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Closed strings from open bosonic strings

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We show that closed strings can be incorporated into a bosonic open-string theory by a suitable enlargement of the open-string Fock space. We present an explicit construction of the states in this space in terms of open-string oscillators. The couplings between closed- and open-string states are those deduced from factorization over the closed-string poles of the nonplanar one-loop open-string diagram. We also construct a field theory of closed strings starting from Witten's field theory of open bosonic strings. We show that the interaction vertex is Becchi-Rouet-Stora-Tyutin invariant and that associativity anomalies are absent. We also calculate the residues of poles in the four-tachyon tree amplitude, finding agreement with dual theory.

I. INTRODUCTION

It is well known that closed-string poles arise in the loop expansion of an open-string theory. This seems to indicate that open-string theories cannot be consistent without the inclusion of closed-string states. Thus it is plausible that closed-string dynamics can be described by an open-string theory. If one could isolate the closed-string sector, one would obtain a theory of closed strings only.

In the early 1970s, a light-cone string field theory was constructed, in which open and closed strings were represented by distinct fields.[1] This was possible, despite the appearance of closed-string poles in an open-string diagram, because the theory was not manifestly Lorentz invariant. The couplings between open- and closed-string states were neatly described by an operator which was a state in the tensor product of the closed- and open-string Fock spaces. This operator was originally deduced from a factorization over the closed-string poles of the nonplanar open-string loop diagram, and gave the overlap of an open- and a closed-string state (in today's terminology).[2,3] [It has recently been brought to a Becchi-Rouet-Stora-Tyutin- (BRST-)invariant form by Shapiro and Thorn.[4]]

In recent days, the question of closed strings arising in open-string theories was considered within the context of Witten's[5] covariant string field theory.[6–8] The enormous difficulties encountered when constructing covariant closed-string field theories[5,9,10] led to the idea that such a theory may be redundant, as closed strings ought to be already present in open-string field theories. In the cubic formulation[6] of Witten's string field theory,[5] a state was identified as the graviton by perturbing about the vacuum solution of the equations of motion. Other closed-string states can also be constructed by considering backgrounds corresponding to various closed-string modes. These states are outside of the open-string Fock space and are not annihilated by the BRST operator. They suffer from associativity anomalies,[11] which led Strominger to propose a field theory based on this anomaly as a closed-string field theory.[12]

Here we explore the consequences of the following point of view, adopted by Strominger[13] as an elementary qualitative way of understanding why closed strings may arise as states in a generalized open-string Fock space. Let $\sigma \in [0,\pi]$ be the parameter along a (closed or open) string. Then a complete basis for the functions defined on the interval $[0,\pi]$ is $\{\cos n\sigma\}$. Although this an appropriate basis for open strings, any smooth function can be written as an infinite series of cosines, provided we only require convergence in the mean. In particular, this is true for a function representing a closed-string configuration. These functions necessarily contain an infinite number of cosines and therefore operate on a generalized Fock space. Thus we obtain a connection between open and closed strings. Closed-string modes and the states in the closed-string Fock space can be expressed entirely in terms of open-string modes, and vice versa. Interestingly enough, the couplings between open- and closed-string states are in agreement with earlier results from dual theory.[2–4]

We apply these ideas to Witten's open-string field theory,[5] in order to obtain a field theory of closed strings. We show that the resulting theory gives the correct spectrum and we also check some physical couplings. We do not attempt to calculate any scattering amplitudes because we are working in the oscillator formalism, in which calculation of diagrams is prohibitively complicated. To this end, a better geometrical understanding is needed. We show that our construction is not inhibited by associativity anomalies.[6,11] However, this naive extension of Witten's open-string field theory is unlikely to lead to the correct scattering amplitudes, even at the tree level.[14] One will have to modify the theory to account for the nonplanar nature of the interaction of closed strings. It should also be pointed out that the perturbation expansion is based on a flat background. Thus the graviton arises as a fluctuation in a flat space-time. Ultimately, the theory should be expressed in a background-independent way. In this respect, it will be of interest to apply the present formalism to the cubic action[6] and then try to express the resulting theory independently of the background. These issues, however, will not concern us here.

The organization of the paper is as follows. In Sec. II we derive a linear relation between the closed- and open-string oscillators, and explicitly construct a generalized Fock space in which these oscillators operate. We limit our discussion to bosonic strings. In Sec. III we introduce the reparametrization ghosts and construct the generalized Fock space for the ghost sector. We also define the BRST operator and discuss physical states in the context of BRST quantization. In Sec. IV we make contact with the old approach of the open- and closed-string overlap operator,[1-4] which will enable us to demonstrate the correctness of the couplings between open- and closed-string states. In Sec. V we discuss Witten's open-string field theory in the generalized Fock space. By restricting this space to the closed-string sector, we shall be able to write a field theory of closed strings only. We show that the interaction vertex is BRST invariant, and that associativity anomalies are absent. We also compare some elementary results with dual theory. Finally, in Sec. VI we discuss our conclusions.

II. CONSTRUCTION OF CLOSED-STRING STATES

We start by introducing the open- and closed-string oscillators in the classical theory of strings. We shall then quantize the theory by postulating the standard commutation relations and constructing the Fock space.

The equation governing the motion of a massless relativistic string is

$$\frac{\partial^2}{\partial\sigma^2}x^\mu(\sigma,\tau)-\frac{\partial^2}{\partial\tau^2}x^\mu(\sigma,\tau)=0 \;, \tag{2.1}$$

where $\mu=1,\ldots,d$ and $\sigma\in[0,\pi]$. The dimension of space-time will be taken to be $d=26$, for conformal invariance. The solution to this equation can be written as

$$x^\mu(\sigma,\tau)=q^\mu+2p^\mu\tau+i\sqrt{2}\sum_{n\neq0}\frac{1}{n}\alpha_n^\mu\cos n\sigma\, e^{-in\tau} \;, \tag{2.2}$$

appropriate for open-string boundary conditions, or

$$x^\mu(\sigma,\tau)=q^\mu+2p^\mu\tau+\frac{i}{\sqrt{2}}\sum_{n\neq0}\frac{1}{n}(A_n^\mu e^{2in\sigma}+\tilde{A}_n^\mu e^{-2in\sigma})e^{-2in\tau} \;, \tag{2.3}$$

if one imposes closed-string boundary conditions. Because x^μ is real, we have $\alpha_{-n}^\mu=(\alpha_n^\mu)^*$ and $A_{-n}^\mu=(A_n^\mu)^*$, $\tilde{A}_{-n}^\mu=(\tilde{A}_n^\mu)^*$. However, as was pointed out in the Introduction, any solution of Eq. (2.1) can be given as one of the two forms, Eqs. (2.2) and (2.3), regardless of the boundary conditions it satisfies, provided we only require convergence of the series in the mean.

