Eikonal Approximation for Three-Particle Scattering*

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The behavior of the 3→3 amplitude in nonrelativistic potential scattering is investigated at high energies and small momentum transfers. An approximation scheme is proposed which involves only on-shell two-body amplitudes. The single- and double-scattering terms are obtained from the Watson-Faddeev equations. An eikonal approximation for the triple-scattering term is developed, and the result is expressed as a one-dimensional integral over the product of the three two-body amplitudes.

I. INTRODUCTION

The eikonal approximation has been used extensively for three-particle systems with a bound state in the incident channel. This paper is an attempt to develop an eikonal approximation for the case in which there is no bound state and all three two-body relative momenta are large. The two main physical differences between this case and the bound-state case are the occurrence of widely separated binary scattering events and the occurrence of triple scattering. The eikonal approximation is not appropriate for describing the first of these, and a different approximation is proposed for the corresponding terms in the T matrix. The eikonal approximation is found to be valid for the triple-scattering term of the T matrix under fairly general conditions. This term is evaluated in terms of the on-shell two-body amplitudes.

Although the present paper deals only with nonrelativistic potential scattering, one of the main motivations for this work was the belief that absorptive corrections to single-particle production reactions in the "multi-Regge region" will ultimately involve the three-particle elastic amplitude. It was hoped that an investigation of potential scattering might shed some light on the problems to be faced in a more realistic calculation. Our results indicate that eikonal-type approximations for these absorptive corrections may encounter difficulties associated with the small transverse momentum of the produced particle.

The remainder of this paper is divided into four sections. In Sec. II we derive an eikonal approximation for the three-particle wave function. An expression for the T matrix is obtained in Sec. III and the terms corresponding to successive binary collisions are separated out. The conditions for the triple-scattering term to correspond to close collisions are discussed. The triple-scattering term is reduced to a simple form in Sec. IV, and Sec. V contains a summary of results.

II. EIKONAL APPROXIMATION FOR THE THREE-PARTICLE WAVE FUNCTION

The wave function for three free particles in the center-of-mass frame is

$$\Psi_F = \exp[i(\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2+\vec{k}_3\cdot\vec{r}_3)]\,, \tag{1}$$

where

$$\vec{k}_1+\vec{k}_2+\vec{k}_3=0\,.$$

The total kinetic energy is

$$E=\frac{k_1^{\,2}}{2m_1}+\frac{k_2^{\,2}}{2m_2}+\frac{k_3^{\,2}}{2m_3}\,. \tag{2}$$

The three-particle system may also be described in terms of the relative motion of particles 1 and 2, and the relative motion of their center of mass and particle 3. The wave function (1) and the energy (2) are then written as

$$\Psi_F=\exp[i(\vec{k}_{12}\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{x}_3)]\,,$$

$$E=\frac{k_{12}^{\,2}}{2\mu_{12}}+\frac{k_3^{\,2}}{2\mu_3}\,,$$

where

$$\vec{r}_{12}=\vec{r}_1-\vec{r}_2\,,$$

$$\vec{x}_3=\vec{r}_3-\frac{m_1\vec{r}_1+m_2\vec{r}_2}{M_{12}}\,,$$

$$\vec{k}_{12}=\mu_{12}\left(\frac{\vec{k}_1}{m_1}-\frac{\vec{k}_2}{m_2}\right)\,,$$

$$M_{12}=m_1+m_2\,,$$

$$M=m_1+m_2+m_3\,,$$

$$\mu_{12}=m_1m_2/M_{12}\,,$$

$$\mu_3=m_3M_{12}/M.$$

The free Hamiltonian in the center-of-mass frame is

$$H_F=-\frac{\nabla_{12}^{\,2}}{2\mu_{12}}-\frac{\nabla_3^{\,2}}{2\mu_3}\,,$$

where

$$\nabla_{12}=\frac{\partial}{\partial\vec{r}_{12}}, \quad \nabla_3=\frac{\partial}{\partial\vec{x}_3}.$$

Now consider a scattering event in which three particles are incident and three emerge. The integral equation for the wave function is

$$\Psi(\vec{r}_{12},\vec{x}_3)=\Psi_F(\vec{r}_{12},\vec{x}_3)+\int d^3r'_{12}d^3x'_3\, G(\vec{r}_{12},\vec{x}_3;\,\vec{r}'_{12},\vec{x}'_3) \times V(\vec{r}'_{12},\vec{x}'_3)\Psi(\vec{r}'_{12},\vec{x}'_3). \tag{3}$$

Here $V(\vec{r}_{12},\vec{x}_3)$ denotes the total potential energy of the system. It will be assumed that only central two-body forces act, so that

$$\begin{aligned}V(\vec{r}_{12},\vec{x}_3)&=V_{12}(\vec{r}_{12})+V_{23}(\vec{r}_{23})+V_{31}(\vec{r}_{31})\\&=V_{12}(\vec{r}_{12})+V_{23}\left(-\frac{m_1}{M_{12}}\vec{r}_{12}-\vec{x}_3\right)\\&\quad+V_{31}\left(-\frac{m_2}{M_{12}}\vec{r}_{12}+\vec{x}_3\right).\end{aligned} \tag{4}$$

The Green's function G satisfies the equation

$$(E-H_F)G(\vec{r}_{12},\vec{x}_3;\,\vec{r}'_{12},\vec{x}'_3)=\delta(\vec{r}_{12}-\vec{r}'_{12})\delta(\vec{x}_3-\vec{x}'_3), \tag{5}$$

subject to outgoing-wave boundary conditions. The solution of (5) is[1]

$$G(\vec{r}_{12},\vec{x}_3;\vec{r}'_{12},\vec{x}'_3)=\frac{-iE}{2\pi^2R^2}\left(\frac{m_1m_2m_3}{M}\right)^{3/2}H_2^{(1)}(R\sqrt{E}), \tag{6}$$

where

$$R=(2\mu_{12}|\vec{r}_{12}-\vec{r}'_{12}|^2+2\mu_3|\vec{x}_3-\vec{x}'_3|^2)^{1/2}.$$

Our first high-energy approximation will be to use the asymptotic form of the Hankel function in Eqs. (6) and (3).

$$\begin{aligned}G(\vec{r}_{12},\vec{x}_3;\vec{r}'_{12},\vec{x}'_3)&\xrightarrow[R\sqrt{E}\to\infty]{}\beta e^{iR\sqrt{E}}R^{-5/2},\\ \beta&=e^{i\pi/4}\left[\frac{E^{3/2}}{2\pi^5}\left(\frac{m_1m_2m_3}{M}\right)^3\right]^{1/2}.\end{aligned} \tag{7}$$

