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Casimir force on a solid ball when $\epsilon(\omega)\mu(\omega)=1$. Finite temperatures

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The Casimir surface force on a solid ball is calculated, assuming the material to be dispersive and to be satisfying the condition $\epsilon(\omega)\mu(\omega)=1$, $\epsilon(\omega)$ being the spectral permittivity and $\mu(\omega)$ the spectral permeability. A recent paper by Brevik and Einevoll [Phys. Rev. D **37**, 2977 (1988)] discussed this problem at zero temperature. In this paper, the case of finite temperatures is covered. The analysis is based upon use of the Debye expansion of the modified Bessel functions. This expansion, being an asymptotic one, is limited in accuracy. At general temperatures, the expansion shows an oscillatory variation on the second-order level. Such a variation is unphysical, so that the physical information that one can extract from second-order theory is limited. On the zeroth-order level, the formalism is, however, found to work well. We discuss the zeroth-order approximation in complete form and comment upon the second-order correction. Although numerical methods are in general necessary, useful analytic approximations can be obtained in the limiting cases of very low, or very high, nondimensional temperatures. At low temperatures, the dispersive effect induces a strong, attractive contribution to the force. At high temperatures, the force becomes small.

I. INTRODUCTION

To gain insight in the nature of the electromagnetic Casimir effect[1]—the change in energy of the vacuum field because of the presence of idealized boundaries or dielectric media—it is desirable to carry out explicit calculations in a variety of situations of physical interest. The present paper is concerned with the Casimir surface force at *finite temperature* acting on a compact spherical ball of radius R, made up of a material satisfying the condition

$$\epsilon(\omega)\mu(\omega)=1\ , \tag{1.1}$$

$\epsilon(\omega)$, $\mu(\omega)$ denoting the spectral permittivity and permeability, respectively. The ball is surrounded by a vacuum. Condition (1.1) is interesting from both a mathematical and a physical point of view. This is so not least in the context of the formally analogous quantum chromodynamics[2] at zero quark-gluon coupling, where (1.1) ensures the gluon velocity to be equal to the velocity of light.

One of us has repeatedly considered the Casimir effect for *nondispersive* media satisfying $\epsilon\mu=1$ at zero temperature.[3,4] The drawback of restricting oneself to a nondispersive material model from the outset is that one loses the strong attractive part of the Casimir force arising from the dispersive effect. In particular, Candelas[5] has stressed the need of taking the dispersive effect into account. In a recent paper[6] we have made an explicit calculation of the surface force on a spherical ball at *zero temperature,* assuming the dispersion relation

$$\mu(\omega)-1=\frac{\mu_0-1}{1-\omega^2/\omega_0^2}\ , \tag{1.2}$$

μ_0 and ω_0 being input parameters. Equation (1.2) is analogous to Sellmeir's dispersion relation for $\epsilon(\omega)$ in ordinary electromagnetism.[7] In the present paper we shall extend the calculations of Ref. 6 to the case of finite temperatures. As before, we make use of Schwinger's source theory,[8] and we also use the Debye expansion for the Riccati-Bessel functions. In general, numerical methods are necessary for the evaluation, although in the limiting cases of very low or very high temperatures, it is possible to give useful analytic approximations for the force.

The main results of the calculations are the following. The *zeroth-order* approximation in the Debye expansion is found to work well, at all temperatures. When the temperature is low, there is a strong, attractive, dispersion-induced contribution to the force. This is the same term as found previously at $T=0$.[6] According to the dispersion relation (1.2), the medium becomes at high temperature very dilute, and the surface force accordingly very weak. In this case, the analytic approximation for the force is most appropriate.

When we proceed to the *second-order* approximation in the Debye expansion we find, however, somewhat surprisingly, that this approximation is in general not reliable; there occur large-amplitude terms, varying with temperature in an oscillatory way. Such behavior is unphysical; it reflects the shortcoming of the Debye expansion at the second-order level. Nevertheless, when used with some care, the second-order results are useful, at least for an order-of-magnitude analysis. The problems are most pronounced at moderate temperatures. At $T=0$, we found previously the approximation to be reliable.[6] Also, at high temperatures, the problems disappear since the second-order contribution is then found to be relatively very small.

In the following section, we give the general theory, starting from the $T=0$ expressions for the force and performing the transition to the finite-temperature case by replacing the integrals over imaginary frequency by discrete series. Thereafter, in Sec. III we consider the zeroth-order theory, which in practice implies calculation of a double series. Because of limited computer capacity, our method is to use the Euler-Maclaurin sum formula first, and thereafter perform the final summation exactly. The results are shown graphically. Section IV is devoted to analogous considerations on the second-order approximation. Analytic expressions in limiting cases are considered in Sec. V. In the theoretical derivations, we put $\hbar$ and c equal to unity.

II. GENERAL THEORY

Quite generally, when dealing with dispersion theory it is simpler to start considering *forces* than *energies*. This is so because the stress components of the electromagnetic energy-momentum tensor, in contradistinction to the energy-density component, gets no additional derivative terms because of dispersion (see Ref. 9). For this reason, we calculated the radial surface force density F in Ref. 6 at zero temperature. It is convenient to recapitulate here the basic expression for F:

$$F=-\frac{1}{4\pi^2R^4}\int_0^\infty dx\,x\sum_{l=1}^{\infty}\frac{\nu\frac{d}{dx}\ln[1-\lambda_l^2(x)]}{\left[1-\frac{\chi^{-1}(x)}{s_l(x)e_l'(x)}\right]\left[1+\frac{\chi^{-1}(x)}{s_l'(x)e_l(x)}\right]} . \tag{2.1}$$

Here $\nu=l+\frac{1}{2}$, and x means the nondimensional frequency

$$x=\hat{k}R=\hat{\omega}R . \tag{2.2}$$

In the derivation of (2.1), we have performed a complex frequency rotation, $k\to i\hat{k}$. Further, we have defined the magnetic susceptibility

$$\chi(x)=\frac{\chi_0}{1+x^2/x_0^2}\quad (x_0=\omega_0R) , \tag{2.3}$$

where $\chi_0=\mu_0-1$. Finally, we have introduced the abbreviation

$$\lambda_l(x)=[s_l(x)e_l(x)]' , \tag{2.4}$$

s_l and e_l being the Riccati-Bessel functions

$$s_l(x)=(\pi x/2)^{1/2}I_\nu(x) ,\qquad e_l(x)=(2x/\pi)^{1/2}K_\nu(x) , \tag{2.5}$$

defined such that their Wronkskian is $W\{s_l,e_l\}=-1$.