Now, initial data can be specified by choosing x^μ and $\partial_\tau x^\mu$ at $\tau=0$. These functions can be expressed in terms of the two bases $\{\cos n\sigma\}$ and $\{e^{2in\sigma}\}$ as

$$\begin{gathered} x^\mu(\tau=0)=q^\mu+i\sqrt{2}\sum_{n\neq0}\frac{1}{n}\alpha_n^\mu\cos n\sigma \;, \\ \partial_\tau x^\mu(\tau=0)=\sqrt{2}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\cos n\sigma \;, \end{gathered} \tag{2.4}$$

and

$$\begin{gathered} x^\mu(\tau=0)=q^\mu+\frac{i}{\sqrt{2}}\sum_{n\neq0}\frac{1}{n}(A_n^\mu-\tilde{A}_{-n}^\mu)e^{2in\sigma} \;, \\ \partial_\tau x^\mu(\tau=0)=\sqrt{2}\sum_{n=-\infty}^{\infty}(A_n^\mu+\tilde{A}_{-n}^\mu)e^{2in\sigma} \;, \end{gathered} \tag{2.5}$$

respectively, where we have defined $\alpha_0^\mu=\frac{1}{2}A_0^\mu=\frac{1}{2}\tilde{A}_0^\mu=\sqrt{2}p^\mu$.

Equating the right-hand sides of Eqs. (2.4) and (2.5), we obtain

$$A_n^\mu=\tfrac{1}{2}\alpha_{2n}^\mu+\tfrac{1}{2}\sum_{m=-\infty}^{\infty}E_{nm}\alpha_{2m+1}^\mu \;, \tag{2.6}$$

$$\tilde{A}_n^\mu=\tfrac{1}{2}\alpha_{2n}^\mu-\tfrac{1}{2}\sum_{m=-\infty}^{\infty}E_{nm}\alpha_{2m+1}^\mu \;, \tag{2.7}$$

where

$$E_{nm}=-\frac{i}{\pi}\left[\frac{1}{2n-2m-1}+\frac{1}{2m+1}\right] . \tag{2.8}$$

These two equations express closed-string oscillator modes in terms of open-string modes. The two sets of modes $(A_n^\mu,\tilde{A}_n^\mu)$ and α_n^μ provide two different pictures of the same string theory, rather than two distinct theories. Equations (2.4) and (2.5) can also be used to express the open-string modes in terms of the closed-string modes. The result is

$$\alpha_{2n}^\mu=A_n^\mu+\tilde{A}_n^\mu,\quad \alpha_{2n+1}^\mu=2\sum_{m=-\infty}^{\infty}F_{nm}(A_m^\mu-\tilde{A}_m^\mu) \;, \tag{2.9}$$

where

$$F_{nm}=\frac{i}{\pi}\frac{1}{2m-2n-1} . \tag{2.10}$$

To quantize this classical theory, we postulate the commutation relations

$$[x^\mu(\sigma,\tau),\partial_\tau x^\nu(\sigma',\tau)]=\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.11}$$

from which it follows that

$$[\alpha_n^\mu,\alpha_m^\nu]=n\eta^{\mu\nu}\delta_{n+m,0} \tag{2.12}$$

and

$$[A_n^\mu,A_m^\nu]=[\tilde{A}_n^\mu,\tilde{A}_m^\nu]=n\eta^{\mu\nu}\delta_{n+m,0},\quad [A_n^\mu,\tilde{A}_m^\nu]=0 \;. \tag{2.13}$$

For consistency, one can check that the commutation relations between the A_n^μ's and $\tilde{A}_n^\mu$'s follow from those between the α_n^μ's and Eqs. (2.6) and (2.7). The proof is straightforward and makes use of the orthogonality of the functions $\cos n\sigma$ and $e^{2in\sigma}$.

We now wish to build the Fock space on which these modes act. We can start by defining a vacuum state $|0_{\rm op}\rangle$ for the open-string modes α_n^μ. This is done in the usual way:

$$\alpha_n^\mu|0_{\rm op}\rangle=0 \quad (n>0), \quad \alpha_0^\mu|0_{\rm op}\rangle=\sqrt{2}p^\mu|0_{\rm op}\rangle \ . \tag{2.14}$$

By acting with α_{-n}^μ on $|0_{\rm op}\rangle$ $(n>0)$, we obtain the open-string Fock space. However, when A_{-n}^μ or $\tilde{A}_{-n}^\mu$ act on the vacuum, they take us out of this Fock space. We therefore have to enlarge this space to accommodate closed-string states. We start by constructing the closed-string vacuum $|0_{\rm cl}\rangle$. It is defined by

$$\begin{aligned}&(A_n^\mu,\tilde{A}_n^\mu)|0_{\rm cl}\rangle=0 \quad (n>0)\ ,\\ &A_0^\mu|0_{\rm cl}\rangle=\tilde{A}_0^\mu|0_{\rm cl}=\frac{1}{\sqrt{2}}p^\mu|0_{\rm cl}\rangle \ .\end{aligned} \tag{2.15}$$

