangle_0 ,\\ &s \equiv \langle S\rangle_0 . \end{aligned} \tag{2.21}$$

The nonvanishing of s, ξ_0, ξ_8 implies, respectively, the noninvariance of the vacuum under F_0^5, $SU_3\times SU_3$, and SU_3 transformations. The dynamics of $\bar{\mathfrak{L}}+\mathfrak{L}_{\mathrm{S.B.}}$ is such that a "spontaneous breakdown" of symmetry is dynamically favored for many choices of the coupling constants.

Once it is known that one or more of the con-

TABLE II. The contribution of the various operators in $\mathfrak{L}$ to the linear and quadratic displaced fields are listed. Here the quantities ξ_0, ξ_8, s are, respectively, $\langle\sigma_0\rangle_0$, $\langle\sigma_8\rangle_0$, $\langle S\rangle_0$. We have omitted several operators whose contributions can be found by inspection.

	$(\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M})^2$	$\mathrm{Tr}(\mathfrak{M}^\dagger\mathfrak{M}\mathfrak{M}^\dagger\mathfrak{M})$	$\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M}(S^2+P^2)$	$S\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M}$	$\det\mathfrak{M}+\mathrm{H.c.}$
S'	0	0	$2s(\xi_0^2+\xi_8^2)$	$\xi_0^2+\xi_8^2$	0
σ_0'	$4\xi_0(\xi_0^2+\xi_8^2)$	$\frac{4}{3}(\xi_0^3-\xi_8^3/\sqrt{2}+3\xi_0\xi_8^2)$	$2\xi_0 s^2$	$2\xi_0 s$	$(2\xi_0^2-\xi_8^2)/\sqrt{3}$
σ_8'	$4\xi_8(\xi_0^2+\xi_8^2)$	$2(\xi_8^3+2\xi_8\xi_0^2-\sqrt{2}\,\xi_0\xi_8^2)$	$2\xi_8 s^2$	$2\xi_8 s$	$-2\xi_8(\xi_0+\xi_8/\sqrt{2})/\sqrt{3}$
$(\sigma_0')^2$	$2(3\xi_0^2+\xi_8^2)$	$2(\xi_0^2+\xi_8^2)$	s^2	s	$2\xi_0/\sqrt{3}$
$(\sigma_8')^2$	$2(3\xi_8^2+\xi_0^2)$	$3\xi_8^2+2\xi_0^2-2\sqrt{2}\,\xi_0\xi_8$	s^2	s	$-(\xi_0+\sqrt{2}\,\xi_8)/\sqrt{3}$
$\sigma_0'\sigma_8'$	$8\xi_0\xi_8$	$2\sqrt{2}\,\xi_8(2\sqrt{2}\,\xi_0-\xi_8)$	0	0	$-2\xi_8/\sqrt{3}$
$\sigma_0'S'$	0	0	$4\xi_0 s$	$2\xi_0$	0
$\sigma_8'S'$	0	0	$4\xi_8 s$	$2\xi_8$	0
$(S')^2$	0	0	$\xi_0^2+\xi_8^2$	0	0
$\sum_{i=1}^{3}\sigma_i^2$	$2(\xi_0^2+\xi_8^2)$	$(\xi_8+\sqrt{2}\,\xi_0)^2$	s^2	s	$-(\xi_0-\sqrt{2}\,\xi_8)/\sqrt{3}$
$\sum_{i=4}^{7}\sigma_i^2$	$2(\xi_0^2+\xi_8^2)$	$\xi_8^2+2\xi_0^2-\sqrt{2}\,\xi_0\xi_8$	s^2	s	$-(\xi_0+\xi_8/\sqrt{2})/\sqrt{3}$
ϕ_0^2	$2(\xi_0^2+\xi_8^2)$	$\frac{2}{3}(\xi_0^2+\xi_8^2)$	s^2	s	$-2\xi_0/\sqrt{3}$
ϕ_8^2	$2(\xi_0^2+\xi_8^2)$	$\frac{1}{3}(3\xi_8^2+2\xi_0^2-2\sqrt{2}\,\xi_0\xi_8)$	s^2	s	$(\xi_0+\sqrt{2}\,\xi_8)/\sqrt{3}$
$\phi_0\phi_8$	0	$\frac{1}{3}2\sqrt{2}\,\xi_8(2\sqrt{2}\,\xi_0-\xi_8)$	0	0	$2\xi_8/\sqrt{3}$
$\sum_{i=1}^{3}\phi_i^2$	$2(\xi_0^2+\xi_8^2)$	$\frac{1}{3}(\xi_8+\sqrt{2}\,\xi_0)^2$	s^2	s	$(\xi_0-\sqrt{2}\,\xi_8)/\sqrt{3}$
$\sum_{i=4}^{7}\phi_i^2$	$2(\xi_0^2+\xi_8^2)$	$\frac{1}{3}(7\xi_8^2+2\xi_0^2-\sqrt{2}\,\xi_0\xi_8)$	s^2	s	$(\xi_0+\xi_8/\sqrt{2})/\sqrt{3}$
P^2	0	0	$\xi_0^2+\xi_8^2$	0	0
ϕ_8P	0	0	0	0	0
ϕ_0P	0	0	0	0	0

stants (2.21) is nonvanishing, it makes sense to rewrite $\mathcal{L}$ in terms of the physically relevant quantities, the fluctuations about the average values of (2.21):

$$\begin{aligned} \sigma_0 &= \sigma_0' + \xi_0\,, \\ \sigma_8 &= \sigma_8' + \xi_8\,, \\ S &= S' + s\,. \end{aligned} \tag{2.22}$$

The primed fields are the ones to be identified asymptotically as those carrying particle excitations.