Expression (2.1) follows from the application of Maxwell's stress tensor and the electromagnetic boundary conditions at radius $r=R$ and does not rest upon any further mathematical approximation. Note that the inclusion of dispersion, which effectively implies that there is a high-frequency cutoff, makes it unnecessary to apply any point-splitting technique to the integrand in (2.1).

If we now apply the Debye expansion of the Riccati-Bessel functions, we can express (2.1) as[6]

$$F=-\frac{1}{8\pi R^4}\left[\frac{\mu_0-1}{\mu_0+1}\right]^2\sum_{l=1}^{\infty}\left[J_l^{(0)}+\frac{1}{\nu^2}J_l^{(2)}-\frac{1}{\nu^2}\frac{\mu_0}{(\mu_0+1)^2}I_l+O(\nu^{-4})\right] . \tag{2.6}$$

At temperature $T=0$, the various summands in (2.6) are given as frequency integrals:

$$J_l^{(0)}=\frac{3}{\pi}\int_0^\infty dz\,z^2\rho^8\left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2 , \tag{2.7}$$

$$J_l^{(2)}=-\frac{3}{4\pi}\int_0^\infty dz\,z^2\rho^8\left[\frac{\alpha^2}{\nu^2+\alpha^2}\right]^2\times(2-36\rho^2+100\rho^4-71\rho^6) , \tag{2.8}$$

$$I_l=\frac{3}{\pi}\frac{(\mu_0+1)^2}{4\mu_0}\times\int_0^\infty dz\,z^2\alpha^4\rho^{14}\times\frac{\left[\nu^2+\frac{2\alpha^2}{\mu_0+1}\right]\left[\nu^2+\frac{2\alpha^2\mu_0}{\mu_0+1}\right]}{(\nu^2+\alpha^2)^4} , \tag{2.9}$$

where

$$z=x/\nu ,\qquad \rho(z)=(1+z^2)^{-1/2} ,\qquad \alpha(z)=(x_0/z)[\tfrac{1}{2}(\mu_0+1)]^{1/2} . \tag{2.10}$$

Reference 6 gives, in tabular form, values of F at $T=0$ for some different sets of input parameters x_0 and μ_0. Actually, the results are given in nondimensional form, with the normalization factor

$$F_0=0.092\,35/8\pi R^4 \tag{2.11}$$

taken from the singular shell calculation of Milton, DeRaad, and Schwinger.[10]

At finite temperatures, we can keep the essential part of the formalism unchanged, the only difference being that the integrals in (2.7)–(2.9) have to be replaced by discrete sums. This corresponds to the following discretization of the frequency $\hat{\omega}$, defined in (2.2), along the imaginary axis,

$$\hat{\omega}\rightarrow\hat{\omega}_n=2\pi n/\beta\ , \tag{2.12}$$

where n is an integer in the range $\langle -\infty,\infty\rangle$ and $\beta=1/k_BT$. Expressions (2.10) become transformed as

$$\begin{aligned} z\rightarrow z_n&=x_n/\nu=\hat{\omega}_nR/\nu\ ,\\ \rho\rightarrow\rho_n&=(1+z_n^2)^{-1/2}\ ,\\ \alpha\rightarrow\alpha_n&=(x_0/z_n)[\tfrac{1}{2}(\mu_0+1)]^{1/2}\ . \end{aligned} \tag{2.13}$$

The rule for going from integral to sum is

$$\int_0^\infty dz\rightarrow\frac{t}{\nu}\sum_{n=0}^{\infty}{}'\ , \tag{2.14a}$$

where

$$t=2\pi R/\beta \tag{2.14b}$$

is a nondimensional temperature. The prime in (2.14a) means that the $n=0$ term is counted with half-weight.

Mathematically, the above properties can be viewed as a consequence of the fact that the analytic extension of the Wightman functions to the complex time plane is periodic in $i\beta$. These relationships have recently been extensively discussed by Fulling and Ruijsenaars.[11] From a practical viewpoint, the only information we need in order to perform calculations is contained in (2.12) and (2.14). These equations are in agreement with Schwinger,[8] Schwinger, DeRaad, and Milton,[12] Lifshitz,[13] and others.

In the following section we consider the zeroth-order term $\sum_l J_l^{(0)}$ in (2.6). We expect beforehand this term to be the dominant one, and explicit calculation will *a posteriori* show this expectation to be true. Recall that the only mathematical approximation introduced so far is the Debye expansion (there is no restriction on the temperature). Since the Debye expansion is an asymptotic expansion in $1/\nu$, we expect that errors arising from the lowest values of ν, in particular, are significant. In the present problem, relatively small errors are in fact quite critical, since we are confronted with large numbers which almost, but not quite, compensate each other. (We have recently met this kind of problem elsewhere, when dealing with Casimir theory in nondispersive media.[3(b)])

III. CALCULATION OF THE ZEROTH-ORDER TERM

A. General remarks

We shall give an extra superscript T to quantities referring to finite temperatures. Using (2.7) and (2.12)–(2.14), we can write the first term in parentheses in the finite-temperature version of (2.6) as

$$J_l^{(0)T}=\frac{3t^3a^4}{\pi}\sum_{n=1}^{\infty}\frac{\nu^5}{(\nu^2+t^2n^2)^4}\frac{n^2}{(n^2+a^2)^2}\ , \tag{3.1}$$

where for convenience we have introduced the symbol

$$a=(x_0/t)[\tfrac{1}{2}(\mu_0+1)]^{1/2}\ . \tag{3.2}$$

Expression (3.1) is to be summed over l.