To find the explicit form of $|0_{\rm cl}\rangle$, let us assume that it is a state of the form

$$|0_{\rm cl}\rangle=\exp\left[\tfrac{1}{2}\sum_{m,n=0}^{\infty}\tilde{\phi}_{mn}\alpha_{-m}^\mu\alpha_{-n}^\mu\right]|0_{\rm op}\rangle \ , \tag{2.16}$$

which does not belong in the open-string Fock space. Since $A_n^\mu+\tilde{A}_n^\mu=\alpha_{2n}^\mu$ [Eq. (2.9)], it follows that α_{2n}^μ has to commute with the exponent in Eq. (2.16). Therefore,

$$\tilde{\phi}_{2r,m}=0 \ , \tag{2.17}$$

i.e., the only nonvanishing components of the matrix $\tilde{\phi}$ are the ones with bond indices odd. Let us conveniently define

$$\phi_{mn}=\tilde{\phi}_{2m+1,2n+1} \ . \tag{2.18}$$

Acting with $A_n^\mu-\tilde{A}_n^\mu$ on $|0_{\rm cl}\rangle$ and using Eq. (2.12), we obtain

$$\sum_{m=0}^{\infty}(2m+1)E_{nm}\phi_{mk}=E_{-n,k} \ , \tag{2.19}$$

which form a system of an infinite number of equations that determine the coefficients ϕ_{mn}. It can be checked that the solution to this system (giving a symmetric matrix ϕ) is

$$\phi_{mn}=\frac{(-)^{m+n}}{2(m+n+1)}C_m^{-1/2}C_n^{-1/2} \ , \tag{2.20}$$

where $(1+x)^a=\sum_{r=0}^{\infty}C_r^ax^r$. We guessed this solution by comparing with the overlap operator[4] which will be discussed in the next section [cf. Eq. (4.5)]. Thus Eq. (2.16) reads

$$|0_{\rm cl}\rangle=\exp\left[\tfrac{1}{2}\sum_{m,n=0}^{\infty}\phi_{mn}\alpha_{-2m-1}^\mu\alpha_{-2n-1}^\mu\right]|0_{\rm op}\rangle \ . \tag{2.21}$$

Proving that Eq. (2.20) is the solution to Eq. (2.19) is straightforward if one uses the identity

$$\sum_{r=0}^{\infty}\frac{1}{r+b}(-)^rC_r^a=\frac{\Gamma(b)\Gamma(a+1)}{\Gamma(a+b+1)} \ , \tag{2.22}$$

and makes repeated use of partial fractions.

By acting with A_{-n}^μ and $\tilde{A}_{-n}^\mu$ on $|0_{\rm cl}\rangle$ $(n>0)$, we build the closed-string Fock space. Together with the open-string sector, they form an enlarged space that includes both open- and closed-string states. This string theory can be described solely in terms of open- or closed-string modes. It is only a question of convenience to choose one set over the other. Thus Eqs. (2.6) and (2.7) provide a transformation from the open- to the closed-string picture. It is also possible to write the open-string vacuum in the closed-string picture. Using Eqs. (2.16) and (2.9), after some algebra we obtain

$$|0_{\rm op}\rangle=\exp\left[\tfrac{1}{2}\sum_{m,n=1}^{\infty}\chi_{mn}(A_{-m}^\mu-\tilde{A}_{-m}^\mu)\times(A_{-n}^\mu-\tilde{A}_{-n}^\mu)\right]|0_{\rm cl}\rangle \ , \tag{2.23}$$

where

$$\chi_{mn}=\frac{(-)^{m+n}}{2(m+n)}C_m^{-1/2}C_n^{-1/2} \ . \tag{2.24}$$

The stress-energy tensor $T_{\alpha\beta}^{(x)}$ (with components $T_{++}^{(x)}=\frac{1}{2}\partial_+x^\mu\partial_+x_\mu$, $T_{--}^{(x)}=\frac{1}{2}\partial_-x^\mu\partial_-x_\mu$, and $T_{+-}^{(x)}=T_{-+}^{(x)}=0$) can also be expressed in terms of two different sets of modes. Thus we obtain the open-string Virasoro generators

$$\begin{aligned}l_n^{(x)}&=\frac{1}{\pi}\int_0^\pi(e^{in\sigma}T_{++}^{(x)}+e^{-in\sigma}T_{--}^{(x)})d\sigma\\ &=\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_m^\mu\alpha_{n-m}^\mu\end{aligned} \tag{2.25}$$

and the closed-string Virasoro generators

$$L_n^{(x)}=\frac{1}{\pi}\int_0^\pi e^{-2in\sigma}T_{--}^{(x)}d\sigma=\tfrac{1}{2}\sum_{m=-\infty}^{\infty}A_m^\mu A_{n-m}^\mu \ , \tag{2.26}$$

$$\tilde{L}_n^{(x)}=\frac{1}{\pi}\int_0^\pi e^{2in\sigma}T_{++}^{(x)}d\sigma=\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\tilde{A}_m^\mu\tilde{A}_{n-m}^\mu \ , \tag{2.27}$$

containing no surprises. They are related by

$$L_n^{(x)}=\tfrac{1}{2}l_{2n}^{(x)}+\tfrac{1}{2}\sum_{m=-\infty}^{\infty}F_{nm}l_{2m+1}^{(x)} \ , \tag{2.28}$$

$$\tilde{L}_n^{(x)}=\tfrac{1}{2}l_{2n}^{(x)}-\tfrac{1}{2}\sum_{m=-\infty}^{\infty}F_{nm}l_{2m+1}^{(x)} \ . \tag{2.29}$$

The zero mode $l_0^{(x)}=L_0^{(x)}+\tilde{L}_0^{(x)}$ is the Hamiltonian of the system. We must also take care of normal ordering. Let $:\Theta:$ $(\ddagger\Theta\ddagger)$ denote normal ordering with respect to $|0_{\rm op}\rangle$ $(|0_{\rm cl}\rangle)$ for an operator Θ. Then, since

$$l_0^{(x)}=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{-n}^\mu\alpha_n^\mu=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(A_{-n}^\mu A_n^\mu+\tilde{A}_{-n}^\mu\tilde{A}_n^\mu) \ , \tag{2.30}$$

we have

$$l_0^{(x)}=:l_0^{(x)}:-1=\ddagger l_0^{(x)}\ddagger-2 \ , \tag{2.31}$$

in $d=26$ dimensions. This also shows that the masses of $|0_{\rm op}\rangle$ and $|0_{\rm cl}\rangle$ are $m^2=-1$ and $m^2=-2$, respectively, as expected. [The cautious reader will notice that Eq. (2.31) appears to contradict the fact that both $:l_0^{(x)}:$ and