If we rewrite $\mathcal{L}$ in terms of the primed fields, we find a new set of interactions. The ideas of small-oscillation theory may now be applied to the non-kinetic-energy part of $\mathcal{L}$. If the constants ξ_0, ξ_8, and s have been correctly chosen, this should be a quadratic form (with anharmonic corrections) about these values. Hence $\mathcal{L} - \mathcal{L}_K$ must be extremal at the proper values of ξ_0, ξ_8, and s. The curvature of the various "potential wells" is to be identified with the (mass)^2 parameter, which must be positive for stability. This procedure does not completely diagonalize H, but hopefully accounts for a large portion of the interaction which accounts for the single-particle spectrum. When appropriate, this semiclassical approach provides a new reference point to which further approximations are referred.

If we substitute (2.22) into (2.3), we find a variety of terms involving the primed fields. The contributions of the nontrivial operators to the terms linear and quadratic in the fields are given in Table II.

The extremal conditions are that the coefficients of the linear terms vanish, i.e.,

$$\begin{aligned} &2sf_3(\xi_0^2+\xi_8^2)+4f_4s^3-sm_0^2+3g_3s^2+g_2(\xi_0^2+\xi_8^2)-\epsilon_9=0, \\ &4f_1\xi_0(\xi_0^2+\xi_8^2)+\tfrac{4}{3}f_2(\xi_0^3-\xi_8^3/\sqrt{3}+3\xi_0\xi_8^2)+2f_3\xi_0s^2-\mu_0^2\xi_0/\sqrt{3}+2g_2\xi_0s+g_1\xi_0(2\xi_0-\xi_8)/\sqrt{3}-\epsilon_0=0, \\ &4f_1\xi_8(\xi_0^2+\xi_8^2)+2f_2(\xi_8^3+2\xi_8\xi_0^2-\sqrt{2}\;\xi_0\xi_8^2)+2\xi_8s^2f_3-\mu_0^2\xi_8/\sqrt{3}+2g_2\xi_8s-(2/\sqrt{3})g_1\xi_8(\xi_0+\xi_8/\sqrt{2})-\epsilon_8=0. \end{aligned} \tag{2.23}$$

The physical content of this set of cubic equations will become more clear when we analyze special cases of particular interest.

III. SPONTANEOUS AND EXPLICIT BREAKDOWN OF SCALE INVARIANCE

A. Introduction

The limit in which $\mathcal{L}$ has $SU_3 \times SU_3$ symmetry seems to resemble the real world in a qualitative way. It is reasonable to ask whether there exists a further limit, that of scale invariance, in which further rough symmetries become exact. If the vacuum is invariant under scale transformations in this limit, all masses have to vanish and the limit of scale invariance might not be useful. The alternative possibility is that of spontaneous breakdown of scale invariance; in this case all masses need not vanish as $\mathcal{L}_{\text{S. B.}} \to 0$ and the limit of scale invariance may bear some crude resemblance to the real world. (As a further consequence the massless "dilaton" required in this case may have a massive counterpart and play a role in determining hadron masses.)

In the present section we investigate the structure of the spontaneously broken theory described by the scale-invariant Lagrangian density $\overline{\mathcal{L}}$, which has canonical dimension four. We consider cases in which one or more of the vacuum expectation values ξ_0, ξ_8, or s are nonzero. Only ratios of these (dimensional) quantities can be determined in this limit, in contrast to the case in the presence of scale-invariance-breaking operators. (Thus scale-invariance "breaking" includes scale "fixing.")

In order to determine whether a solution exhibiting spontaneous breakdown of scale invariance is a reasonable physical limit, we have to see whether the scale-invariance-breaking theory can go over smoothly to such a limit. This requirement turns out to be very strong, and special procedures are required to achieve a smooth connection. Conversely, "turning on" a weak scale-invariance-breaking interaction, in general, will destroy the Goldstone nature of the ground state unless done in a very specific way.

Following a study of the spontaneously broken solutions of the theory defined by $\overline{\mathcal{L}}$ (in Sec. III B), we study the problem of "turning on" the interactions which violate scale invariance. Rather than deal with the entire set of interactions appearing in $\mathcal{L}_{\text{S. B.}}$, we only consider the violation of scale invariance by linear terms (i.e., we deal with models having both operator PCAC and PCDC):

$$\mathcal{L} = \overline{\mathcal{L}} - \epsilon_0\sigma_0 - \epsilon_8\sigma_8 - \epsilon_9 S\,. \tag{3.1}$$

Section III C treats the case of fixed quartic couplings in the 18-meson model. Since none of these solutions is stable in the presence of scale-invariance breaking, we consider in Sec. III D properties of the solution in which the quartic couplings are allowed to vary with ϵ_0 and ϵ_8. Some of the questions raised in this section are clarified in Sec. IV, in which the relations between couplings and stability conditions are studied.