As in the two-body eikonal approximation, we let

$$\Psi=\Psi_F\phi$$

and obtain an integral equation for ϕ,

$$\phi(\vec{r}_{12},\vec{x}_3)=1+\beta\int d^3r'_{12}d^3x'_3\left(\frac{1}{R'}\right)^{5/2}\exp[i(R'\sqrt{E}-\vec{k}_{12}\cdot\vec{r}'_{12}-\vec{k}_3\cdot\vec{x}'_3)]\,V(\vec{r}_{12}-\vec{r}'_{12},\vec{x}_3-\vec{x}'_3)\phi(\vec{r}_{12}-\vec{r}'_{12},\vec{x}_3-\vec{x}'_3), \tag{8}$$

where

$$R'=(2\mu_{12}r'^2_{12}+2\mu_3x'^2_3)^{1/2}.$$

Equation (8) may be simplified by defining six-dimensional generalized position and momentum vectors.

$$\begin{aligned}\vec{r}&=(\sqrt{2\mu_{12}}\,\vec{r}_{12},\sqrt{2\mu_3}\,\vec{x}_3),\\ \vec{k}&=(\vec{k}_{12}/\sqrt{2\mu_{12}},\vec{k}_3/\sqrt{2\mu_3}).\end{aligned} \tag{9}$$

These vectors have the following useful properties:

$$\vec{k}\cdot\vec{r}=\vec{k}_{12}\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{x}_3,$$

$$r'=R',$$

$$k=\sqrt{E}.$$

In this six-dimensional space Eq. (8) becomes

$$\phi(\vec{r})=1+\beta'\int d^6r'\left(\frac{1}{r'}\right)^{5/2}e^{i(kr'-\vec{k}\cdot\vec{r}')}\times V(\vec{r}-\vec{r}')\phi(\vec{r}-\vec{r}'), \tag{10}$$

$$\beta'=\frac{\beta}{8}\left(\frac{M}{m_1m_2m_3}\right)^{3/2}.$$

Further simplification of Eq. (10) is obtained by choosing new coordinates in six-space. First choose Cartesian axes such that $\vec{k}$ lies along the first axis. Then, instead of (9), we have

$$\vec{k}=(k,0,0,0,0,0),$$

$$\vec{r}=(r_1,r_2,r_3,r_4,r_5,r_6).$$

Next define six-dimensional spherical coordinates for the position vector.

$$\begin{aligned}r_1&=r\cos\theta,\\ r_2&=r\sin\theta\cos\phi_1,\\ r_3&=r\sin\theta\sin\phi_1\cos\phi_2,\\ r_4&=r\sin\theta\sin\phi_1\sin\phi_2\cos\phi_3,\\ r_5&=r\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3\cos\phi_4,\\ r_6&=r\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4.\end{aligned}$$

The volume element and ranges of integration in spherical coordinates are

$$d^6r=r^5\sin^4\theta\sin^3\phi_1\sin^2\phi_2\sin\phi_3\,dr\,d\theta\,d\phi_1d\phi_2d\phi_3d\phi_4,$$

$$0<r<\infty,$$

$$0<\theta,\,\phi_1,\,\phi_2,\,\phi_3<\pi,$$

$$-\pi<\phi_4<\pi.$$

When the integral in Eq. (10) is written in spherical coordinates, the following integral over θ' is obtained:

$$\int_0^\pi d\theta' \sin^4\theta' \, e^{ikr'(1-\cos\theta')} V(\vec{r}-\vec{r}\,')\phi(\vec{r}-\vec{r}\,') . \qquad (11)$$

We will assume that k is sufficiently large so that the main contribution to this integral comes from the stationary points of the exponential and evaluate the integral accordingly. Furthermore, the contribution of the stationary point at $\theta'=\pi$ will be ignored because it contains the rapidly oscillating factor $e^{2ikr'}$. Our approximate form of (11) is

$$[V(\vec{r}-\vec{r}\,')\phi(\vec{r}-\vec{r}\,')]_{\theta'=0}\int_0^\infty d\theta'\,\theta'^4 e^{ikr'\theta'^2/2}$$

$$=\frac{3\sqrt{\pi/2}\,e^{-3\pi i/4}}{(kr')^{5/2}}\,V(\vec{r}-r'\hat{k})\phi(\vec{r}-r'\hat{k}) .$$

Since $V(\vec{r}-r'\hat{k})\phi(\vec{r}-r'\hat{k})$ is independent of ϕ_1', ϕ_2', ϕ_3', and ϕ_4', the rest of the angular integration in (10) just contributes a numerical factor which turns out to be $8\pi^2/3$. Thus, Eq. (10) becomes

$$\phi(\vec{r})=1-\frac{i}{2k}\int_0^\infty dr'\,V(\vec{r}-r'\hat{k})\phi(\vec{r}-r'\hat{k}) \qquad (12)$$

and this is easily solved to give

$$\phi(\vec{r})=\exp\left[-\frac{i}{2k}\int_0^\infty dr'\,V(\vec{r}-r'\hat{k})\right] . \qquad (13)$$

The solution (13) of Eq. (12) is valid only if $\lim_{r'\to\infty}V(\vec{r}-r'\hat{k})=0$. To check that this is so, and to see just what (13) represents, we transform back to ordinary three-space. Using (4) and (9) it is readily verified that

$$V(\vec{r}-r'\hat{k})=V_{12}(\vec{r}_{12}-\vec{v}_{12}t)+V_{23}(\vec{r}_{23}-\vec{v}_{23}t)$$
$$+V_{31}(\vec{r}_{31}-\vec{v}_{31}t) ,$$

where

$$\vec{v}_{12}=\vec{k}_{12}/\mu_{12} , \text{ etc.},$$
$$t=r'/2k .$$

Note that the limit of $V(\vec{r}-r'\hat{k})$ vanishes as long as all the relative velocities are nonzero. We may now write the eikonal approximation for the three-body wave function as[2]

$$\Psi=\exp[i(\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2+\vec{k}_3\cdot\vec{r}_3)]\exp\left\{-i\int_0^\infty dt[V_{12}(\vec{r}_{12}-\vec{v}_{12}t)+V_{23}(\vec{r}_{23}-\vec{v}_{23}t)+V_{31}(\vec{r}_{31}-\vec{v}_{31}t)]\right\} . \qquad (14)$$

For two particles alone, say 1 and 2, the eikonal approximation for the wave function is[3]

$$\exp[i(\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2)]\exp\left[-i\int_0^\infty dt\,V_{12}(\vec{r}_{12}-\vec{v}_{12}t)\right] . \qquad (15)$$

This two-body approximation is accurate when $r_{12}\lesssim$(potential range) and typical scattering angles are small compared to $(kd)^{-1/2}$, where k is the relative momentum and

$$d=\min\begin{cases}\text{potential range}\\ \text{(relative velocity)/(potential strength)}.\end{cases}$$

Glauber has shown that this restriction on the scattering angles is satisfied quite generally if the wavelength is short compared to the potential range and the energy is large compared to the potential strength.