$\ddagger l_0^{(x)}\ddagger$ are positive-definite operators. However, in order to obtain Eq. (2.31), one has to regulate the theory, so that diverging sums of the form $\sum n$ become finite (and negative). Thus Eq. (2.31) follows assuming the validity of ζ-function regularization. A careful manipulation of these diverging sums could entail a discretization of the string. As one goes to the continuum limit, one should obtain the same results as with ζ-function regularization. We have not checked this in detail, but no results on physical quantities seem to be affected by the regulator.]

This completes the discussion of the Fock space and the spectrum of the theory. It remains to be seen that the theory is unitary. We shall discuss this issue in the context of BRST quantization.

III. REPARAMETRIZATION GHOSTS AND PHYSICAL STATES

In this section we introduce the reparametrization ghosts $b_{\alpha\beta}$ and c^{α} and define the BRST operator Q. We then build the Fock space of the ghost sector and identify the physical sector of the theory. The BRST operator is defined by

$$Q=\frac{1}{\pi}\int_0^{\pi}d\sigma(J_+ + J_-) , \tag{3.1}$$

where J_α is the BRST current with components $J_\pm$:

$$J_\pm = c^{\pm}(T^{(x)}_{\pm\pm}+\tfrac{1}{2}T^{(\mathrm{gh})}_{\pm\pm}) . \tag{3.2}$$

The stress-energy tensor for the ghost degrees of freedom is given by

$$T^{(\mathrm{gh})}_{++}=\tfrac{1}{2}c^+\partial_+ b_{++}+\partial_+ c^+ b_{++} , \tag{3.3}$$

and similarly for $T^{(\mathrm{gh})}_{--}$.

Physical states are annihilated by Q: $Q(\text{phys})=0$; and we identify states differing by $Q|\phi\rangle$, for some state $|\phi\rangle$.

The ghost fields $b_{\alpha\beta}$ and c^{α} can be written in the open- and closed-string pictures as

$$\begin{aligned} b_{\pm\pm}(\sigma,\tau)&=\sum_{n=-\infty}^{\infty} b_n e^{-in(\tau\pm\sigma)} , \\ c^{\pm}(\sigma,\tau)&=\sum_{n=-\infty}^{\infty} c_n e^{-in(\tau\pm\sigma)} , \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} b_{++}(\sigma,\tau)&=\sqrt{2}\sum_{n=-\infty}^{\infty} \tilde{B}_n e^{-2in(\tau+\sigma)} , \\ b_{--}(\sigma,\tau)&=\sqrt{2}\sum_{n=-\infty}^{\infty} B_n e^{-2in(\tau-\sigma)} , \end{aligned} \tag{3.5}$$

$$\begin{aligned} c^{+}(\sigma,\tau)&=\sqrt{2}\sum_{n=-\infty}^{\infty} \tilde{C}_n e^{-2in(\tau+\sigma)} , \\ c^{-}(\sigma,\tau)&=\sqrt{2}\sum_{n=-\infty}^{\infty} C_n e^{-2in(\tau-\sigma)} , \end{aligned} \tag{3.6}$$

respectively.

The Fourier modes in the two pictures are related by

$$\begin{aligned} B_n&=\frac{1}{\sqrt{2}}b_{2n}-\frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty} F_{nm} b_{2m+1} , \\ \tilde{B}_n&=\frac{1}{\sqrt{2}}b_{2n}+\frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty} F_{nm} b_{2m+1} , \end{aligned} \tag{3.7}$$

where F_{mn} is given by Eq. (2.10), and similarly for C_n and $\tilde{C}_n$. Equation (3.7) is the map from the open- to the closed-string picture for the ghost sector. Its inverse is

$$\begin{aligned} b_{2n}&=\frac{1}{\sqrt{2}}(B_n+\tilde{B}_n) , \\ b_{2n+1}&=-\frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty} F_{mn}(B_m-\tilde{B}_m) , \end{aligned} \tag{3.8}$$

and similarly for c_{2n} and c_{2n+1}.

To derive the above equations, one can follow the discussion leading to similar relations between the Fourier modes of the bosonic coordinates x^{μ} [cf. Eqs. (2.6), (2.7), and (2.9)]. These modes obey the usual anticommutation relations

$$\{b_n,c_m\}=\{B_n,C_m\}=\{\tilde{B}_n,\tilde{C}_m\}=\delta_{m+n,0} , \tag{3.9}$$

with all other anticommutators vanishing.

We can also construct the ghost Fock space in a similar manner. Starting from the open-string ghost vacua $|\pm\rangle$ defined by $b_0|-\rangle=c_0|+\rangle=0$, $b_0|+\rangle=|-\rangle$, $c_0|-\rangle=|+\rangle$, we build a Fock space by acting with the modes b_{-n}, c_{-n} $(n>0)$ on $|\pm\rangle$. We then introduce the closed-string vacua: $|\pm,\pm\rangle$, where $\{B_n,C_n\}$ $(\{\tilde{B}_n,\tilde{C}_n\})$ act on the first (second) index. The requirement that $(B_n,\tilde{B}_n,C_n,\tilde{C}_n)|\pm,\pm\rangle=0$ $(n>0)$ leads to the following identifications:

$$\begin{aligned} &|++\rangle=e^{\Omega}\Theta_c|+\rangle, \quad |+-\rangle=e^{\Omega}(|+\rangle+\Theta_c|-\rangle) , \\ &|-+\rangle=e^{\Omega}(|+\rangle-\Theta_c|-\rangle), \quad |--\rangle=e^{\Omega}|-\rangle , \end{aligned} \tag{3.10}$$

where

$$\Omega=\sum_{m,n=0}^{\infty}\theta_{mn}c_{-2m-1}b_{-2n-1} \tag{3.11}$$

and

$$\begin{aligned} \theta_{mn}&=\frac{(-)^{m+n+1}}{2(m+n+1)}C_m^{-3/2}C_n^{-1/2} , \\ \Theta_c&=i\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}C_n^{-3/2}c_{-2n-1} . \end{aligned} \tag{3.12}$$