When the Lagrangian has the form of Eq. (3.1), the extremal conditions (2.23) simplify to the form

$$\begin{aligned}\epsilon_0 &= 2[2f_1\xi_0(\xi_0^2+\xi_8^2) \\ &\quad + \tfrac{2}{3}(\xi_0^3+3\xi_0\xi_8^2-\xi_8^3/\sqrt{2})f_2+\xi_0 s^2 f_3], \\ \epsilon_8 &= 2\xi_8[2f_1(\xi_0^2+\xi_8^2) \\ &\quad +(2\xi_0^2+\xi_8^2-\sqrt{2}\,\xi_0\xi_8)f_2+s^2f_3], \\ \epsilon_9 &= 2s[f_3(\xi_0^2+\xi_8^2)+2s^2f_4].\end{aligned} \tag{3.2}$$

We shall often find it convenient to use the parameters a and b of Ref. 20,

$$a=\frac{\epsilon_8}{\sqrt{2}\,\epsilon_0}, \quad b=\frac{\xi_8}{\sqrt{2}\,\xi_0}. \tag{3.3}$$

B. Analysis of Solutions Exhibiting Spontaneous Breakdown of Scale Invariance

Our first goal is to classify the solutions to Eqs. (3.2) which have one or more of (ξ_0, ξ_8, s) nonvanishing even when $\epsilon_0=\epsilon_8=\epsilon_9=0$. It is remarkable that we can obtain a relation between ξ_0 and ξ_8 which is independent of s and the couplings f_i. Multiplying the first and second equations of Eqs. (3.2) by ξ_8 and ξ_0, respectively, we find (denoting by a bar quantities belonging to the scale-invariant limit)

$$\bar{\xi}_8(-\sqrt{2}\,\bar{\xi}_8^3+3\bar{\xi}_0\bar{\xi}_8^2+3\sqrt{2}\,\bar{\xi}_0^2\bar{\xi}_8-4\bar{\xi}_0^3)=0. \tag{3.4}$$

This relation is entirely due to the coupling $\mathrm{Tr}\,\mathfrak{M}^\dagger\mathfrak{M}\mathfrak{M}^\dagger\mathfrak{M}$ and is absent if f_2 is zero. Equation (3.4) shows that if $\xi_0=0$, then ξ_8 also must vanish. However, it is possible to have $\xi_0\neq 0$ when ξ_8 vanishes. Hence we can have spontaneously broken solutions of $SU_3\times SU_3$ for which the vacuum is SU_3-invariant. However, $\xi_8\neq 0$ automatically implies $\xi_0\neq 0$.

Dividing (3.3) by $\bar{\xi}_0^3$ and introducing the parameters of Eq. (3.3), Eq. (3.4) has the form

$$\bar{b}(\bar{b}^3-\tfrac{3}{2}\bar{b}^2-\tfrac{3}{2}\bar{b}+1)=0. \tag{3.5}$$

The cubic term is easily factored, giving

$$\bar{b}(\bar{b}-2)(\bar{b}-\tfrac{1}{2})(\bar{b}+1)=0. \tag{3.6}$$

It is very interesting that *the roots of (3.6) coincide with the horizontal lines bounding the allowed domains of Okubo and Mathur*.[17] In Sec. IV we shall exhibit in detail how the full boundaries are related to the positivity of (mass)^2 in a model with $\epsilon_0, \epsilon_8\neq 0$.

Next we consider the dependence of the full solution on the quantity s.

Case 1. $\bar{s}=0$, $\bar{\xi}_0\neq 0$. For this value of s the third equation of Eqs. (3.2) disappears. The mesons S and P decouple and one is essentially left with the 18-meson model. From Eq. (3.2) we see that for each value of $\bar{b}$ we can determine the ratio of f_1/f_2. *Only when the quartic couplings assume these ratios can the indicated solution exist.*

$$\begin{array}{lcccc}\bar{b}= & 0 & \tfrac{1}{2} & 2 & -1 \\ f_1/f_2= & -\tfrac{1}{3} & -\tfrac{1}{2} & -\tfrac{1}{3} & -1.\end{array} \tag{3.7}$$

Case 2. $\bar{s}\neq 0$, $\bar{\xi}_0\neq 0$, $\bar{\xi}_8\neq 0$. In this case the third equation of Eqs. (3.2) may be used (with $\epsilon_9=0$) to determine the ratio $\bar{s}/\bar{\xi}_0$ in terms of $\bar{b}$ and f_3/f_4. To conform to the notation of our previous paper,[15] we introduce the notation

$$b'\equiv s/\xi_0, \quad \bar{b}'\equiv\bar{s}/\bar{\xi}_0, \tag{3.8}$$

and obtain from (3.2)

$$(\bar{b}')^2=-(1+2\bar{b}^2)f_3/2f_4. \tag{3.9}$$

Equation (3.9) shows that f_3/f_4 must be negative in order that this solution exist. We now substitute Eq. (3.9) into the first two equations of Eqs. (3.2) (with $\epsilon_0=\epsilon_8=0$) and obtain for each allowed $\bar{b}$ a relation among the coupling constants f_i which must hold if a spontaneously broken solution of the assumed type is to exist (Table III).

For illustration we exhibit in Table IV the "masses" of the 18 mesons belonging to $(3,\bar{3})+(\bar{3},3)$ in the case $\bar{s}=0$. In every case we have a massless dilaton, and for $\bar{b}\neq 0$ a massless κ. [The vanishing of the κ mass is an automatic consequence of the second equation of Eqs. (3.2), with $\epsilon_8=0$.] The relation of various domain boundaries

TABLE III. The existence of solutions exhibiting spontaneous breakdown only occurs when certain coupling-constant relations are satisfied. In each case there is a definite value of the vacuum-expectation-value ratios $b=\xi_8/\sqrt{2}\,\xi_0$ and $b'=s/\xi_0$. These relations are given by the columns of Table III.

$\bar{b}$	0	$\frac{1}{2}$	2	−1
$\bar{b}'^2$	$-\frac{1}{2}f_3/f_4$	$-\frac{3}{4}f_3/f_4$	$-\frac{9}{2}f_3/f_4$	$-\frac{3}{2}f_3/f_4$
Consistency relation	$4(3f_1+f_2)f_4=3f_3^2$	$2(2f_1+f_2)f_4=f_3^2$	$4(3f_1+f_2)f_4=3f_3^2$	$4(f_1+f_2)f_4=f_3^2$

to subsymmetries of $SU_3 \times SU_3$ and their relation to particle masses has been analyzed by Okubo and Mathur.[17] Inspection of Table IV shows that one can choose f_2 so that all m^2 are non-negative except for the case $\bar{b} = \frac{1}{2}$. The case of SU_3 invariance ($\bar{b} = 0$) is characterized by a massless pseudoscalar nonet, with the scalars massive except for the SU_3 singlet (the dilaton). This solution seems most attractive of those listed.