It is clear that (14) represents a simple physical generalization of (15), and that we may expect similar criteria for accuracy of these two approximations. It will be assumed here that (14) is an acceptable representation of the three-body wave function if each two-body subsystem obeys the restrictions stated in the preceding paragraph.

III. APPROXIMATION FOR THE THREE-PARTICLE T MATRIX

Before considering the three-body T matrix, it will be necessary to make some preliminary remarks concerning the two-body case. The two-particle T matrix of particles 1 and 2 in their center-of-mass frame is

$$(\vec{k}_{12}'|T_{12}|\vec{k}_{12})=\frac{1}{(2\pi)^3}\int d^3r_{12}\,e^{-i\vec{k}_{12}'\cdot\vec{r}_{12}}\,V_{12}(\vec{r}_{12})\psi_{12}(\vec{r}_{12}) , \qquad (16)$$

where the eikonal approximation for ψ_{12} is given by (15). Substitution of (15) into (16) gives a T matrix which violates detailed balance, i.e., is asymmetric in $\vec{k}_{12}$ and $\vec{k}_{12}'$. The conventional procedure for removing this asymmetry is to replace (15) by[3]

$$e^{i\vec{K}_{12}\cdot\vec{r}_{12}}\exp\left[-i\int_0^\infty dt\,V_{12}(\vec{r}_{12}-\hat{K}_{12}v_{12}t)\right] , \qquad (17)$$

where

$$\vec{K}_{12}=\tfrac{1}{2}(\vec{k}_{12}'+\vec{k}_{12}) .$$

The T matrix then reduces to

$$\frac{iv_{12}}{(2\pi)^2}\int_0^\infty db\,b\,J_0(\kappa_{12}b)\,[\,e^{i\chi_{12}'(b,\,v_{12})}-1] , \qquad (18)$$

where

$$\vec{\kappa}_{12}=\vec{k}_{12}'-\vec{k}_{12} ,$$
$$\chi_{12}'(b,v_{12})=-\int_{-\infty}^\infty dt\,V_{12}(\vec{b}+\hat{K}_{12}v_{12}t) . \qquad (19)$$

The direction of $\vec{b}$ in (19) is perpendicular to $\vec{K}_{12}$.

Another way to make the T matrix symmetric is to replace (15) by

$$e^{i\vec{k}_{12}\cdot\vec{r}_{12}}\exp\left[-i\int_0^\infty dt\,V_{12}(\vec{r}_{12}-\vec{K}_{12}t/\mu_{12})\right] \quad (20)$$

rather than (17). The two-particle T matrix then reduces to

$$\frac{iK_{12}}{(2\pi)^2\mu_{12}}\int_0^\infty db\,b\,J_0(\kappa_{12}b)[e^{i\chi_{12}(b,K_{12})}-1], \quad (21)$$

where

$$\chi_{12}(b,K_{12})=(\mu_{12}v_{12}/K_{12})\chi'_{12}(b,v_{12})$$
$$=-\int_{-\infty}^\infty dt\,V_{12}(\vec{b}+\vec{K}_{12}t/\mu_{12}). \quad (22)$$

Note that when the scattering angle θ_{12} is small,

$$K_{12}/\mu_{12}v_{12}=1+O(\theta_{12}^2),$$

so that (18) and (21) are essentially identical when

$$\theta_{12}\ll(v_{12}/V_0a)^{1/2}, \quad (23)$$

where a = (potential range) and V_0 = (average potential strength). This inequality can only break down when[4] $v_{12}/V_0a\ll1$ (i.e., in the classical limit) and even then violation of (23) requires that

$$(V_0/E_{12})(V_0a/v_{12})^{1/2}\gtrsim1, \quad (24)$$

since typical scattering angles in the classical limit are of order V_0/E_{12}, where $E_{12}=k_{12}^2/2\mu_{12}$. When (18) and (21) differ appreciably, it is likely that (18) is the better approximation of the two because it satisfies the optical theorem (for real V_{12}) and (21) does not. In spite of this we are going to use (21) in the following because the criteria for its validity are rather weak, and it will simplify the relationship between the two- and three-body T matrices. We must therefore avoid situations in which any two-body subsystem scatters classically under conditions such that (24) holds for that sybsystem.

For future reference we write the scattering amplitude corresponding to (21) as

$$f_{12}(\kappa_{12},K_{12})=-iK_{12}\int_0^\infty db\,b\,J_0(\kappa_{12}b)\,[e^{i\chi_{12}(b,K_{12})}-1]. \quad (25)$$

The three-body T matrix is

$$(\vec{k}'|T|\vec{k})\equiv(\vec{k}'_1,\vec{k}'_2,\vec{k}'_3|T|\vec{k}_1,\vec{k}_2,\vec{k}_3),$$
$$=\frac{1}{(2\pi)^6}\int d^3r_{12}d^3x_3\exp[-i(\vec{k}'_{12}\cdot\vec{r}_{12}+\vec{k}'_3\cdot\vec{x}_3)]\,V(\vec{r}_{12},\vec{x}_3)\Psi(\vec{r}_{12},\vec{x}_3). \quad (26)$$

Rather than using (14) for Ψ we will use a three-body generalization of (20). We take

$$\Psi(\vec{r}_{12},\vec{x}_3)=\exp[i(\vec{k}_{12}\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{x}_3)]\exp\left\{-i\int_0^\infty dt[V_{12}(\vec{r}_{12}-\vec{K}_{12}t/\mu_{12})+V_{23}(\vec{r}_{23}-\vec{K}_{31}t/\mu_{23})+V_{31}(\vec{r}_{31}-\vec{K}_{31}t/\mu_{31})]\right\}$$

or, in the notation of (13),

$$\Psi(\vec{r})=e^{i\vec{k}\cdot\vec{r}}\exp\left[-\frac{i}{2K}\int_0^\infty dr'\,V(\vec{r}-r'\hat{K})\right], \quad (27)$$

where

$$\vec{K}=\tfrac{1}{2}(\vec{k}'+\vec{k}),$$
$$K=\left[\sum_{i=1}^3\frac{|(\vec{k}'_i+\vec{k}_i)/2|^2}{2m_i}\right]^{1/2}.$$

The eikonal approximation obtained by substituting (27) into (26) is

$$(\vec{k}'|T|\vec{k})=\frac{1}{(8\pi^2)^3}\left(\frac{M}{m_1m_2m_3}\right)^{3/2}\int d^6r\,e^{-i\vec{\kappa}\cdot\vec{r}}\,V(\vec{r})\exp\left[-\frac{i}{2K}\int_0^\infty dr'\,V(\vec{r}-r'\hat{K})\right]$$
$$=\frac{2iK}{(8\pi^2)^3}\left(\frac{M}{m_1m_2m_3}\right)^{3/2}\int d^5B\,e^{-i\vec{\kappa}\cdot\vec{B}}\,(e^{i\chi(\vec{B},\vec{K})}-1). \quad (28)$$