Again, these expressions for the closed-string ghost vacua were guessed at after comparing with the BRST-invariant version of the overlap operator.[4] (It should be noted that we do not completely agree with Ref. 4 in the ghost sector. However, we have not been able to track down the origin of the discrepancy.) For completeness, we express the open-string vacua in terms of the closed-string oscillators:

$$|+\rangle=e^{\bar{\Omega}}(|+-\rangle+|-+\rangle), \quad |-\rangle=e^{\bar{\Omega}}|--\rangle , \tag{3.13}$$

where

$$\overline{\Omega}=\sum_{m,n=1}^{\infty}\lambda_{mn}(C_{-m}-\tilde{C}_{-m})(B_{-n}-\tilde{B}_{-n})\ ,\qquad(3.14)$$

and $\lambda_{mn}=[(-)^{m+n}/4(m+n)]C_{m-1}^{-3/2}C_n^{-1/2}$.

Care should be taken in defining the inner product in the Hilbert space in which the zero modes act. In open-string theory, such an inner product is defined by $\langle +|-\rangle=\langle -|+\rangle=1$, $\langle +|+\rangle=\langle -|-\rangle=0$. Thus $\langle +|$ ($\langle -|$) is the bra that is dual to the ket $|-\rangle$ ($|+\rangle$). With this understanding, we may define

$$\langle +|=\langle ++|e^{\overline{\Omega}^\dagger},\quad \langle -|=(\langle +-|+\langle -+|)e^{\overline{\Omega}^\dagger}\ .\qquad(3.15)$$

Notice that the closed-string ghost numbers of $|+\rangle$, $|-\rangle$, and their duals, $\langle -|$, $\langle +|$, are 0, -1, and 0, 1, respectively. We shall make use of these assignments in Sec. V in discussing closed-string field theory. In a similar way, one defines the duals of the closed-string vacua, $|\pm,\pm\rangle$. Explicitly,

$$\langle ++|=\langle +|e^{\Omega^\dagger},\quad \langle +-|=(\langle -|+\langle +|\Theta_b)e^{\Omega^\dagger}\ ,$$
$$\langle -+|=(\langle -|-\langle +|\Theta_b)e^{\Omega^\dagger},\quad \langle --|=\langle -|\Theta_b e^{\Omega^\dagger}\ ,\qquad(3.16)$$

where $\Theta_b=-i\sum[(-)^n/(2n+1)]C_n^{-1/2}b_{2n+1}$. It is now an easy exercise to show that Q can be written as

$$Q=\sum_{n=-\infty}^{\infty}(l_{-n}^{(x)}+\tfrac{1}{2}l_{-n}^{(\mathrm{gh})})c_n$$
$$=\sum_{n=-\infty}^{\infty}[(L_{-n}^{(x)}+\tfrac{1}{2}L_{-n}^{(\mathrm{gh})})C_n+(\tilde{L}_{-n}^{(x)}+\tfrac{1}{2}\tilde{L}_{-n}^{(\mathrm{gh})})\tilde{C}_n]\qquad(3.17)$$

in the open- and closed-string pictures, respectively. These are exactly the forms that Q takes in the ordinary open- and closed-string theories. Therefore all states in the two sectors that are physical in the respective ordinary theories are still physical by the same token. In particular, the closed-string vacuum, $|0_{\mathrm{cl}},\pm\pm\rangle$, is a physical state. Although $l_n^{(x)}|0_{\mathrm{cl}}\rangle\neq 0$ ($n>0$), the closed-string vacuum $|0_{\mathrm{cl}},\pm\pm\rangle$ is a physical state, because it contains open-string ghost oscillators which allow for $Q|0_{\mathrm{cl}},\pm\pm\rangle=0$. This contrasts the results of Refs. 8, 12, and 13 where a closed-string state is identified but is not annihilated by Q. In our approach, it is straightforward to write any closed-string state in terms of open-string oscillators. Moreover, all physical states in the standard closed-string theory are annihilated by the BRST operator which reduces to its standard forms in both the open- and the closed-string sectors [cf. Eq. (3.17)].

IV. SCATTERING AMPLITUDES FOR CLOSED AND OPEN STRINGS

To compute scattering amplitudes for external open-string states, we can express all operators in terms of the open-string modes α_n^μ. It is then clear that we obtain the same results as in the ordinary open-string theory. The same is true for the closed-string sector, because of Eqs. (2.13), (2.26), and (2.27).

It remains to be seen that mixed diagrams containing both closed- and open-string external states are as expected. To answer this question, we shall make use of the overlap operator Υ, introduced to describe transitions between closed- and open-string states.[1-4] This operator led to the correct couplings of open- and closed-string states. These couplings can be deduced by the factorization over the closed-string poles of a nonpolar one-loop open-string diagram. The operator Υ was recently brought into a BRST-invariant form by Shapiro and Thorn,[4] which we shall use. Explicitly,

$$_{12}\langle\Upsilon|={}_1\langle +,0_{\mathrm{op}}|\,{}_2\langle ++,0_{\mathrm{cl}}|e^{W_x^{12}}e^{W_{\mathrm{gh}}^{12}}\ ,\qquad(4.1)$$

where

$$W_x^{12}=\tfrac{1}{2}\sum_{m,n=0}^{\infty}\phi_{mn}\alpha_{2m+1}^1\alpha_{2n+1}^1-\sum_{m=1}^{\infty}\alpha_{2m}^1(A_m^2+\tilde{A}_m^2)$$
$$+\sum_{m,n=0}^{\infty}\psi_{mn}\alpha_{2m+1}^1(A_n^2-\tilde{A}_n^2)$$
$$+\tfrac{1}{2}\sum_{m,n=1}^{\infty}\chi_{mn}(A_m^2-\tilde{A}_m^2)(A_n^2-\tilde{A}\,n^2)\qquad(4.2)$$