The main points of interest in the foregoing analysis are: (1) Only certain definite values of the ratio $\bar{b} = \bar{\xi}_8/\sqrt{2}\,\bar{\xi}_0$ are allowed when scale invariance is spontaneously broken; (2) for each of these values of $\bar{b}$, the coupling constants must have a definite relation [given in Eq. (3.7) or Table III] for the solution to exist; (3) the values of $\bar{b}$ obtained in this model are closely related to the positivity conditions of Ref. 17.

Since the value of $\bar{b}$ is restricted to one of four values, we see that in order that a scale-invariance-broken theory go over smoothly to one of the spontaneously scale-broken limits, it is necessary that the scale-invariance-breaking parameters ϵ_i be adjusted in such a manner that $b \to \bar{b}$ as $\epsilon_i \to 0$. A further physical requirement is that all (mass)^2 be positive throughout this limit.

C. Turning on Scale Breaking in the 18-Meson Model; Case of Fixed Quartic Couplings

Before considering the intricacies of the complete equations (3.2), we analyze in detail the behavior of the 18-meson model as we switch on the linear scale-invariance-breaking interaction [(3.1) with $\epsilon_9 = 0$]. In this subsection we keep the quartic couplings f_1, f_2 of $\bar{\mathcal{L}}$ fixed, although we shall later permit them to vary as the ϵ_i are varied. The case $\langle\sigma_8\rangle = 0$ ($b = 0$) was already studied in Ref. 7, where it was concluded (using a cubic scale-invariance-breaking term that if the ratio f_1/f_2 was held fixed at that value ($-\frac{1}{3}$) for which spontaneous breakdown is possible, then the solution goes over to the normal vacuum in the limit of scale invariance. The relevant equation follows from (3.2), with $\xi_8 = 0$:

$$\epsilon_0 = \tfrac{4}{3}\xi_0^{\,3}(3f_1 + f_2). \tag{3.10}$$

If $3f_1 + f_2$ vanishes, then no solution exists with $\epsilon_0 \neq 0$.

Using the definition of b given in Eq. (3.4), we rewrite (3.2) as

$$\begin{aligned} \frac{\epsilon_0}{4\xi_0^{\,3}} &= f_1(1+2b^2) + \tfrac{1}{3}f_2(1+6b^2-2b^3), \\ \frac{\epsilon_8}{4\sqrt{2}\,\xi_0^{\,3}} &= b[f_1(1+2b^2) + f_2(1+b^2-b)]. \end{aligned} \tag{3.11}$$

TABLE IV. The masses of the SU_2 multiplets in the scale-invariant 18-meson model are given for the four allowed solutions to the spontaneous-breakdown equations. The mass unit is $\xi_0^{\,2}f_2/3$. Note that in each case there is a 0^+ dilaton. The case $\bar{b} = \frac{1}{2}$ is unphysical since no choice of f_2 makes every m^2 positive.

$\bar{b}$	0	$\frac{1}{2}$	2	-1
f_1/f_2	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	-1
$m_{\sigma_1}^{\,2}$	0	0	0	0
$m_{\sigma_2}^{\,2}$	-8	9	-72	36
$m_{\pi_N}^{\,2}$	-8	-18	-72	36
$m_{\kappa}^{\,2}$	-8	0	0	0
$m_{\eta_1}^{\,2}$	0	0	0	0
$m_{\eta_2}^{\,2}$	0	9	0	36
$m_{\pi}^{\,2}$	0	0	0	36
$m_{K}^{\,2}$	0	0	-72	0

Taking the ratio of these equations yields for the parameter a [cf. Eq. (3.3)]

$$a \equiv \frac{\epsilon_8}{\sqrt{2}\,\epsilon_0} = \frac{b[(1+2b^2)f_1 + (1-b+b^2)f_2]}{(1+2b^2)f_1 + \frac{1}{3}(1+6b^2-2b^3)f_2}. \tag{3.12}$$

We know that the right-hand side has a limit of the form 0/0 as we approach the scale-invariant limit.

If we hold f_1/f_2 fixed at the values given in Eq. (3.7) and analyze (3.12) as $b \to \bar{b}$, we find the following ratios:

$$\begin{aligned} \bar{b} &= \quad \tfrac{1}{2} \quad\; 2 \quad -1 \\ \bar{a} &= \; -1 \quad -\tfrac{1}{4} \quad \tfrac{1}{2}. \end{aligned} \tag{3.13}$$

We can obtain a more general result, which gives b as a function of a, by substituting the allowed values of f_1/f_2 in (3.12) and simplifying the cubics (one-term factors).