The available computer was an IBM PC, XT, model 286; the programming language was Turbo-Pascal-87. It turned out that the computer capacity was insufficient to handle the double series directly. [It is of interest to go into some detail at this point. Consider the n series of the last factor in (3.1) separately, and consider first the limit of low t. The main contribution occurs when n is of the same order as a. Because of weak convergence, one has to carry out the sum at least until $n\sim 2a$. Taking x_0,μ_0 to be of order unity, the main contribution occurs, according to (3.2), when $n\sim 1/t$. Thus $n\gg 1$ when $t\ll 1$, in accordance with the general properties of low-temperature Casimir theory.[14] It is mainly this need of including a large number of n terms that is time consuming for the computer. In the opposite limit, for $t\gg 1$, we can analogously consider the l series of the first factor in (3.1) separately. The main contribution occurs when $l\sim tn\gg 1$, so that the problems arise in this limit from the l series. For the intermediate moderate-temperature cases we have to face an extensive double sum.]

In our calculations, the following procedure is followed.

(1) The series over l is calculated first, by means of the approximate Euler-Maclaurin formula.[15] In many applications, such as in the Casimir calculations of Ref. 4(b), for instance, it is possible to use the following simplified version of this formula:

$$\sum_{l=1}^{\infty}f(l)=\int_0^\infty dl\,f(l)+\tfrac{1}{2}[f(\infty)-f(0)]+\tfrac{1}{12}[f'(\infty)-f'(0)]+\cdots\ . \tag{3.3a}$$

However, in the present case this simple formula becomes inaccurate for low values of t. We therefore have to take into account the more complete version

$$\sum_{l=1}^{\infty}f(l)=\sum_{l=1}^{L}f(l)+\int_{L+1}^{\infty}dl\,f(l)+\tfrac{1}{2}[f(\infty)+f(L+1)]+\sum_{k=1}^{\infty}\frac{1}{(2k)!}B_{2k}[f^{(2k-1)}(\infty)-f^{(2k-1)}(L+1)]\ , \tag{3.3b}$$

where the B_{2k} are the Bernoulli numbers. We shall limit ourselves to including only the first correction term, corresponding to $B_{2k}=B_2=\frac{1}{6}$. Moreover, numerical trials indicate that if we choose for the parameter L a value as high as $L=4$, we obtain satisfactory accuracy also in the limit of low t.

(2) In the final series over n, we decompose the summand into partial fractions and perform the various summations exactly.

B. Calculation

Using formula (3.3b) in the summation of (3.1) over l, we can, after some algebra, write the result as a sum over five terms X_I:

$$\sum_{l=1}^{\infty} J_l^{(0)T} = \sum_{I=1}^{5} X_I \ , \tag{3.4}$$

where

$$X_1 = \frac{a^4 t}{2\pi} \sum_{n=1}^{\infty} \frac{n^6}{(n^2+a^2)^2(n^2+b^2)^3} \ , \tag{3.5a}$$

$$X_2 = \frac{3a^4 b^2 t}{2\pi} \sum_{n=1}^{\infty} \frac{n^2}{(n^2+a^2)^2(n^2+b^2)^2} \ , \tag{3.5b}$$

$$X_3 = \frac{3a^4}{\pi t^5} \sum_{l=1}^{4} \nu^5 X_3' \left[a ; \frac{\nu}{t} \right] , \tag{3.5c}$$

with

$$X_3' \left[a ; \frac{\nu}{t} \right] = \sum_{n=1}^{\infty} \frac{n^2}{(n^2+a^2)^2(n^2+\nu^2/t^2)^4} \ ,$$

$$X_4 = \frac{7a^4 b^4}{\pi t} \sum_{n=1}^{\infty} \frac{n^2}{(n^2+a^2)^2(n^2+b^2)^4} \ , \tag{3.5d}$$

$$X_5 = \frac{2a^4 b^6}{\pi t} \sum_{n=1}^{\infty} \frac{n^2}{(n^2+a^2)^2(n^2+b^2)^5} \ , \tag{3.5e}$$

with the definition

$$b = 11/2t \ . \tag{3.6}$$

The terms have been arranged according to increasing inverse powers in n. The term X_3 is special, in that it is a sum over the lowest values of l, up to $l=L=4$.

The next step is to carry out the summations over n by resolving the summands into partial fractions. The following series occurs repeatedly:

$$S(a;k) = \sum_{n=1}^{\infty} (n^2+a^2)^{-k} \ , \tag{3.7}$$

where k is a positive integer. Various values of $S(a;k)$ up to $k=5$ are collected in Appendix A. After a lengthy calculation we can express the terms in (3.5) as follows:

$$X_1 = \frac{a^4 t}{2\pi} \left[\frac{a^6}{(a^2-b^2)^3} S(a;2) + \frac{3a^4 b^2}{(a^2-b^2)^4} S(a;1) - \frac{b^6}{(a^2-b^2)^2} S(b;3) + \frac{b^4(3a^2-b^2)}{(a^2-b^2)^3} S(b;2) - \frac{3a^4 b^2}{(a^2-b^2)^4} S(b;1) \right] , \tag{3.8a}$$

$$X_2 = \frac{3a^4 b^2 t}{2\pi} \left[-\frac{a^2}{(a^2-b^2)^2} S(a;2) - \frac{a^2+b^2}{(a^2-b^2)^3} S(a;1) - \frac{b^2}{(a^2-b^2)^2} S(b;2) + \frac{a^2+b^2}{(a^2-b^2)^3} S(b;1) \right] , \tag{3.8b}$$