and

$$W_{\mathrm{gh}}^{12}=\sum_{m,n=0}^{\infty}\theta_{mn}c_{2m+1}^1b_{2n+1}^1-\sum_{m=1}^{\infty}c_{2m}^1b_0^1+\sum_{m,n=1}^{\infty}\kappa_{mn}b_{2m+1}^1(C_n^2-\tilde{C}_n^2)+\tfrac{1}{4}\sum_{m=1}^{\infty}b_0^1(C_m^2+\tilde{C}_m^2)$$
$$+\sum_{m,n=1}^{\infty}\xi_{mn}c_{2m+1}^1(B_n^2-\tilde{B}_n^2)-\tfrac{1}{4}\sum_{m=1}^{\infty}b_{2m}^1(C_m^2+\tilde{C}_m^2)-2\sum_{m=1}^{\infty}c_{2m}^1(B_m^2+\tilde{B}_m^2)$$
$$+\sum_{m,n=1}^{\infty}\lambda_{mn}(C_m^2-\tilde{C}_m^2)(B_n^2-\tilde{B}_n^2)-\tfrac{1}{2}\sum_{m=1}^{\infty}(C_m^2+\tilde{C}_m^2)(B_0^2+\tilde{B}_0^2)\ .\qquad(4.3)$$

The coefficients ϕ_{mn}, χ_{mn}, θ_{mn}, and λ_{mn} are given by Eqs. (2.20), (2.24), (3.12), and (3.14), respectively. We also have

$$\psi_{mn}=i\frac{(-)^{m+n}}{2n-2m-1}C_m^{-1/2}C_n^{-1/2}\ ,$$
$$\kappa_{mn}=\frac{i}{4}\frac{(-)^{m+n+1}}{2n-2m-1}C_m^{-1/2}C_{n-1}^{-3/2}\ . \tag{4.4}$$
$$\xi_{mn}=2i\frac{(-)^{m+n}}{2n-2m-1}C_m^{-3/2}C_n^{-1/2}\ .$$

Thus the open- and closed-string modes are treated as independent oscillators. If $|\phi\rangle$ is an open- (closed-) string state, then ${}_1\langle\phi|\Upsilon\rangle_{12}$ (${}_2\langle\phi|\Upsilon\rangle_{12}$) is a closed- (open-) string state.

There is a direct correspondence between the bosonic part of Υ and the relation between open- and closed-string oscillators [Eqs. (2.6) and (2.7)]: If $|\phi\rangle$ is a state written in terms of closed-string oscillators, then ${}_2\langle\phi|\Upsilon\rangle_{12}$ is the same state written in terms of open-string oscillators. So Υ is essentially a picture-changing operator. (The same is not quite true for the ghost sector, as pointed out earlier.) We shall verify this for the tachyon $|0_{\rm cl}\rangle$ and the graviton $\zeta_{\mu\nu}A_1^\mu\tilde{A}\,{}^\nu_{-1}|0_{\rm cl}$. We can easily see that, ignoring ghost modes,

$${}_2\langle 0_{\rm cl}|\Upsilon\rangle_{12}=\exp\left[\sum_{k,l=0}^{\infty}\phi_{kl}\alpha^\mu_{-2k-1}\alpha^\mu_{-2l-1}\right]|0_{\rm op}\rangle \tag{4.5}$$

which is identical to Eq. (2.21). In fact, it is because of this correspondence that we were able to guess the solution to Eq. (2.19) and find the coefficients ϕ_{mn}.

For the graviton we have

$${}_2\langle 0_{\rm cl}|\zeta_{\mu\nu}A_1^\mu\tilde{A}\,{}^\nu_1|\Upsilon\rangle_{12}=\zeta_{\mu\nu}\left[\tfrac{1}{2}\alpha^\mu_{-2}+\sum_{k=0}^{\infty}\psi_{k1}\alpha^\mu_{-2k-1}\right]\left[\tfrac{1}{2}\alpha^\mu_{-2}-\sum_{l=0}^{\infty}\psi_{l1}\alpha^\nu_{-2l-1}\right]\exp\left[\tfrac{1}{2}\sum\phi_{mn}\alpha_{-2m-1}\alpha_{-2n-1}\right]|0_{\rm op}\rangle\ , \tag{4.6}$$

where we used the fact that $\zeta^\mu_\mu=0$. Using Eqs. (2.6), (2.7), (2.16), and (2.20), we obtain

$$A^\mu_{-m}|0_{\rm cl}\rangle=\left[\tfrac{1}{2}\alpha^\mu_{-2m}+\sum_{n=0}^{\infty}m\psi_{nm}\alpha^\mu_{-2n-1}\right]\times\exp\left[\tfrac{1}{2}\sum\phi_{kl}\alpha_{-2k-1}\cdot\alpha_{-2l-1}\right]|0_{\rm op}\rangle\ ,$$
$$\tilde{A}\,{}^\mu_{-m}|0_{\rm cl}\rangle=\left[\tfrac{1}{2}\alpha^\mu_{-2m}-\sum_{n=0}^{\infty}m\psi_{nm}\alpha^\mu_{-2n-1}\right]\times\exp\left[\tfrac{1}{2}\sum\phi_{kl}\alpha_{-2k-1}\cdot\alpha_{-2l-1}\right]|0_{\rm p}\rangle\ . \tag{4.7}$$

Therefore $\zeta_{\mu\nu}A^\mu_{-1}\tilde{A}\,{}^\nu_{-1}|0_{\rm cl}\rangle$ is identical to the right-hand side of Eq. (4.6). We can also show that the same correspondence exists for all other states, by making use of Eq. (4.7). Because of this correspondence, the scattering amplitudes for one closed-string state and an arbitrary number of open-string states are the same as those predicted by the factorization of nonplanar open-string loop graphs.[4]

V. A CLOSED-STRING FIELD THEORY

Although a satisfactory covariant field theory has been found for open strings,[5] it has not yet been possible to develop a similar theory for closed strings. However, since closed strings inevitably appear as intermediate states in the interaction of open strings, they also appear in open-string field theory. In this section we show how the ideas we developed above can lead to a closed-string field theory. Our discussion is based on Witten's string field theory.