The resulting equations are

$$\begin{aligned} a &= \frac{3b}{2b^2-2b-1}, & \frac{f_1}{f_2} &= -\frac{1}{2}, \\ a &= \frac{1-b}{2b}, & \frac{f_1}{f_2} &= -\frac{1}{3}, \\ a &= \frac{3}{2}\,\frac{b^2}{b^2-b+1}, & \frac{f_1}{f_2} &= -1; \end{aligned} \tag{3.14}$$

and the appropriate inverse relations are

$$\begin{aligned} b &= \frac{2\alpha+3-[(2\alpha+3)^2+8\alpha^2]^{1/2}}{4\alpha}, & \frac{f_1}{f_2} &= -\frac{1}{2}, \\ b &= 1/(1+2\alpha), & \frac{f_1}{f_2} &= -\frac{1}{3}, \\ b &= \frac{2\alpha-[4\alpha^2-8\alpha(2\alpha-3)]^{1/2}}{2(2\alpha-3)} & \frac{f_1}{f_2} &= -1. \end{aligned} \tag{3.15}$$

We now see that those solutions which connect smoothly with the spontaneously broken limit are characterized by

$$\begin{aligned}(\bar{a},\bar{b}) &= (-1,\tfrac{1}{2}),\quad f_1/f_2=-\tfrac{1}{2},\\ &= (-\tfrac{1}{4},2),\quad f_1/f_2=-\tfrac{1}{3},\\ &= (\tfrac{1}{2},-1),\quad f_1/f_2=-1,\end{aligned}\tag{3.16}$$

and hence lie *outside the allowed domains* of Okubo and Mathur.[17] The significance of this result is that when the breaking of scale invariance is turned on, one (mass)^2 (or more) becomes negative. This means that the spontaneously broken solution is unstable to the type of external perturbation considered here, and hence is not an acceptable solution.

Hence in the 18-meson model, we find that if the structure of $\bar{\mathfrak{L}}$ (specifically the ratio f_1/f_2) is *fixed* by the existence of a spontaneous breakdown of scale invariance, there is no physical limit $\epsilon_0\to 0$, $\epsilon_8\to 0$ which goes over smoothly to any of the spontaneous-breakdown solutions.

However, if one is willing to vary f_1/f_2 in addition to the other parameters, it is possible to achieve the smooth and physical connection denied us in the foregoing analysis.

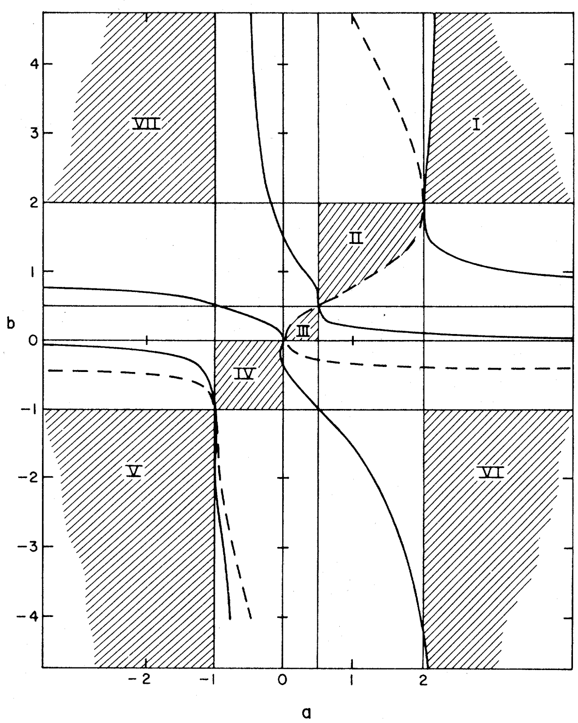

FIG. 1. The allowed domains for the parameters $a=\epsilon_8/\sqrt{2}\,\epsilon_0$, $b=\xi_8/\sqrt{2}\,\xi_0$ resulting from the positivity of squared masses are shown. When $\epsilon_0\xi_0<0$ the positivity of the κ, η, π, K (mass)^2 gives the squares labeled I–V. For $\epsilon_0\xi_0>0$ these same restrictions lead to domains VI and VII. The remaining masses lead to positivity conditions related to the solid and dashed curves. These conditions eliminate parts of the domains I–VII previously given.

D. Turning on Scale-Invariance Breaking in the 18-Meson Model; Variable Quartic Couplings

In the previous section we showed that although the 18-meson model has spontaneously broken solutions, they were all unstable under small perturbations. The point of view was that we should take $\bar{\mathfrak{L}}$, i.e., f_1, f_2, to be fixed once and for all by its form in the absence of explicit breaking terms. If we relax this constraint and allow f_1, f_2 to vary smoothly with the scale-invariance-breaking parameters, we can find stable solutions, as we now show.

As we approach a spontaneously broken solution, b must approach one of its allowed values $\bar{b}$, and f_1/f_2 must approach its corresponding value (3.16) (call it $\bar{f}_1/\bar{f}_2$). Let these quantities approach their limits together as follows:

$$\frac{f_1}{f_2}-\frac{\bar{f}_1}{\bar{f}_2}=\rho(b-\bar{b}).\tag{3.17}$$

If we allow this freedom, then a is no longer determined uniquely but depends on ρ.

Going back to Eq. (3.12) and substituting (3.17), we find for the three solutions the following cases as $b\to\bar{b}$:

$$\begin{aligned}&b\to\frac{1}{2},\quad \frac{f_1}{f_2}\to-\frac{1}{2},\quad a\to\frac{1}{2}\frac{(3\rho-2)}{(3\rho+1)},\\ &b\to 2,\quad \frac{f_1}{f_2}\to-\frac{1}{3},\quad a\to 2\frac{(27\rho+1)}{(27\rho-8)},\\ &b\to -1,\quad \frac{f_1}{f_2}\to -1,\quad a\to-\frac{3\rho+1}{3\rho-2}.\end{aligned}\tag{3.18}$$

In addition, $a\to 0$ as $b\to 0$ (which case requires $f_1/f_2\to\frac{1}{3}$). These results reduce to those of the previous section, (3.13), if we let $\rho=0$.

It is clear from Eq. (3.18) that a can take on any value by choosing ρ appropriately (real). This is the freedom that saves the model from instability. In terms of the Okubo-Mathur plot (Fig. 1), it means that as we approach one of the horizontal boundary lines $\bar{b}$, we must adjust ρ so that a lies in one of the allowed regions.