$$\begin{aligned} X_3' \left[a ; \frac{\nu}{t} \right] = & -\frac{a^2}{(a^2-\nu^2/t^2)^4} S(a;2) - \frac{3a^2+\nu^2/t^2}{(a^2-\nu^2/t^2)^5} S(a;1) - \frac{\nu^2/t^2}{(a^2-\nu^2/t^2)^2} S\left[\frac{\nu}{t};4 \right] \\ & + \frac{a^2+\nu^2/t^2}{(a^2-\nu^2/t^2)^3} S\left[\frac{\nu}{t};3 \right] - \frac{2a^2+\nu^2/t^2}{(a^2-\nu^2/t^2)^4} S\left[\frac{\nu}{t};2 \right] + \frac{3a^2+\nu^2/t^2}{(a^2-\nu^2/t^2)^5} S\left[\frac{\nu}{t};1 \right] , \end{aligned} \tag{3.8c}$$

$$\begin{aligned} X_4 = \frac{7a^4 b^4}{\pi t} \Bigg[& -\frac{a^2}{(a^2-b^2)^4} S(a;2) - \frac{3a^2+b^2}{(a^2-b^2)^5} S(a;1) - \frac{b^2}{(a^2-b^2)^2} S(b;4) + \frac{a^2+b^2}{(a^2-b^2)^3} S(b;3) \\ & - \frac{2a^2+b^2}{(a^2-b^2)^4} S(b;2) + \frac{3a^2+b^2}{(a^2-b^2)^5} S(b;1) \Bigg] , \end{aligned} \tag{3.8d}$$

$$\begin{aligned} X_5 = \frac{2a^4 b^6}{\pi t} \Bigg[& \frac{a^2}{(a^2-b^2)^5} S(a;2) + \frac{4a^2+b^2}{(a^2-b^2)^6} S(a;1) - \frac{b^2}{(a^2-b^2)^2} S(b;5) + \frac{a^2+b^2}{(a^2-b^2)^3} S(b;4) \\ & - \frac{2a^2+b^2}{(a^2-b^2)^4} S(b;3) + \frac{3a^2+b^2}{(a^2-b^2)^5} S(b;2) - \frac{4a^2+b^2}{(a^2-b^2)^6} S(b;1) \Bigg] \end{aligned} \tag{3.8e}$$

(recall that $\nu = l + \frac{1}{2}$). Adding these expressions for X_I, we can now find the zeroth-order sum $\sum_l J_l^{(0)T}$ in (3.4). Figure 1 shows how this sum varies versus the nondimensional temperature t defined in (2.14b). The variation is shown for two different sets of input parameters x_0 and μ_0. One expects that the application of the theory to the hadronic world[6] corresponds to a value of ω_0 that is of order $1/R$. That means, $x_0 = \omega_0 R \sim 1$. This is the physical motivation for our choice of input values of x_0 in Fig. 1. The values of μ_0, on the other hand, are arbitrarily chosen, since there is no obvious analogous physical guidance for this parameter.

As mentioned at the end of Sec. II, we expect the zeroth-order result as shown in Fig. 1 to represent the

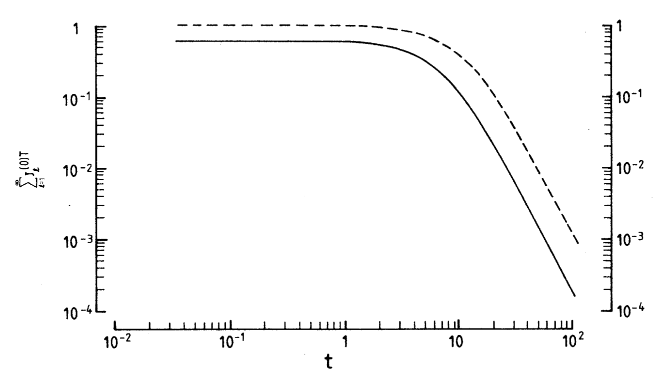

FIG. 1. Zeroth-order contribution vs the nondimensional temperature t defined in Eq. (2.14b). Solid line: $x_0=4$, $\mu_0=3$. Dashed line: $x_0=8$, $\mu_0=1.5$.

real, physical force in a reasonably accurate way. The force is seen to tend to zero at high values of t. This is physically obvious from the dispersion relation (1.2), since $\mu(\omega\to\infty)\to 1$ so that the influence from the medium must tend to zero at the high frequencies that are associated with the high temperatures. At low temperatures, $t \lesssim 1$, the curves exhibit a quite flat plateau, so that the force in this region is almost temperature independent. In dimensional terms, we have according to (2.14b), $t=(2\pi R/\hbar c)k_B T$. For a hadronic "ball" of radius $R=1$ fm, the condition $t=1$ corresponds to the very high temperature of $T=3.63\times 10^{11}$ K. By contrast, if $R=1$ μm (aerosol particle, for instance), we find that $t=1$ corresponds only to a moderate temperature of $T=363$ K. As a rule of thumb, it is helpful to remember that in order to employ the low-temperature approximation for the *nondimensional* temperature t, the radius R in an experiment carried out at room temperature must be smaller than about 1 μm.

From Fig. 1 it is apparent that the low-temperature plateau is definitely passed when t becomes of the order of 10. It is interesting to notice that this behavior is supported by the following physical argument. From Eq. (1.2), we expect that the influence from dispersion becomes important when the temperature is so high that the most significant frequencies in the radiation field are of the same order as ω_0. It is natural to identify the most significant frequency with the value ω_m that corresponds to the *maximum* of the blackbody energy distribution. We know that ω_m is related to temperature through Wien's displacement law $\omega_m=2.8/\beta$. From our initial order of magnitude condition $\omega\sim\omega_0$ we thus derive $2.8/\beta\sim\omega_0$ or, using (2.14b), $t\sim 2x_0$. With the values of x_0 used in Fig. 1, we in fact obtain $t\sim 10$ as a typical transition value, in agreement with the curves. In view of the crudeness of the argument, the agreement is actually better than we might expect.