Here $\vec{\kappa}$ is a generalized momentum transfer

$$\vec{\kappa}=\vec{k}'-\vec{k},$$

and by energy conservation

$$\vec{\kappa}\cdot\vec{K}=0.$$

In (28) we have split $\vec{r}$ into a part perpendicular to $\vec{K}$ and a part parallel to $\vec{K}$,

$$\vec{r} = z\hat{K} + \vec{B},$$

$$z = \hat{K}\cdot\vec{r}, \quad \vec{B}\cdot\vec{K} = 0.$$

Finally, $\chi(\vec{B},\vec{K})$ is

$$\chi(\vec{B},\vec{K}) = -\frac{1}{2K}\int_{-\infty}^{\infty} dz\, V(\vec{B}+z\hat{K}) = \chi_{12}(b_{12}, K_{12}) + \chi_{23}(b_{23}, K_{23}) + \chi_{31}(b_{31}, K_{31}), \tag{29}$$

where the two-body impact parameters b_{ij} may be specified in terms of $\vec{B}$ by writing $\vec{B}$ in the coordinate system of Eqs. (9) as

$$\vec{B} = (\sqrt{2\mu_{12}}\,\vec{b}'_{12}, \sqrt{2\mu_3}\,\vec{b}'_3),$$

and defining

$$\vec{b}'_{23} = -\vec{b}'_3 - (m_1/M_{12})\vec{b}'_{12},$$

$$\vec{b}'_{31} = \vec{b}'_3 - (m_2/M_{12})\vec{b}'_{12},$$

then

$$\vec{b}_{ij} = \vec{b}'_{ij} - \hat{K}_{ij}\hat{K}_{ij}\cdot\vec{b}'_{ij}. \tag{30}$$

The two-body eikonals χ_{ij} are defined as in (22).

In its present form the T matrix in Eq. (28) contains terms that depend on the asymptotic behavior of the eikonal wave function (27) at arbitrarily large particle separations. Since (27) is not accurate for particle separations larger than the range of forces, these terms must be separated out and calculated more accurately. This may be done by performing a multiple-scattering expansion of the type used in the eikonal approximation for scattering by a compound system.[3] We define profile functions

$$\Gamma(\vec{B},\vec{K}) = 1 - e^{i\chi(\vec{B},\vec{K})},$$

$$\Gamma_{ij}(b_{ij}, K_{ij}) = 1 - e^{i\chi_{ij}(b_{ij},K_{ij})}.$$

Using (29), we find that

$$\Gamma = \Gamma_{12} + \Gamma_{23} + \Gamma_{31} - \Gamma_{12}\Gamma_{23} - \Gamma_{23}\Gamma_{31} - \Gamma_{31}\Gamma_{12} + \Gamma_{12}\Gamma_{23}\Gamma_{31}. \tag{31}$$

The first three terms in (31) correspond to single scattering of one two-particle subsystem, the next three to double scattering, and the last to triple scattering. The single-scattering term Γ_{12}, for example, is finite for arbitrarily large values of b_{23} and b_{31}. The double-scattering term $\Gamma_{12}\Gamma_{23}$, for example, is finite for arbitrarily large values of b_{31}. Only the triple-scattering term $\Gamma_{12}\Gamma_{23}\Gamma_{31}$ vanishes when any one of the three impact parameters becomes large, and it is therefore the only term for which the eikonal approximation may be reliable.

It is not difficult to think of an improved approximation for the single- and double-scattering terms. Consideration of the Glauber approximation for scattering off a two-particle bound state in the limit that the bound-state wave function approaches a plane wave indicates that the following prescription should be valid: Replace the single- and double-scattering terms in (28) by the corresponding terms in the Watson-Faddeev[5] multiple-scattering expansion with only the on-energy-shell parts of the intermediate-state propagators in the double-scattering terms retained. This prescription gives an expression which involves only on-energy-shell two-body T matrices and these may be replaced by their eikonal approximations if desired. It is perhaps worth mentioning that a similar expression is obtained by substituting the single- and double-scattering terms of (31) into (28) and evaluating the resulting integral by the methods of Sec. IV. The main difference between the expressions is that the one obtained from (28) contains a rather poor approximation to the on-energy-shell parts of the intermediate-state propagators.

We have seen that only those configurations in which all three impact parameters are simultaneously small contribute to the triple-scattering term. This still leaves open the possibility that the triple-scattering term receives important contributions from configurations corresponding to three widely separated binary collisions; such a configuration would have small impact parameters but arbitrarily large particle separations between collisions, and would not be well represented by the eikonal approximation. Under certain circumstances this possibility can be ruled out by the following kinematical argument. Suppose that particles 1 and 2 collide at a point P which lies far from the trajectory of particle 3 (which will be called T) and acquire momenta $\vec{q}_1$ and $\vec{q}_2$. A further collision is possible if the extension of $\vec{q}_1$ intersects T, but $\vec{q}_1 + \vec{q}_2$ is parallel to T (it is equal to $-\vec{k}_3$) so that if 1 is headed for T, 2 must be headed away from T. In the limit of small-angle scattering, it will be impossible for either 1 or 3 to have a further collision with 2. Thus triple scattering can only occur when all three particles are simultaneously within the range of forces. The situation is illustrated in Fig. 1.

This argument obviously breaks down if $\vec{q}_1$ and $\vec{q}_2$ are parallel. In fact, if $\vec{q}_1$ and $\vec{q}_2$ make an angle δ then three collisions separated by distances of order a/δ are possible, and the eikonal approximation cannot be trusted if δ is small. In Sec. IV we will consider only the situation in which $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ are sufficiently far from collinearity for the eikonal approximation to be valid and for triple

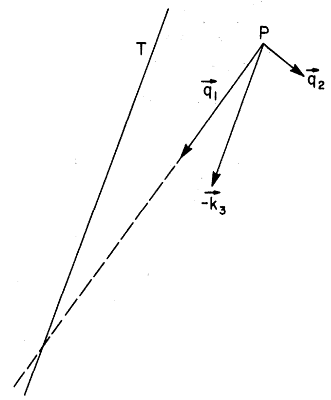

FIG. 1. A momentum configuration for which three widely separated binary collisions cannot occur.

scattering to correspond to close collisions. It should be pointed out, however, that amplitudes with three nearly parallel momenta may be an important part of the absorptive correction to high-energy single-particle production processes because the produced particle tends to have small transverse momentum. This is a somewhat delicate point because the more collinearly produced particles will also tend to be more peripherally produced and therefore may not be greatly affected by absorption. It is possible that absorption involving triple scattering will only be important at momentum transfers for which this collinearity problem does not arise.