Witten has developed a covariant formulation of open-string field theory,[5] by defining an interaction vertex, which in terms of oscillators is of the form[15]

$${}_{123}\langle V_3|={}_1\langle 0_{\rm op}|\,{}_1\langle +|\,{}_2\langle 0_{\rm op}|\times{}_2\langle +|\,{}_3\langle 0_{\rm op}|\,{}_3\langle +|e^{N_x}e^{N_{\rm gh}}\ , \tag{5.1}$$

where

$$N_x=\tfrac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=0}^{\infty}\alpha^{\mu,r}_m N^{rs}_{mn}\alpha^{\mu,s}_n\ ,$$
$$N_{\rm gh}=\sum_{r,s=1}^{3}\sum_{m,n=0}^{\infty}b^r_m\tilde{N}\,{}^{rs}_{mn}nc^s_n\ . \tag{5.2}$$

The coefficients N^{rs}_{mn} and $\tilde{N}\,{}^{rs}_{mn}$ are Fourier modes of Neumann functions and have been computed in Ref. 15. An important property of this vertex is that it is BRST invariant:

$${}_{123}\langle V_3|(Q_1+Q_2+Q_3)=0\ . \tag{5.3}$$

An integration is also defined by

$$\int A=\langle I|A\rangle\ , \tag{5.4}$$

where the "identity" state $\langle I|$ satisfies ${}_{123}\langle V_3|I\rangle_2|A\rangle_3={}_1\langle A|$. It is also convenient to define a two-state vertex $\langle V_2|$ by ${}_{12}\langle V_2|={}_{123}\langle V_3|I\rangle_3$. Explicitly,

$$\langle I|=\langle +|\exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\alpha^\mu_n\alpha^\mu_n\right]\times\exp\left[\sum_{n=1}^{\infty}(-)^n b_n c_n\right]b_-b_+\ , \tag{5.5}$$

where $b_\pm=\sum(\pm)^n b_n$ and

$$_{12}\langle V_2| = ({}_1\langle 0_{\rm op}, +|\,{}_2\langle 0_{\rm op}, -| + {}_1\langle 0_{\rm op}, -|\,{}_2\langle 0_{\rm op}, +|)$$
$$\times \exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\alpha_n^{\mu,1}\alpha_n^{\mu,2}\right]$$
$$\times \exp\left[\sum_{n=1}^{\infty}(-)^n(b_n^1c_n^2+b_n^2c_n^1)\right]. \tag{5.6}$$

The action

$$S=\tfrac{1}{2}\,{}_{12}\langle V_2|Q_1|A\rangle_1|A\rangle_2+\tfrac{1}{3}\,{}_{123}\langle V_3|A\rangle_1|A\rangle_2|A\rangle_3 \tag{5.7}$$

is invariant under the transformation

$$\delta|A\rangle_1=Q|\epsilon\rangle_1+{}_2\langle A|\,{}_3\langle\epsilon|V_3\rangle_{123}-\,{}_2\langle\epsilon|{}_3\langle A|V_3\rangle_{123}\,. \tag{5.8}$$

The state $|A\rangle$ has ghost number $-\frac{1}{2}$ and the ghost numbers of $\langle V_3|$, $\langle I|$, and $\langle V_2|$ are $\frac{3}{2}$, $-\frac{3}{2}$, and 0, respectively.

The vertex operator contains factors that are exponentials of quadratic forms of open-string oscillators. In general, these operators take us out of the open-string Fock space. It is therefore not surprising that closed-string states can couple to open strings even though in the functional integral one integrates only over open-string states. It appears that, for consistency, one ought to extend the integration region over the enlarged Fock space that we described in Sec. II. This way the theory will describe both open and closed strings.

We can now go one step further and restrict the enlarged Fock space to the closed-string sector. If we express the vertex operator in terms of closed-string oscillators, we obtain a closed-string field theory defined over the ordinary closed-string Fock space. Explicitly, the vertex operator is now

$$_{123}\langle V_3|={}_1\langle 0_{\rm cl},++|\,{}_2\langle 0_{\rm cl},++|\,{}_3\langle 0_{\rm cl},++|\exp\left[\tfrac{1}{2}\sum_{r=1}^{3}\sum_{m,n=1}^{\infty}\chi_{mn}(A_m^{\mu,r}-\tilde{A}_m^{\mu,r})(A_n^{\mu,r}-\tilde{A}_n^{\mu,r})\right]$$
$$\times\exp\left[\sum_{r=1}^{3}\sum_{m,n=1}^{\infty}\lambda_{mn}(B_m^r-\tilde{B}_m^r)(C_n^r-\tilde{C}_n^r)\right]e^{N_x}e^{N_{\rm gh}}P^1P^2P^3\,, \tag{5.9}$$

where we used Eqs. (2.23) and (3.15) to write the open-string vacuum in terms of closed-string oscillators. Using Eqs. (2.9) and (3.8), we can also write N_x and $N_{\rm gh}$ in terms of closed-string oscillators. The answer is

$$N_x=\tfrac{1}{2}(A_m^r+\tilde{A}_m^r)N_{2m,2n}^{rs}(A_n^s+\tilde{A}_n^s)+F_{lm}(A_l^r-\tilde{A}_l^r)N_{2m+1,2n}^{rs}(A_n^s+\tilde{A}_n^s)$$
$$+\tfrac{1}{2}F_{lm}(A_l^r-\tilde{A}_l^r)N_{2m+1,2n+1}^{rs}F_{kn}(A_k^2-\tilde{A}_k^s)\,, \tag{5.10}$$