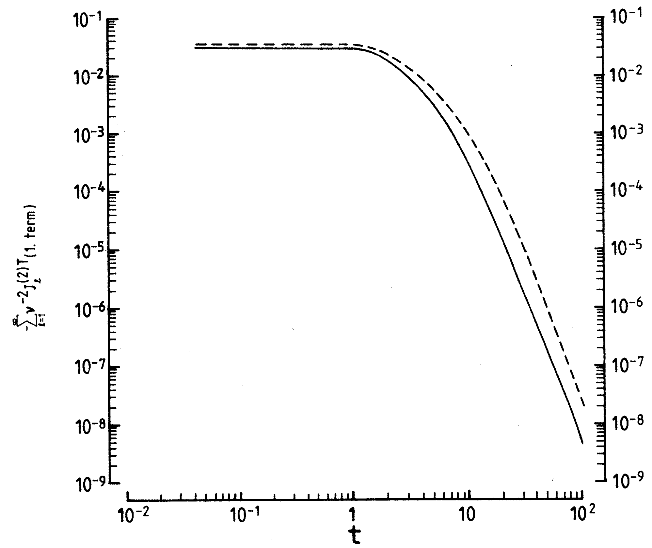

FIG. 2. Second-order contribution (first term) vs the nondimensional temperature t. Solid line: $x_0=4$, $\mu_0=3$. Dashed line: $x_0=8$, $\mu_0=1.5$.

IV. ON THE SECOND-ORDER CORRECTION

When we go to the second-order terms in the Debye expansion (2.6), viz., $\sum_l \nu^{-2}J_l^{(2)T}$ and $\sum_l \nu^{-2}I_l^{T}$, the calculations become considerably more laborious. The calculational technique is the same as listed earlier in Sec. III A. We shall abstain from giving a presentation of the complete second-order expressions here. In part, this is due to the complexity of the formalism. There is, however, also another and more fundamental reason, as we have already mentioned: The second-order expressions become physically *unreasonable* at arbitrary finite temperatures. In general, one obtains the expression for the Casimir force to some order of approximation after collecting several terms that almost, but not quite, compensate each other. When the order of approximation is increased, there is in our case a general tendency for the individual alternating terms to increase in magnitude, and the expected near cancellation does not any longer take place. We have actually carried out a complete calculation of all the contributions to the second-order term $\sum_l \nu^{-2}J_l^{(2)T}$ (not shown here), and we found an unphysical, oscillatory behavior with respect to temperature. The oscillatory behavior becomes more pronounced as the temperature increases. At $T=0$, the second-order Debye approximation leads to reasonable results, as found previously,[6] and also at low temperatures we find that this approximation is reliable.

We find it worthwhile to present below the calculation of the first, and most significant, term in $\sum_l \nu^{-2}J_l^{(2)T}$. Using (2.8), together with (2.12)–(2.14), we first obtain, at arbitrary temperatures,

$$\frac{1}{\nu^2}J_l^{(2)T}=-\frac{3t^3a^4}{4\pi}\sum_{n=1}^{\infty}\frac{\nu^3}{(t^2n^2+\nu^2)^4}\frac{n^2}{(n^2+a^2)^2}\left[2-\frac{36\nu^2}{t^2n^2+\nu^2}+\frac{100\nu^4}{(t^2n^2+\nu^2)^2}-\frac{71\nu^6}{(t^2n^2+\nu^2)^3}\right]. \tag{4.1}$$

We concentrate henceforth on the first term here. Summing it over l by means of the Euler-Maclaurin formula (3.3b) (with $L=4$), we obtain an expression that can be written as a sum over five terms Y_I:

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)T}(1\text{ term})=\sum_{I=1}^{5}Y_I\ . \tag{4.2}$$

Here

$$Y_1=-\frac{a^4}{8\pi t}\sum_{n=1}^{\infty}\frac{n^4}{(n^2+a^2)^2(n^2+b^2)^3}\ , \tag{4.3a}$$

$$Y_2=-\frac{3a^4b^2}{8\pi t}\sum_{n=1}^{\infty}\frac{n^2}{(n^2+a^2)^2(n^2+b^2)^3}\ , \tag{4.3b}$$

$$Y_3=-\frac{3a^4}{2\pi t^5}\sum_{l=1}^{4}\nu^3X_3'\left[a;\frac{\nu}{t}\right]\ , \tag{4.3c}$$

$$Y_4=-\tfrac{27}{1694}X_4\ , \tag{4.3d}$$

$$Y_5=-\tfrac{4}{121}X_5\ . \tag{4.3e}$$

In analogy to the zeroth-order calculation, the Y_I have been arranged according to increasing inverse powers in n. Recall that X_3' is defined in (3.5c).

The task now is to decompose the summands in Y_1 and Y_2 into partial fractions and sum over n exactly:

$$Y_1=\frac{a^4}{8\pi t}\left[\frac{a^4}{(a^2-b^2)^3}S(a;2)+\frac{a^4+2a^2b^2}{(a^2-b^2)^4}S(a;1)-\frac{b^4}{(a^2-b^2)^2}S(b;3)+\frac{2a^2b^2}{(a^2-b^2)^3}S(b;2)-\frac{a^4+2a^2b^2}{(a^2-b^2)^4}S(b;1)\right]\ , \tag{4.4a}$$

$$Y_2=\frac{3a^4b^2}{8\pi t}\left[-\frac{a^2}{(a^2-b^2)^3}S(a;2)-\frac{2a^2+b^2}{(a^2-b^2)^4}S(a;1)+\frac{b^2}{(a^2-b^2)^2}S(b;3)-\frac{a^2+b^2}{(a^2-b^2)^3}S(b;2)+\frac{2a^2+b^2}{(a^2-b^2)^4}S(b;1)\right]\ . \tag{4.4b}$$

With all the terms Y_I known, we can calculate the expression (4.2). The result is shown in Fig. 2, with the same input parameters x_0 and μ_0 as in Fig. 1. The temperature variation for this selected term in the second-order correction is seen to be reasonable; there is a similar low-temperature plateau as we found earlier in the zeroth-order calculation. On the plateau, the magnitude of the selected second-order term is seen to be less than 10% of that of the zeroth-order term in Fig. 1. However, we ought to be cautious at this stage in drawing definite conclusions as regards the relative magnitude of the second-order contribution at low temperatures. First, and most importantly, we have not included the contribution from the other second-order terms. As we shall see in the next section, these terms will (at low temperatures) reduce the magnitude of the second-order contribution significantly. Second, the figures, of course, refer to a specific choice of material parameters x_0 and μ_0. The numerical information that we have indicates that the zeroth-order approximation is on the whole quite accurate. In the high-temperature region, in particular, it is easy to show that the second-order correction is very small in comparison to the zeroth-order term. We shall return to this point in the next section in connection with analytic approximations. The mentioned oscillatory problem for the second-order term is thus no problem in this temperature region; the zeroth-order theory is fully adequate.