IV. EIKONAL APPROXIMATION FOR THE TRIPLE-SCATTERING TERM

In Sec. III we developed an approximation scheme for three-particle scattering in which the triple-scattering term is given by

$$(\vec{k}'|T_3|\vec{k}) = -\frac{2iK}{(8\pi^2)^3}\left(\frac{M}{m_1 m_2 m_3}\right)^{3/2} \times \int d^5B\, e^{-i\vec{\kappa}\cdot\vec{B}}\Gamma_{12}(b_{12}, K_{12}) \times \Gamma_{23}(b_{23}, K_{23})\Gamma_{31}(b_{31}, K_{31}). \tag{32}$$

In this section we will reduce this expression to a more manageable form which involves only the two-particle amplitudes as given by (25).

We begin by constructing a basis for the space $\mathcal{B}$ of vectors $\vec{B}$ perpendicular to $\vec{K}$. To do this we note that despite the asymmetric appearance of $\vec{r}$ in Eq. (9), it has a magnitude

$$r = (2/M)^{1/2}(m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2)^{1/2}$$

which is symmetric under permutations of particle indices. Thus we may expect that the vectors

$$\begin{aligned} R_{23}\vec{r} &= (\sqrt{2\mu_{23}}\,\vec{r}_{23}, \sqrt{2\mu_1}\,\vec{x}_1), \\ R_{31}\vec{r} &= (\sqrt{2\mu_{31}}\,\vec{r}_{31}, \sqrt{2\mu_2}\,\vec{x}_2), \end{aligned} \tag{33}$$

can be obtained from $\vec{r}$ by rotations in six-dimensional space. In (33) we have indicated the rotation matrices by R_{23} and R_{31}. It is readily verified that these matrices do in fact exist and are given explicitly by

$$R_{23} = \frac{1}{\sqrt{M_{12}M_{23}}}\begin{pmatrix} -\sqrt{m_3 m_1} & -\sqrt{m_2 M} \\ \sqrt{m_2 M} & -\sqrt{m_3 m_1} \end{pmatrix},$$

$$R_{31} = \frac{1}{\sqrt{M_{12}M_{31}}}\begin{pmatrix} -\sqrt{m_2 m_3} & \sqrt{m_1 M} \\ -\sqrt{m_1 M} & -\sqrt{m_2 m_3} \end{pmatrix}.$$

These matrices satisfy the orthogonality relations

$$R_{23}{}^T = R_{23}{}^{-1},$$
$$R_{31}{}^T = R_{31}{}^{-1},$$

and have determinants equal to +1.

In order to avoid any confusion, we point out that each element in R_{23} and R_{31} is itself a multiple of the 3×3 unit matrix. Also, the matrices act directly on the vectors in (33), and the same coordinate system is used in (9) and (33).

Since $\vec{k}\cdot\vec{r} = \sum \vec{k}_i\cdot\vec{r}_i$ is a scalar which is invariant under particle permutations, it is not surprising that relations similar to (33) hold for $\vec{k}$,

$$R_{23}\vec{k} = (\vec{k}_{23}/\sqrt{2\mu_{23}}, \vec{k}_1/\sqrt{2\mu_1}),$$
$$R_{31}\vec{k} = (\vec{k}_{31}/\sqrt{2\mu_{31}}, \vec{k}_2/\sqrt{2\mu_2}),$$

and also for $\vec{K}$ and $\vec{\kappa}$.

The rotation matrices may be used to construct a basis for $\mathcal{B}$. First, we define

$$\vec{B}_{12} = (\sqrt{2\mu_{12}}\,\vec{b}_{12}, 0),$$

where $\vec{b}_{12}$ is the impact-parameter vector of particles 1 and 2 as given by (30). The vector $\vec{B}_{12}$ is perpendicular to $\vec{K}$ and spans a plane in $\mathcal{B}$. An orthonormal basis for this plane is given by the two vectors,

$$\hat{\Gamma}_{12} = (\hat{\gamma}, 0),$$
$$\hat{\Sigma}_{12} = (\hat{\sigma}_{12}, 0),$$

where $\hat{\gamma}$ is a unit vector perpendicular to the plane of $\vec{K}_{12}$, $\vec{K}_{23}$, and $\vec{K}_{31}$ (note that $\vec{K}_{12}/\mu_{12} + \vec{K}_{23}/\mu_{23} + \vec{K}_{31}/\mu_{31} = 0$) and $\hat{\sigma}_{12}$ is a unit vector lying in the plane of $\vec{K}_{12}$, $\vec{K}_{23}$, and $\vec{K}_{31}$ and is perpendicular to $\vec{K}_{12}$. We adopt the sign convention that $\hat{\gamma}$ points in the same direction as $\vec{K}_{12}\times\vec{K}_{23}$, $\vec{K}_{23}\times\vec{K}_{31}$, and $\vec{K}_{31}\times\vec{K}_{12}$ and that

$$\hat{\sigma}_{12} = \hat{\gamma}\times\hat{K}_{12}.$$

For later use we also define

$$\hat{\sigma}_{23} = \hat{\gamma} \times \hat{K}_{23},$$
$$\hat{\sigma}_{31} = \hat{\gamma} \times \hat{K}_{31}.$$

To continue with the construction of a basis, define

$$R_{23}\vec{B}_{23} = (\sqrt{2\mu_{23}}\,\vec{b}_{23}, 0)$$

and note that $R_{23}\vec{B}_{23}$ is perpendicular to $R_{23}\vec{K}$. Thus $\vec{B}_{23}$ is perpendicular to $\vec{K}$. The vector $R_{23}\vec{B}_{23}$ spans a plane with the orthonormal basis

$$R_{23}\hat{\Gamma}_{23} = (\hat{\gamma}, 0),$$
$$R_{23}\hat{\Sigma}_{23} = (\hat{\sigma}_{23}, 0),$$

so $\vec{B}_{23}$ spans a plane in $\mathcal{B}$ with the orthonormal basis

$$\hat{\Gamma}_{23} = \frac{1}{\sqrt{M_{12}M_{23}}}(-\sqrt{m_1 m_3}\,\hat{\gamma}, -\sqrt{m_2 M}\,\hat{\gamma}),$$
$$\hat{\Sigma}_{23} = \frac{1}{\sqrt{M_{12}M_{23}}}(-\sqrt{m_1 m_3}\,\hat{\sigma}_{23}, -\sqrt{m_2 M}\,\hat{\sigma}_{23}).$$

Similarly, the vector $\vec{B}_{31}$ defined by

$$R_{31}\vec{B}_{31} = (\sqrt{2\mu_{31}}\,\vec{b}_{31}, 0)$$

spans a plane in $\mathcal{B}$ with orthonormal basis

$$\hat{\Gamma}_{31} = \frac{1}{\sqrt{M_{12}M_{31}}}(-\sqrt{m_2 m_3}\,\hat{\gamma}, \sqrt{m_1 M}\,\hat{\gamma}),$$
$$\hat{\Sigma}_{31} = \frac{1}{\sqrt{M_{12}M_{31}}}(-\sqrt{m_2 m_3}\,\hat{\sigma}_{31}, \sqrt{m_1 M}\,\hat{\sigma}_{31}).$$