We note that if $\rho\to\infty$, then for the three cases in Eq. (3.18) $\bar{a}\to\bar{b}$. These are the points of "normal" solutions corresponding to all masses zero.

In summary, this model exhibits the following properties. If we start from a spontaneously broken solution characterized by $\bar{b}$, and perturb it with linear terms whose strengths are in the ratio a, then we must first be sure that we choose a in an allowed domain. Having done so, a connection between the vacuum expectation values and the parameters of the scale-invariant Lagrangian is established in order that the solution not go normal (i.e., all masses zero) as we turn off the scale-invariance-breaking terms.

IV. STABILITY CONDITIONS AND ALLOWED DOMAINS FOR SYMMETRY-BREAKING PARAMETERS IN THE 18-MESON MODEL

A. Introduction

The significance of many of the results of Sec. III is made more transparent if we compute the masses. At the same time we are able to give a physical interpretation to the allowed domains for a and b found in a formal (and more general) analysis by Okubo and Mathur.[17] It is not completely surprising that the results of our approximation must satisfy these more general conditions because the tree approximation is in essence a semiclassical one in which the fluctuations are considered small. The classical limit must surely obey the general requirements. The specific features of the model under consideration lead to slightly smaller domains than those of Ref. 17.

The 18-meson model with a linear $\mathcal{L}_{\text{S.B.}}$ is sufficiently simple that we can write the m_i^2 as simple explicit functions of a and b. The stability requirement $m_i^2 \geq 0$ then gives constraints on the parameters. Four of these conditions give precisely the $U_3 \times U_3$ domain structure of Ref. 17 while the remaining conditions constrict the allowed domains somewhat more.

B. Positivity Conditions for Squared Masses

In order to compute the masses, we need the coefficients of the quadratic terms in $\mathcal{L}$, expressed in terms of the displaced fields. These are given in the first column of Table V. If we eliminate the f_i in favor of the other parameters, we obtain the expressions given in column two of Table V.

The desired relations for f_i are obtained by solving the extremal conditions

$$\begin{aligned}\epsilon_0 &= 2\xi_0^{\,3}[2f_1(1+2b^2)+\tfrac{2}{3}f_2(1+6b^2-2b^3)],\\ \epsilon_8 &= 4\xi_0^{\,3}\sqrt{2}\,b[f_1(1+2b^2)+f_2(1-b+b^2)]\end{aligned} \tag{4.1}$$

for f_1 and f_2 (recall $a=\epsilon_8/\sqrt{2}\,\epsilon_0$);

$$\begin{aligned}f_1 &= \frac{\epsilon_0}{\xi_0^{\,3}}\,\frac{a(2b^3-6b^2-1)+3b(b^2-b+1)}{8(1+2b^2)b(b-\frac{1}{2})(b+1)(b-2)},\\ f_2 &= \frac{\epsilon_0}{\xi_0^{\,3}}\,\frac{3(a-b)}{8b(b-\frac{1}{2})(b+1)(b-2)}.\end{aligned} \tag{4.2}$$

Mixing occurs for the pairs (ϕ_0, ϕ_8) and (σ_0', σ_8'). For ϕ_0, ϕ_8 we define physical fields η, η' in the usual way by

$$\begin{aligned}\eta &= \phi_8\cos\theta - \phi_0\sin\theta,\\ \eta' &= \phi_8\sin\theta + \phi_0\cos\theta.\end{aligned} \tag{4.3}$$

It turns out that the $\eta-\eta'$ mixing problem has a very simple (and wrong) solution in this model.

TABLE V. The quadratic terms in the displaced fields are given in the first column. When the quartic coupling constants are eliminated using Eq. (4.2), the masses are expressible in terms of the parameters a and b as indicated in the second column.

$\sigma_0'^2$	$2[f_1(3+2b^2)+f_2(1+2b^2)]$	$m_\sigma^{\,2}$ see text [Eq. (4.6)]
$\sigma_8'^2$	$2[f_1(1+6b^2)+f_2(3b^2-2b+1)]$	$m_{\sigma'}^{\,2}$ see text [Eq. (4.6)]
$\sigma_0'\sigma_8'$	$4\sqrt{2}\,b[2f_1+(2-b)f_2]$	
$\sum_{i=1}^{3}\sigma_i^2$	$2[f_1(1+2b^2)+f_2(1+b)^2]$	$m_{\pi_N}^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left[\dfrac{a}{b}+\dfrac{9(a-b)}{(2b-1)(b+1)(b-2)}\right]$
$\sum_{i=4}^{7}\sigma_i^2$	$2[f_1(1+2b^2)+f_2(b^2-b+1)]$	$m_\kappa^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left(\dfrac{a}{b}\right)$
$\phi_0^{\,2}$	$\frac{2}{3}(1+2b^2)(3f_1+f_2)$	$m_\eta^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left(\dfrac{a-\frac{1}{2}}{b-\frac{1}{2}}\right)$
$\phi_8^{\,2}$	$\frac{2}{3}[3f_1(1+2b^2)+f_2(3b^2-2b+1)]$	$m_{\eta'}^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left(\dfrac{a+1}{b+1}\right)$
$\phi_0\phi_8$	$\frac{4}{3}\sqrt{2}\,bf_2(2-b)$	
$\sum_{i=1}^{3}\phi_i^{\,2}$	$\frac{2}{3}[3f_1(1+2b^2)+f_2(1+b)^2]$	$m_\pi^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left(\dfrac{a+1}{b+1}\right)$
$\sum_{i=4}^{7}\phi_i^{\,2}$	$\frac{2}{3}[3f_1(1+2b^2)+f_2(7b^2-b+1)]$	$m_K^{\,2}=-\dfrac{\epsilon_0}{\xi_0}\left(\dfrac{a-2}{b-2}\right)$