V. ANALYTIC APPROXIMATIONS IN LIMITING CASES

It is desirable to give analytic expressions for the surface force when the temperature is either very low or very high. Note that "low" or "high" in the present context refer to the value of the *nondimensional* temperature t, as defined by (2.14b). As mentioned in Sec. III B, the physical temperature T, corresponding to a value of $t=1$, can actually be very high if the radius R is small, of hadronic dimensions. The accessibility to analytic approximations in limiting cases is valuable in cases of physical interest; under many circumstances it is unnecessary to take the full formalism into account. Besides, it is reassuring to check the numerical work against the analytic approximations in limiting cases.

A. Low temperatures

We shall always assume that the input parameters x_0 and μ_0 are chosen such that at low temperatures, the quantity a, defined by (3.2), is a large quantity. By low temperatures we thus mean

$$t\ll 1\ , \tag{5.1a}$$

$$a\gg 1\ . \tag{5.1b}$$

Mainly, we follow the same calculational procedure as in the general case. The low-temperature approximation is not applied until the final stage, viz., in the evaluation of the sum over n. This means that we insert analytic approximations for the sums $S(a;k)$ appearing in (3.8) and (4.4). For this purpose, the Poisson sum formula is most useful. This topic is treated in Appendix B. The important formula is given in (B8). Essential simplifications are moreover obtained in the final expressions if we make the further assumption that x_0 is a large quantity (the usefulness of this assumption was discussed in Ref. 6). From (3.2) and (3.6) we have

$$\frac{a}{b}=\frac{2x_0}{11}[\tfrac{1}{2}(\mu_0+1)]^{1/2} , \tag{5.2}$$

showing that the condition $x_0 \gg 1$ implies $a/b \gg 1$. Calculating the various terms in (3.5) we obtain

$$\sum_{l=1}^{\infty} J_l^{(0)T}=\frac{x_0}{8}\left[\frac{\mu_0+1}{2}\right]^{1/2}-\frac{3}{32}+O(t^{-3}e^{-3\pi/t}) , \tag{5.3}$$

valid when $t \ll 1$, $a \gg 1$, $x_0 \gg 1$. In the limit when $t \to 0$, only the two first terms survive on the right-hand side of (5.3). This limiting expression agrees exactly with the zero-temperature result derived in Eq. (6.1) in Ref. 6.

The first term to the right in (5.3) is obviously due to dispersion. This term corresponds to a strong, attractive part of the surface force; cf. (2.6). It was precisely this term that was lost in the earlier Casimir calculations carried out on the assumption of a *nondispersive* dielectric model from the outset.

Calculating the second-order terms Y_l in (4.2) in the same way we obtain, in the limit as $t \to 0$,

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)T}(1\ \text{term})=-\frac{25}{2\times 11^3}-\frac{4561}{3\times 210^2}$$
$$=-0.043\,866,\quad t\to 0 . \tag{5.4}$$

This specific term is seen to be quite appreciable; it is almost half of the finite term $-\frac{3}{32}$ (not related to dispersion) in (5.3). However, Eq. (5.4) does not alone give a quantitatively correct picture of the second-order correction. We have to take into account the other second-order terms also. At the low temperatures under investigation here there are no problems in principle in evaluating the remaining terms in (4.1). We have performed this calculation explicitly; although we do not go into detail here it is of interest to write down the resulting values obtained for the successive four terms in (4.1):

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)T}=-0.043\,866+0.493\,494$$
$$-0.959\,571+0.510\,972$$
$$=0.001\,028,\quad t\to 0 \tag{5.5}$$

(the apparent discrepancy in the last digit in the second equation is due to rounding off errors). This example in fact demonstrates in a striking way a characteristic feature of the Casimir calculations in general: Several individual terms conspire so as to give a small final number. The sum (5.5) is seen to be much less in magnitude than its first term (5.4). We note that (5.5) agrees well with the second-order correction calculated earlier[3(b),6] at zero temperature:

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)}=(\tfrac{1}{2}\pi^2-4)\frac{9}{2^{13}}=0.001\,027,\quad T=0 . \tag{5.6}$$

To give an idea about the magnitude of the second-order correction from I_l, cf. (2.9), we also quote from Refs. 3(b) and 6:

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}I_l=(\tfrac{1}{2}\pi^2-4)\frac{63}{2^{11}}=0.028\,756,\quad T=0 . \tag{5.7}$$

The correction term shown to the utmost right in (5.3) is, for reasonable values of t, much smaller than the second-order correction (5.5) and we therefore abstain from giving this term explicitly.

B. High temperatures

When the nondimensional temperature satisfies the condition

$$t \gg 1 , \tag{5.8}$$

the formalism becomes simplified considerably. (No restriction will now be made on the value of a.) In the preceding discussion on low temperatures we did not apply the simplifying approximation until the final n summation; in the present case we introduce the simplification at an earlier stage, viz., in the l summation. The final sum over n can easily be done exactly.