The $\hat{\Gamma}$'s and $\hat{\Sigma}$'s provide us with a basis for $\mathcal{B}$ The $\hat{\Gamma}$'s are linearly dependent,

$$\frac{\hat{\Gamma}_{12}}{\sqrt{2\mu_{12}}} + \frac{\hat{\Gamma}_{23}}{\sqrt{2\mu_{23}}} + \frac{\hat{\Gamma}_{31}}{\sqrt{2\mu_{31}}} = 0, \tag{34}$$

and span a two-dimensional subspace of $\mathcal{B}$ which we will call Γ. The $\hat{\Sigma}$'s are linearly independent and span a three-dimensional subspace of $\mathcal{B}$, called Σ, which is perpendicular to Γ. Note that the scalar product of $\vec{r}$ with the left-hand side of (34) is $\hat{\gamma}\cdot(\vec{r}_{12}+\vec{r}_{23}+\vec{r}_{31})$ and that the linear dependence of the $\hat{\Gamma}$'s is therefore a manifestation of the geometrical relation $\vec{r}_{12}+\vec{r}_{23}+\vec{r}_{31}=0$.

The use of this basis for the evaluation of (32) is facilitated by obtaining an expression for the unit dyadic in $\mathcal{B}$. This is clearly the sum of the unit dyadics in Γ and Σ,

$$\overleftrightarrow{I}_{\mathcal{B}} = \overleftrightarrow{I}_{\Gamma} + \overleftrightarrow{I}_{\Sigma}.$$

The unit dyadic in Γ may be written

$$\overleftrightarrow{I}_{\Gamma} = c_{12}\hat{\Gamma}_{12}\hat{\Gamma}_{12} + c_{23}\hat{\Gamma}_{23}\hat{\Gamma}_{23} + c_{31}\hat{\Gamma}_{31}\hat{\Gamma}_{31},$$

where the coefficients are to be determined. This may be done, for example, by solving the equations

$$\hat{\Gamma}_{12}\cdot\overleftrightarrow{I}_{\Gamma}\cdot\hat{\Gamma}_{12} = \hat{\Gamma}_1\cdot\overleftrightarrow{I}_{\Gamma}\cdot\hat{\Gamma}_1 = 1,$$
$$\hat{\Gamma}_{12}\cdot\overleftrightarrow{I}_{\Gamma}\cdot\hat{\Gamma}_1 = 0,$$

where

$$\hat{\Gamma}_1 = (0, \hat{\gamma}).$$

The result of this is

$$\overleftrightarrow{I}_{\Gamma} = (1/M)(M_{12}\hat{\Gamma}_{12}\hat{\Gamma}_{12} + M_{23}\hat{\Gamma}_{23}\hat{\Gamma}_{23} + M_{31}\hat{\Gamma}_{31}\hat{\Gamma}_{31}). \tag{35}$$

The general form of the unit dyadic in Σ is

$$\begin{aligned}\overleftrightarrow{I}_{\Sigma} = {} & a_{12}\hat{\Sigma}_{12}\hat{\Sigma}_{12} + a_{23}\hat{\Sigma}_{23}\hat{\Sigma}_{23} + a_{31}\hat{\Sigma}_{31}\hat{\Sigma}_{31} \\ & + a_1(\hat{\Sigma}_{31}\hat{\Sigma}_{12} + \hat{\Sigma}_{12}\hat{\Sigma}_{31}) \\ & + a_2(\hat{\Sigma}_{12}\hat{\Sigma}_{23} + \hat{\Sigma}_{23}\hat{\Sigma}_{12}) \\ & + a_3(\hat{\Sigma}_{23}\hat{\Sigma}_{31} + \hat{\Sigma}_{31}\hat{\Sigma}_{23}).\end{aligned} \tag{36}$$

As before, the a's may be obtained by evaluating the matrix elements of $\overleftrightarrow{I}_{\Sigma}$. To avoid solving several simultaneous equations, we first note that the vectors

$$\vec{\Sigma}_a = \hat{\Sigma}_{23} + \hat{\Sigma}_{31},$$
$$\vec{\Sigma}_b = \hat{\Sigma}_{23} - \hat{\Sigma}_{31}$$

form an orthogonal basis for the plane spanned by $\hat{\Sigma}_{23}$ and $\hat{\Sigma}_{31}$. Thus, the vector

$$\vec{\Sigma}_c = \hat{\Sigma}_{12} - \hat{\Sigma}_{12}\cdot(\hat{\Sigma}_a\hat{\Sigma}_a + \hat{\Sigma}_b\hat{\Sigma}_b)$$

is orthogonal to both $\hat{\Sigma}_{23}$ and $\hat{\Sigma}_{31}$. Hence,

$$\begin{aligned}&\vec{\Sigma}_c\cdot\vec{\Sigma}_c = \vec{\Sigma}_c\cdot\overleftrightarrow{I}_{\Sigma}\cdot\vec{\Sigma}_c = a_{12}(\vec{\Sigma}_c\cdot\hat{\Sigma}_{12})^2, \\ &a_{12} = \frac{\vec{\Sigma}_c\cdot\vec{\Sigma}_c}{(\vec{\Sigma}_c\cdot\hat{\Sigma}_{12})^2} = \frac{1}{\vec{\Sigma}_c\cdot\hat{\Sigma}_{12}},\end{aligned} \tag{37}$$

and a_{23} and a_{31} may be obtained similarly. To calculate a_1 we construct the vector

$$\vec{\Sigma}_d = \hat{\Sigma}_{12} - \hat{\Sigma}_{23}\hat{\Sigma}_{23}\cdot\hat{\Sigma}_{12}$$

which is orthogonal to $\hat{\Sigma}_{23}$. Then,

$$\begin{aligned}\vec{\Sigma}_d\cdot\vec{\Sigma}_d &= \vec{\Sigma}_d\cdot\overleftrightarrow{I}_{\Sigma}\cdot\vec{\Sigma}_d \\ &= a_{12}(\vec{\Sigma}_d\cdot\hat{\Sigma}_{12})^2 + a_{31}(\vec{\Sigma}_d\cdot\hat{\Sigma}_{31})^2 \\ &\quad + 2a_1(\vec{\Sigma}_d\cdot\hat{\Sigma}_{12})(\vec{\Sigma}_d\cdot\hat{\Sigma}_{31}).\end{aligned}$$

Having already found a_{12} and a_{31} this gives a_1 directly, and a_2 and a_3 are found in the same way.