The mixing formulas are

$$\tan 2\theta = \frac{2m_{08}{}^2}{m_0{}^2 - m_8{}^2},$$

$$m_\eta{}^2 = m_8{}^2\cos^2\theta + m_0{}^2\sin^2\theta - m_{08}{}^2\sin 2\theta, \qquad (4.4)$$

$$m_{\eta'}{}^2 = m_8{}^2\sin^2\theta + m_0{}^2\cos^2\theta + m_{08}{}^2\sin 2\theta,$$

where the quantities $m_0{}^2$, $m_8{}^2$, $m_{08}{}^2$ enter in $-\mathcal{L}_{\text{S.B.}}$ as

$$\tfrac{1}{2}m_0{}^2\phi_0{}^2 + \tfrac{1}{2}m_8{}^2\phi_8{}^2 + m_{08}{}^2\phi_0\phi_8.$$

From Table V we see that provided b and f_2 are nonvanishing, we get the joint results

$$\begin{aligned} &\tan\theta = 1/\sqrt{2}\,, \\ &m_{\eta'}{}^2 = m_\pi{}^2, \\ &\eta' = (\tfrac{1}{3})^{1/2}\phi_8 + (\tfrac{2}{3})^{1/2}\phi_0, \\ &\eta = (\tfrac{2}{3})^{1/2}\phi_8 - (\tfrac{1}{3})^{1/2}\phi_0. \end{aligned} \qquad (4.5)$$

This result is remarkable not because it disagrees with experiment (it was evident from the beginning that the η' would give trouble in a model of this type), but rather that the mixing angle θ and the π-η' degeneracy are independent of the values of the coupling parameters of the model. This feature suggests the existence of extra symmetries in the model beyond those assumed.

The σ_0-σ_8 mixing is not so simple. In this case the mixing angle depends on b and on f_1/f_2. In order to analyze the positivity conditions, it is convenient to use the combinations $m_\sigma{}^2 + m_{\sigma'}{}^2$ and $m_\sigma{}^2 m_{\sigma'}{}^2$ rather than the m^2 of the separate eigenstates σ, σ'. These quantities are somewhat lengthy as given by

$$\begin{aligned} m_\sigma{}^2 + m_{\sigma'}{}^2 &= -\frac{\epsilon_0}{2\xi_0}\,\frac{a(8b^3 - 9b^2 - 6b + 2) - 3b(b^2 + 2b - 2)}{b(b - \frac{1}{2})(b+1)(b-2)}, \\ m_\sigma{}^2 m_{\sigma'}{}^2 &= \frac{3}{2}\,\frac{\epsilon_0{}^2}{\xi_0{}^2}\,\frac{2a^2b(2b^3 + 3b - 4) + 3b^2(-2b^2 + 2b + 1) + a(-6b^4 - 4b^3 + 3b^2 - 6b + 2)}{(2b^2+1)b(b - \frac{1}{2})(b+1)(b-2)}. \end{aligned} \qquad (4.6)$$

Now we note from Table V that the conditions that the K, $\pi(\eta')$, η, and κ (mass)^2 be positive are exactly the same as given by Okubo and Mathur[17] for the case in which $\overline{\mathcal{L}}$ has $\text{U}_3\times\text{U}_3$ symmetry:

$$\begin{aligned} m_\kappa{}^2 &= \frac{-(\epsilon_0\xi_0)(ab)}{\xi_0{}^2 b^2} \geqslant 0, \\ m_\eta{}^2 &= \frac{(-\epsilon_0\xi_0)(a - \frac{1}{2})(b - \frac{1}{2})}{\xi_0{}^2(b - \frac{1}{2})^2} \geqslant 0, \\ m_\pi{}^2 = m_{\eta'}{}^2 &= \frac{(-\epsilon_0\xi_0)(1+a)(1+b)}{\xi_0{}^2(1+b)^2} \geqslant 0, \\ m_K{}^2 &= \frac{(-\epsilon_0\xi_0)(a-2)(b-2)}{\xi_0{}^2(b-2)^2} \geqslant 0. \end{aligned} \qquad (4.7)$$

Hence when $\epsilon_0\xi_0 < 0$, we get the domains labeled I–V in Fig. 1, while domains V and VI result when $\epsilon_0\xi_0 > 0$.

In the particular model considered here, there are three further conditions arising from $m_{\pi_N}{}^2 \geqslant 0$, $m_\sigma{}^2 + m_{\sigma'}{}^2 \geqslant 0$, and $m_\sigma{}^2 m_{\sigma'}{}^2 \geqslant 0$. The appropriate positivity conditions are most easily found by plotting the zero curves. For the scalar pion π_N this yields the dashed curve of Fig. 1. The solid curve corresponds to the vanishing of the second equation of Eqs. (4.6). The other condition leads to no new restrictions.

Spontaneously broken solutions are easily recognized from the mass column in Table V. In order for a mass to survive when we turn off the linear driving terms, we must be near one of the zeros in the denominator. That is, b must approach one of the special values $\bar{b}$ $(=-1,\ 0,\ \frac{1}{2},\ 2)$. The masses that remain nonzero are just those of Table IV. For the special case $a=b$, we can easily verify that all masses go to zero when $\epsilon_0 \to 0$; hence these are normal solutions. Noting this we then can see from Fig. 1 that $b = \frac{1}{2}$ has no stable spontaneously broken solution, i.e., if $b \to \frac{1}{2}$ the solution must go normal.