We first consider the zeroth-order term $\sum_l J_l^{(0)T}$. The basic expression is (3.1), which is to be summed over l. Using the Maclaurin formula (3.3b) (with $L=4$) and expanding thereafter in powers of $1/t$, we obtain

$$\sum_{l=1}^{\infty}\frac{\nu^5}{(\nu^2+t^2n^2)^4}=\frac{1}{6t^2n^2}[1+O(t^{-4})] . \tag{5.9}$$

Using (A2) we then have

$$\sum_{l=1}^{\infty} J_l^{(0)T}=-\frac{t}{4\pi}+\frac{at}{8}\coth\pi a$$
$$+\frac{\pi a^2 t}{8}\frac{1}{\sinh^2\pi a}+O(t^{-3}) . \tag{5.10}$$

The numerical values calculated from this expression are small, although the individual terms themselves are relatively large. As an example, inserting $x_0=4$, $\mu_0=3$ we obtain, at a temperature of $t=100$,

$$\sum_{l=1}^{\infty} J_l^{(0)T}=1.7533\times 10^{-4} . \tag{5.11}$$

As we have already mentioned, the relative importance of the second-order Debye correction is small at high temperatures. For completeness, we give the result of a calculation of all the terms appearing in $\sum_l \nu^{-2}J_l^{(2)T}$, cf. (4.1):

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)T}=\frac{\pi}{480t}+\frac{1}{80\pi a^2 t}-\frac{3}{320at}\coth\pi a$$
$$-\frac{\pi}{320t}\frac{1}{\sinh^2\pi a},\quad t\gg 1 . \tag{5.12}$$

Again inserting $x_0=4$, $\mu_0=3$, $t=100$, we obtain

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}J_l^{(2)T}=4.132\times 10^{-10} , \tag{5.13}$$

which is seen to be much smaller than (5.11).

We stress again that the case of high temperatures corresponds to a medium that is *dilute*. The dominant frequencies in the radiation field are high, and the electric and magnetic susceptibilities accordingly small, corresponding to weak surface forces.

It ought to be observed that at *macroscopic* dimensions, the high-temperature approximation in t is legitimate even in cases where the physical temperature T is low. For instance, when R is as large as 1 cm, we find according to (2.14b) that $t=275 \gg 1$ even at the very low physical temperature of $T=10$ K.

VI. FINAL REMARKS

Using the Debye expansion, and assuming the dispersion relation (1.2), we have calculated the temperature dependence of the surface force on the ball, assuming that the constituent material satisfies Eq. (1.1). At $T=0$, the Debye expansion is found to work to the second order in $1/\nu$. The zeroth-order formalism actually turns out to be quite insensitive to variations in temperature in the low-temperature region, and we expect the $T=0$ theory to be applicable even up to $t \sim 1$. This feature is apparent from the existence of the flat plateau in Fig. 1, and also from formula (5.3), which holds in the simplifying case of $x_0 \gg 1$. Thus, the force is given with reasonable accuracy from Fig. 1 [or eventually from Eq. (5.3)], with its second-order correction at low temperatures being calculable from Eqs. (5.6) and (5.7).

When the nondimensional temperature is high, $t \gg 1$, formula (5.10) is useful. The fact that the Debye expansion is not generally applicable beyond the zeroth order is no problem in the region of high t, since the second-order correction, as we have seen, is negligible in this region.

A remark is called for, as regards the accuracy of our method of summing over l by means of the Euler-Maclaurin formula, choosing $L=4$. We checked the errors numerically, in zeroth order as well as second-order Debye theory, for a variety of input values of the parameter tn. We found the errors to be less than 1.5% in any of the cases investigated.

At low temperatures, we made use of the Poisson sum formula derived in Appendix B. The accuracy here is better the higher is the value of a. For instance, taking $k=8$, we found for $a=3$ an error of about 0.6%. This value of a is in the low-temperature region actually a low value, cf. the definition equation (3.2). Higher values of a will thus (for the same value of k) improve the accuracy. This example illustrates the usefulness of the use of the Poisson formula at low temperatures.

At high temperatures, we also made numerical checks on the approximate formulas. For the zeroth-order term, Eq. (5.10), the error was found to be about 0.1%, whereas for the second-order term in (5.12) the error was considerably smaller, about 0.001%.

Further, there are technical "overflow" and "underflow" problems that ought to be mentioned. The overflow problem, present when the magnitude of numbers (in practice, exponentials) exceeded the computer capacity, limited the range of available values for the input parameters x_0 and μ_0 in our work, but created otherwise no serious problems since error messages were displayed. The underflow problem was more intricate; when summing a series of terms, among which at least one was beneath detectable computer limits, it could happen that the computer converted the *whole sum* to zero. No error message was given. A defect of this kind in the system is, of course, disturbing when doing a summation of small numbers and the problem ought to be pointed out as a warning to other workers. One can overcome the problem by inserting explicit tests, requiring the computer to put all numbers below a given limit equal to zero.

We round off this work by stressing again one of the chief advantages of using the dispersion formalism in the handling of the electromagnetic Casimir problem: We manage in this way to introduce a soft, physically related, high-frequency cutoff, and do not have to draw into consideration artificial mathematical methods like time splitting in order to obtain finite results [recall the discussion following Eq. (2.5)]. We find it reasonable that this method can be used with advantage also in related field theoretical problems.

APPENDIX A: SOME EXACT EXPRESSIONS FOR $S(a;k)$

The sum $S(a;k)$ is defined by Eq. (3.7). For the calculation as far as it is shown in this paper we need the explicit expressions up to $k=5$. For $k=1$ and $k=2$ we have, from Ref. 16, for instance,

$$S(a;1)=-\frac{1}{2a^2}+\frac{\pi}{2a}\coth\pi a\ , \tag{A1}$$

$$S(a;2)=-\frac{1}{2a^4}+\frac{\pi}{4a^3}\coth\pi a+\frac{\pi^2}{4a^2}\frac{1}{\sinh^2\pi a}\ . \tag{A2}$$