The results obtained by this process are

$$\begin{aligned}&a_{ij} = \alpha\left(1 - \frac{m_i m_j}{M_{ki}M_{jk}}\eta_k^{\,2}\right), \\ &a_i = \alpha\left(\frac{m_j m_k}{M_{ij}M_{ki}}\right)^{1/2}\left(\eta_i + \frac{m_i}{M_{jk}}\eta_j\eta_k\right),\end{aligned} \tag{38}$$

where i, j, k is here (and from now on) a cyclic permutation of 1, 2, 3 and

$$\eta_i = \hat{K}_{ij}\cdot\hat{K}_{ki},$$
$$\frac{1}{\alpha} = 1 - \sum_{i=1}^{3}\left(\frac{m_j m_k}{M_{ij}M_{ki}}\right)\eta_i^{\,2} - 2\left(\frac{m_1 m_2 m_3}{M_{12}M_{23}M_{31}}\right)\eta_1\eta_2\eta_3.$$

Note that $\alpha > 0$ since $a_{12} > 0$ by (37) and $a_{12}/\alpha > 0$ by (38), and that $1/\alpha = 0$ when the average momenta are collinear, i.e., when $|\eta_1| = |\eta_2| = |\eta_3| = 1$.

It is now possible to express the volume element d^5B in terms of the two-body impact-parameter vectors. We do this in two steps. First, we expand $\vec{B}$ in components,

$$\vec{B} = z_1\hat{\Sigma}_{23} + z_2\hat{\Sigma}_{31} + z_3\hat{\Sigma}_{12} + z_4\hat{\Gamma}_{12} + z_5\hat{\Gamma}_{23}, \tag{39}$$

and write the volume element as

$$d^5B = |\hat{\Gamma}_{12}\times\hat{\Gamma}_{23}|\,|\hat{\Sigma}_{12}\cdot(\hat{\Sigma}_{23}\times\hat{\Sigma}_{31})|\,dz_1\cdots dz_5.$$

The factor $|\hat{\Gamma}_{12}\times\hat{\Gamma}_{23}|$ is easily obtained:

$$|\hat{\Gamma}_{12}\times\hat{\Gamma}_{23}| = [1-(\hat{\Gamma}_{12}\cdot\hat{\Gamma}_{23})^2]^{1/2} = (m_2M/M_{12}M_{23})^{1/2}.$$

The factor $|\hat{\Sigma}_{12}\cdot(\hat{\Sigma}_{23}\times\hat{\Sigma}_{31})|$ is most easily evaluated by using an orthogonal basis in Σ. One can use the basis $\vec{\Sigma}_a$, $\vec{\Sigma}_b$, $\vec{\Sigma}_c$ set up for the calculation of a_{12}. Then,

$$|\hat{\Sigma}_{12}\cdot(\hat{\Sigma}_{23}\times\hat{\Sigma}_{31})| = \left|\det\begin{pmatrix} \hat{\Sigma}_{12}\cdot\hat{\Sigma}_a & \hat{\Sigma}_{12}\cdot\hat{\Sigma}_b & \hat{\Sigma}_{12}\cdot\hat{\Sigma}_c \\ \hat{\Sigma}_{23}\cdot\hat{\Sigma}_a & \hat{\Sigma}_{12}\cdot\hat{\Sigma}_b & 0 \\ \hat{\Sigma}_{31}\cdot\hat{\Sigma}_a & \hat{\Sigma}_{12}\cdot\hat{\Sigma}_b & 0 \end{pmatrix}\right|$$

$$= |\hat{\Sigma}_{12}\cdot\hat{\Sigma}_c|\,|\hat{\Sigma}_{23}\times\hat{\Sigma}_{31}| = 1/\sqrt{\alpha}.$$

So we have

$$d^5B = \left(\frac{m_2M}{\alpha M_{12}M_{23}}\right)^{1/2} dz_1\cdots dz_5. \tag{40}$$

The second step is to relate the z's to the two-body impact parameters by defining variables B_{ij}, β_{ij}, where

$$\begin{aligned} \vec{B}\cdot\hat{\Gamma}_{ij} &= \sqrt{2\mu_{ij}}\,\vec{b}_{ij}\cdot\hat{\gamma} = B_{ij}\cos\beta_{ij}, \\ \vec{B}\cdot\hat{\Sigma}_{ij} &= \sqrt{2\mu_{ij}}\,\vec{b}_{ij}\cdot\hat{\sigma}_{ij} = B_{ij}\sin\beta_{ij}. \end{aligned} \tag{41}$$

Because of the linear dependence of the $\hat{\Gamma}$'s, we have the relation

$$\frac{B_{12}\cos\beta_{12}}{\sqrt{2\mu_{12}}} + \frac{B_{23}\cos\beta_{23}}{\sqrt{2\mu_{23}}} + \frac{B_{31}\cos\beta_{31}}{\sqrt{2\mu_{31}}} = 0 \tag{42}$$

and the B's and β's thus constitute a set of five independent variables. The z's may be expressed in terms of these new variables by using the unit dyadic to obtain the coefficients in (39),

$$\vec{B} = \vec{B}\cdot(\overleftrightarrow{I}_\Gamma + \overleftrightarrow{I}_\Sigma).$$

The resulting relations are

$$z_i = a_{jk}B_{jk}\sin\beta_{jk} + a_jB_{ij}\sin\beta_{ij} + a_kB_{ki}\sin\beta_{ki},$$

$$z_4 = \frac{M_{12}}{M}B_{12}\cos\beta_{12} - \left(\frac{M_{12}M_{31}m_3}{m_2M^2}\right)^{1/2} B_{31}\cos\beta_{31},$$

$$z_5 = \frac{M_{23}}{M}B_{23}\cos\beta_{23} - \left(\frac{M_{12}M_{31}m_1}{m_2M^2}\right)^{1/2} B_{31}\cos\beta_{31}.$$

It will be convenient to introduce a sixth independent variable

$$z_6 = \frac{B_{12}\cos\beta_{12}}{\sqrt{2\mu_{12}}} + \frac{B_{23}\cos\beta_{23}}{\sqrt{2\mu_{23}}} + \frac{B_{31}\cos\beta_{31}}{\sqrt{2\mu_{31}}}$$

and to take account of relation (42) by including a factor $\delta(z_6)dz_6$ in (40). In terms of the new variables, the volume element is

$$d^5B = \left(\frac{m_2M}{\alpha M_{12}M_{23}}\right)^{1/2} |J|\,\delta(z_6)\,dB_{12}dB_{23}dB_{31}d\beta_{12}d\beta_{23}d\beta_{31},$$

where J is the Jacobian determinant

$$J = \det\begin{bmatrix} \partial z_1/\partial B_{12} & \partial z_1/\partial B_{23} & \cdots & \partial z_1/\partial\beta_{31} \\ \partial z_2/\partial B_{12} & \partial z_2/\partial B_{23} & \cdots & \partial z_2/\partial\beta_{31} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \partial z_6/\partial B_{12} & \partial z_6/\partial B_{23} & \cdots & \partial z_6/\partial\beta_{31} \end{bmatrix}.$$