In Sec. III B we noted that for $\bar{b} = \frac{1}{2}$ no stable solution existed having $m^2 \neq 0$. The result of Fig. 1 shows clearly that only the "normal" solution $(\bar{a} = \bar{b})$ can exist and for this case all masses vanish.

Finally, we mention the case in which $a = b$, i.e., the "response" ratio $\langle\sigma_8\rangle/\langle\sigma_0\rangle$ is the same as the "driving" ratio ϵ_8/ϵ_0. In this case all masses are the same $(m^2 = -\epsilon_0/\xi_0)$ except for one mixed scalar $(m_{\sigma'}{}^2 = -3\epsilon_0/\xi_0)$.

C. 20-Meson Model With Linear Scale-Invariance Breaking

The analogous 20-meson model can be solved in the same way. The Lagrangian now becomes

$$\begin{aligned} \mathcal{L}_{\text{int}} = {} & f_1(\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M})^2 + f_2\operatorname{Tr}(\mathfrak{M}^\dagger\mathfrak{M})^2 \\ & + f_3(S^2 + P^2)\operatorname{Tr}\mathfrak{M}^\dagger\mathfrak{M} + f_4(S^2+P^2)^2 \\ & - \epsilon_0\sigma_0 - \epsilon_8\sigma_8 - \epsilon_9 S. \end{aligned} \qquad (4.8)$$

The extra terms in this model fail to split the η'-π degeneracy; hence we do not give the results in detail.

All the masses in Table V for the 18-meson model remain unchanged except the σ and σ'. The $\eta\eta'$ mixing also remains the same. The P meson does not mix with η and η', and its mass has the simple form

$$m_P{}^2 = -\frac{\epsilon_0}{\xi_0}\frac{a'}{b'}\,. \tag{4.9}$$

The S meson does mix with the σ and σ', further complicating the mass formulas. These masses were not calculated.

D. Nature of the Perturbation Expansion About the Symmetry Limit

An example of a perturbation series with a finite radius of convergence can be seen in the model worked out in Sec. IV B. This model has the defect of π-η' degeneracy so we do not try to fit this model to experimental numbers. However, the general feature emerges that if we expand about a spontaneously broken solution, then the radius of convergence is determined by the closeness of other spontaneously broken solutions.

Let us look at two masses; for illustration, we choose the following:

$$\begin{aligned} m_\pi{}^2 = m_{\eta'}{}^2 &= -\frac{\epsilon_0}{\xi_0}\frac{(a+1)}{(b+1)}, \\ m_\kappa{}^2 &= -\frac{\epsilon_0}{\xi_0}\frac{a}{b}\,. \end{aligned} \tag{4.10}$$

A spontaneously broken solution is one in which one or more masses survive when the driving terms $\epsilon_0\sigma_0 + \epsilon_8\sigma_8$ are turned off. If we turn them off in a fixed ratio then a is constant. If the κ mass survives in this limit, then b must be proportional to ϵ_0 for small ϵ_0,

$$b = \epsilon_0\lambda(\epsilon_0), \tag{4.11}$$

$\lambda(0)$ finite. Then the π mass is of the form

$$m_\pi{}^2 = -\frac{\epsilon_0}{\xi_0}\frac{(a+1)}{(\lambda\epsilon_0+1)}\,. \tag{4.12}$$

This mass now has a pole as a function of ϵ_0 when $[\epsilon_0\lambda(\epsilon_0)+1]$ vanishes. The vanishing of this denominator is just the condition for a different spontaneously broken solution.

In the foregoing illustration, b was required to depend on ϵ_0. However, if the system instead goes normal in the symmetry limit, then b need not depend on ϵ_0, and we cannot conclude that masses must have singularities as functions of ϵ_0, although the possibility is not ruled out.

---

*Supported in part by the National Science Foundation.

[1]S. Gasiorowicz and D. Geffen, Rev. Mod. Phys. 41, 531 (1969). This article summarizes much work in the field and has an extensive bibliography.

[2]C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D 2, 685 (1970).

[3]J. Ellis, Nucl. Phys. B22, 478 (1970).

[4]B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory,* 1970 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970).

[5]J. Schechter and Y. Ueda, Phys. Rev. D 3, 2874 (1971). These authors allow nonpolynomial interactions, but use linear realizations. Our point of view is similar to that of this work.

[6]P. Carruthers, Phys. Rev. D 2, 2265 (1970).

[7]W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969).

[8]M. M. Lévy, Nuovo Cimento 52, 23 (1967).

[9]S. K. Bose and W. D. McGlinn, Phys. Rev. D 3, 2962 (1971).

[10]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[11]P. Carruthers, Phys. Reports 1C, 1 (1971). This review contains many references to previous work on scale invariance.

[12]G. Mack, Nucl. Phys. B5, 499 (1968); G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).

[13]P. Carruthers, Phys. Rev. D 3, 959 (1971).

[14]E. Golowich and E. Lasley, Phys. Rev. D 3, 3114 (1971), have presented results somewhat different from those of Ref. 13.

[15]P. Carruthers and Richard W. Haymaker, Phys. Rev D 4, 406 (1971).

[16]P. Carruthers, Phys. Rev. Letters 26, 1346 (1971).

[17]S. Okubo and V. Mathur, Phys. Rev. D 1, 2046 (1970); 1, 3468 (1970).

[18]P. Carruthers and Richard W. Haymaker, preceding paper, Phys. Rev. D 4, 1808, (1971).

[19]R. Dashen, Phys. Rev. 183, 1245 (1969).

[20]M. Gell-Mann, Physics 1, 63 (1964).

[21]M. Gell-Mann, in *Proceedings of the Hawaii Topical Conference in Particle Physics* (Western Periodicals, Los Angeles, 1969).