Successive differentiations of (A2) with respect to a yield

$$\begin{aligned}S(a;3)=&-\frac{1}{2a^6}+\frac{3\pi}{16a^5}\coth\pi a\\&+\frac{3\pi^2}{16a^4}\frac{1}{\sinh^2\pi a}+\frac{\pi^3}{8a^3}\frac{\coth\pi a}{\sinh^2\pi a}\ ,\end{aligned} \tag{A3}$$

$$\begin{aligned}S(a;4)=&-\frac{1}{2a^8}+\frac{5\pi}{32a^7}\left[\coth\pi a+\frac{\pi a}{\sinh^2\pi a}\right]\\&+\frac{\pi^3}{24a^5}\frac{\coth\pi a}{\sinh^2\pi a}(3+\pi a\coth\pi a)\\&+\frac{\pi^4}{48a^4}\frac{1}{\sinh^4\pi a}\ ,\end{aligned} \tag{A4}$$

$$\begin{aligned}S(a;5)=&-\frac{1}{2a^{10}}+\frac{35\pi}{256a^9}\left[\coth\pi a+\frac{\pi a}{\sinh^2\pi a}\right]\\&+\frac{15\pi^3}{128a^7}\frac{\coth\pi a}{\sinh^2\pi a}\\&+\frac{\pi^4}{96a^6}\frac{\coth^2\pi a}{\sinh^2\pi a}(5+\pi a\coth\pi a)\\&+\frac{\pi^4}{192a^6}\frac{1}{\sinh^4\pi a}(5+4\pi a\coth\pi a)\ .\end{aligned} \tag{A5}$$

For a complete calculation of the quantity $\sum_l \nu^{-2}J_l^{(2)T}$, we would need the terms up to $k=8$.

APPENDIX B: APPLICATION OF THE POISSON SUM FORMULA

There exists an interesting approximate method for evaluating the sum $S(a;k)$, valid for all integers k, and especially suitable in the low-temperature region. This is to apply the Poisson sum formula.[12,17] Because of the usefulness of the method and the transparency of the obtained results we shall present the method briefly here, although strictly speaking the approximations for the lower values of k are derivable already from the exact expressions in Appendix A.

The Poisson formula reads

$$\sum_{n=-\infty}^{\infty} f(n)=2\pi \sum_{n=-\infty}^{\infty} c(2\pi n)\ , \tag{B1}$$

where (α is an arbitrary number here)

$$c(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx\ . \tag{B2}$$

We insert $f(x)=(x^2+a^2)^{-k}$ in (B2), and write $c(\alpha)$ as

$$c(\alpha)=\frac{1}{\sqrt{\pi}\Gamma(k)}\left[\frac{\alpha}{2a}\right]^{k-1/2} K_{k-1/2}(\alpha a)\ , \tag{B3}$$

where K is a modified Bessel function. We take into account that $f(n)$ is symmetric in n, and also that

$$c(0)=\lim_{\alpha\to 0} c(\alpha)=\frac{\Gamma(k-\frac{1}{2})}{2\sqrt{\pi}\Gamma(k)a^{2k-1}}\ , \tag{B4}$$

which is derivable from (B3) using the approximate form[15]

$$K_\nu(x)\simeq \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}x)^{-\nu} \tag{B5}$$

for small x. Then (B1) yields

$$S(a;k)=\frac{\sqrt{\pi}\Gamma(k-\frac{1}{2})}{2\Gamma(k)a^{2k-1}}-\frac{1}{2a^{2k}} +\frac{2\sqrt{\pi}}{\Gamma(k)(2a)^{k-1/2}} \times \sum_{n=1}^{\infty}(2\pi n)^{k-1/2}K_{k-1/2}(2\pi n a)\ . \tag{B6}$$

The quantity a, as defined in (3.2), is so far arbitrary. As mentioned in the main text, we assume that the low-temperature condition $t \ll 1$ implies that also $a \gg 1$, cf. Eqs. (5.1). This simplifies the expansion (B6) significantly. Namely, for great arguments x we have[15]

$$K_\nu(x)\simeq \left[\frac{\pi}{2x}\right]^{1/2} e^{-x}\ , \tag{B7}$$

so that only the first term ($n=1$) in the sum in (B6) may be of any significance at low temperatures. We thus finally obtain

$$S(a;k)\simeq\frac{\sqrt{\pi}\Gamma(k-\frac{1}{2})}{2\Gamma(k)a^{2k-1}}-\frac{1}{2a^{2k}}+\left[\frac{\pi}{a}\right]^k\frac{e^{-2\pi a}}{\Gamma(k)}\ , \tag{B8}$$

valid for any integer k.

[1]H. B. G. Casimir, Proc. K. Ned. Akad. Wet. Ser. B: **51**, 793 (1948).

[2]T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic, New York, 1981).

[3](a) I. Brevik, J. Phys. A **15**, L369 (1982); (b) **20**, 5189 (1987); Can. J. Phys. **61**, 493 (1983).

[4](a) I. Brevik and H. Kolbenstvedt, Ann. Phys. (N.Y.) **143**, 179 (1982); (b) **149**, 237 (1983); (c) Can. J. Phys. **62**, 805 (1984); (d) **63**, 1409 (1985).

[5]P. Candelas, Ann. Phys. (N.Y.) **143**, 241 (1982).

[6]I. Brevik and G. Einevoll, Phys. Rev. D **37**, 2977 (1988).

[7]M. Born and E. Wolf, *Principles of Optics,* 5th ed. (Pergamon, Oxford, 1975), p. 97.

[8]J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, MA, 1970), Vol. I.

[9]L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media,* 2nd ed. (Pergamon, Oxford, 1984), Sec. 81.

[10]K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) **115**, 388 (1978).

[11]S. A. Fulling and S. N. M. Ruijsenaars, Phys. Rep. **152**, 135 (1987).

[12]J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton, Ann. Phys. (N.Y.) **115**, 1 (1978).

[13]E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1955) [Sov. Phys. JETP **2**, 73 (1956)].

[14]E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon, Oxford, 1980), Sec. 81.

[15]*Handbook of Mathematical Functions,* Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1964) (reprinted by Dover, New York, 1972).

[16]A. D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions* (Holden-Day, San Francisco, 1968).

[17]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).