After a somewhat lengthy calculation, one finds that

$$|J| = \frac{\alpha}{\sqrt{2}}\,\frac{M_{12}M_{23}}{m_2M}\left(\frac{M_{31}}{m_3m_1}\right)^{1/2} B_{12}B_{23}B_{31}$$

and

$$\begin{aligned} d^5B = {}& \left(\frac{32\alpha(m_1m_2m_3)^3}{MM_{12}M_{23}M_{31}}\right)^{1/2} \\ & \times\delta(b_{12}\cos\beta_{12} + b_{23}\cos\beta_{23} + b_{31}\cos\beta_{31}) \\ & \times b_{12}b_{23}b_{31}\,db_{12}db_{23}db_{31}d\beta_{12}d\beta_{23}d\beta_{31}, \end{aligned} \tag{43}$$

where

$$b_{ij} = B_{ij}/\sqrt{2\mu_{ij}}$$

is the impact parameter for subsystem ij.

It remains to express the integrand in (32) in terms of the b's and β's. This has already been done for the profile functions. The exponential factor is evaluated using the following definitions of λ_{ij} and ϕ_{ij}:

$$\lambda_{ij} = |\vec{\kappa}_{ij} - \hat{K}_{ij}\hat{K}_{ij}\cdot\vec{\kappa}_{ij}|,$$

$$\vec{\kappa}\cdot\hat{\Gamma}_{ij} = \vec{\kappa}_{ij}\cdot\hat{\gamma}/\sqrt{2\mu_{ij}} = \lambda_{ij}\cos\phi_{ij}/\sqrt{2\mu_{ij}},$$

$$\vec{\kappa}\cdot\hat{\Sigma}_{ij} = \vec{\kappa}_{ij}\cdot\hat{\sigma}_{ij}/\sqrt{2\mu_{ij}} = \lambda_{ij}\sin\phi_{ij}/\sqrt{2\mu_{ij}},$$

then

$$\begin{aligned} \vec{\kappa}\cdot\vec{B} &= \vec{\kappa}\cdot(\overleftrightarrow{I}_\Gamma + \overleftrightarrow{I}_\Sigma)\cdot\vec{B} \\ &= \sum_{k=1}^{3} b_{ij}(\xi_{ij}\cos\beta_{ij} + \zeta_{ij}\sin\beta_{ij}), \end{aligned} \tag{44}$$

where

$$\xi_{ij} = (M_{ij}/M)\lambda_{ij}\cos\phi_{ij},$$

$$\zeta_{ij} = a_{ij}\lambda_{ij}\sin\phi_{ij} + (\mu_{ij}/\mu_{ki})^{1/2}a_i\lambda_{ki}\sin\phi_{ki} + (\mu_{ij}/\mu_{jk})^{1/2}a_j\lambda_{jk}\sin\phi_{jk}.$$

Substituting (43) and (44) into (32) and Fourier transforming the δ function, we obtain

$$(\vec{k}'|T_3|\vec{k}) = -\frac{iKM}{(2\pi)^7}\left(\frac{2\alpha}{M_{12}M_{23}M_{31}}\right)^{1/2}\int_{-\infty}^{\infty} d\lambda \int_0^{\infty} b_{12}b_{23}b_{31}db_{12}db_{23}db_{31}\int_0^{2\pi} d\beta_{12}d\beta_{23}d\beta_{31}$$
$$\times\exp\left\{-i\sum_{k=1}^{3} b_{ij}[(\xi_{ij}+\lambda)\cos\beta_{ij}+\zeta_{ij}\sin\beta_{ij}]\right\}\Gamma_{12}(b_{12}, K_{12})\Gamma_{23}(b_{23}, K_{23})\Gamma_{31}(b_{31}, K_{31}). \tag{45}$$

For the final reduction we define

$$\rho_{ij} = [(\xi_{ij}+\lambda)^2+\zeta_{ij}^{\,2}]^{1/2},$$

$$\cos\psi_{ij} = \frac{\xi_{ij}+\lambda}{\rho_{ij}},$$

$$\sin\psi_{ij} = \zeta_{ij}/\rho_{ij};$$

then,

$$(\xi_{ij}+\lambda)\cos\beta_{ij}+\zeta_{ij}\sin\beta_{ij} = \rho_{ij}\cos(\beta_{ij}-\psi_{ij})$$

and the integrals over β_{ij} in (45) become Poisson integrals for $J_0(\rho_{ij}b_{ij})$. Comparing with (25) we find that

$$(\vec{k}'|T_3|\vec{k}) = \frac{1}{(2\pi)^4}\frac{KM}{K_{12}K_{23}K_{31}}\left(\frac{2\alpha}{M_{12}M_{23}M_{31}}\right)^{1/2}\int_{-\infty}^{\infty} d\lambda\, f_{12}(\rho_{12}, K_{12})f_{23}(\rho_{23}, K_{23})f_{31}(\rho_{31}, K_{31}).$$

V. SUMMARY AND CONCLUSIONS

We have obtained a high-energy, small-scattering-angle approximation for the $3\to 3$ amplitude in nonrelativistic potential scattering. The single- and double-scattering terms are the on-shell parts of the corresponding terms in the Watson-Faddeev multiple-scattering expansion. The triple-scattering term is given by an eikonal approximation which reduces to a one-dimensional momentum-transfer integral over the product of the three on-shell two-body amplitudes. This term is accurate only when the momenta are not too close to being parallel. It may be necessary to enlarge this domain of validity before a similar approximation can be applied to the calculation of absorptive corrections for single-particle production processes.

The momentum-transfer integral in the triple-scattering term involves two-body amplitudes evaluated at the same average momenta, K_{ij}, as the actual two-body subsystems. The squares of the momentum transfers appearing in the two-body amplitudes are quadratic functions of the variable of integration with coefficients which depend on all three initial and final momenta. The integration carries the two-body amplitudes outside the physical region. However, the exit from the physical region is by way of large momentum transfers, so the unphysical contributions to the integral will be negligible for forward-peaked two-body amplitudes.

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[1]Arnold Sommerfeld, *Partial Differential Equations in Physics* (Academic, New York, 1949), p. 233.

[2]A similar result has been obtained by Sugar and Blankenbecler using different methods. R. L. Sugar and R. Blankenbecler, Phys. Rev. 183, 1387 (1969).

[3]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. I, p. 315.

[4]It is assumed throughout this discussion that the criteria stated at the end of Sec. II are satisfied, i.e., $V_0/E_{12}\ll 1$ and $k_{12}a\gg 1$.

[5]K. M. Watson, Phys. Rev. 89, 575 (1953); L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [Soviet Phys. JETP 12, 1014 (1961)].