$$N_{\rm gh}=(B_m^r+\tilde{B}_m^r)\tilde{N}_{2m,2n}^{rs}2n(C_n^s+\tilde{C}_n^s)+F_{ml}(B_l^r-\tilde{B}_l^r)\tilde{N}_{2m+1,2n}^{rs}2n(C_n^s+\tilde{C}_n^s)$$
$$+(B_m^r+\tilde{B}_m^r)\tilde{N}_{2m,2n+1}^{rs}(2n+1)F_{nk}(C_k^s-\tilde{C}_k^2)+F_{ml}(B_l^r-\tilde{B}_l^r)\tilde{N}_{2m+1,2n+1}^{rs}(2n+1)F_{nk}(C_k^s-\tilde{C}_k^s)\,, \tag{5.11}$$

where we have dropped the summation symbols. (It is understood that the indices r and s take on the values 1, 2, and 3, whereas the other latin indices assume all integer values in F, and non-negative values in N and $\tilde{N}$.) In Eq. (5.9) we included projection operators

$$P=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi(L_0-\tilde{L}_0)}\,, \tag{5.12}$$

to project out the states that are not annihilated by $L_0-\tilde{L}_0$. Notice that the ghost number of the vertex operator $\langle V_3|$ is 3, which is as it should be, because closed-string physical states have ghost number -1. The ghost numbers of the "identity" state $\langle I|$ [Eq. (5.5)] and the two-string vertex $\langle V_2$ [Eq. (5.6)] are -1 and 1, respectively, because the respective vacua in $\langle I|$ and $\langle V_2|$ are $\langle ++|$ and ${}_1\langle ++|\,{}_2(\langle +-|+\langle -+|)+{}_1(\langle +-|+\langle -+|)\,{}_2\langle ++|$ [cf. Eqs. (5.5), (5.6), and (3.15)]. Thus in the closed-string field theory, a state $|A\rangle$ and its conjugate ${}_1\langle A|={}_{12}\langle V_2|A\rangle_2$ do not have the same ghost number. Although, e.g., $|A\rangle$, $\langle V_3|$, and $\langle I|$ have ghost numbers -1, 3, and -1, respectively, their conjugates $\langle A|$, $|V_3\rangle$, and $|I\rangle$ have ghost numbers 0, 0, and -2, respectively. Because of the ghost-number assignments, a geometrical interpretation of the interaction vertex is not obvious for closed strings.

The interaction vertex seems to correspond to the bilocal interaction discussed first by Witten,[5] and subsequently by various groups.[9] The kinetic energy part differs from the one considered by Lykken and Raby,[9] as a comparison of the ghost-number assignments of the integration vertex shows. In our case, $|I\rangle$ has ghost number -2, which does not lend itself to a geometrical interpretation, unlike in the case of Lykken and Raby, where $|I\rangle$ has ghost number -3. It can be seen that our definition of $|I\rangle$ contains an insertion of $C_0-\tilde{C}_0$ [cf. Eq. (5.14)], in agreement with de Alwis and Ohta.[9] However, the precise connection cannot be conclusively established before we bring the vertices into a manageable form containing only annihilation closed-string modes.

BRST invariance is easily established, because the BRST operator Q is the same as in the open-string theory [cf. Eq. (3.17)]. The only novel element is the projection operator onto the kernel of $L_0-\tilde{L}_0$. But this is not a problem because Q commutes with this operator. This can be checked explicitly, but it follows immediately from the fact that Q is the integral over σ of a charge-density operator [Eq. (3.1)] and $L_0-\tilde{L}_0$ generates translations in the σ direction.

The Witten action for closed strings is therefore

$$S=\tfrac{1}{2}\,{}_{12}\langle V_2|QP|A\rangle_1|A\rangle_2+\tfrac{1}{3}\,{}_{123}\langle V|A\rangle_1|A\rangle_2|A\rangle_3\,, \tag{5.13}$$

where $|A\rangle$ is a state in the closed-string Fock space. By choosing the gauge $b_0|A\rangle=0$, or, in closed-string modes $(B_0+\tilde{B}_0)|A\rangle=0$ [notice that this gauge is equivalent to $B_0|A\rangle=\tilde{B}_0|A\rangle=0$, because the vacuum $|-\rangle$ is proportional to $|--\rangle$, cf. Eq. (3.14)], we obtain a kinetic-energy term of the form

$$\begin{aligned}S_0&\equiv\tfrac{1}{2}\langle\bar{A}|B_0\tilde{B}_0(C_0-\tilde{C}_0)QPB_0\tilde{B}_0|\bar{A}\rangle\\&=\tfrac{1}{2}\langle\bar{A}|(L_0+\tilde{L}_0)PB_0\tilde{B}_0)|\bar{A}\rangle\,,\end{aligned} \tag{5.14}$$

where $|A\rangle=B_0\tilde{B}_0|\bar{A}\rangle$ and we define a conjugate state $\langle A|$ of the same ghost number as $|A\rangle$. It follows that the propagator is

$$\Delta=\frac{1}{L_0+\tilde{L}_0}P=\int d^2z\, z^{L_0}\bar{z}^{\tilde{L}_0}\,, \tag{5.15}$$

in agreement with dual theory, and there are insertions of the operators $B_0=\oint b_{++}$ and $\tilde{B}_0=\oint b_{--}$. These insertions are precisely the ones needed in order that the integrand in a scattering amplitude be a density in moduli space.[14]

Thus our procedure has led to the correct propagator for close strings and therefore also to the correct spectrum. We can now proceed to calculate the residues of poles in the scattering amplitude of four tachyons. In the following, we shall ignore the ghost degrees of freedom.

Consider on-shell tachyons $|p_i\rangle$, where $p_i^2=2$ $(i=1,2,3,4)$. Define the Mandelstam variables $s=-(p_1+p_2)^2$, $t=-(p_2+p_3)^2$, and $u=-(p_1+p_3)^2$. The Virasoro-Shapiro amplitude is

$$S^{(\mathrm{VS})}=\frac{\Gamma(-\frac{1}{2}s-1)\Gamma(1-\frac{1}{2}t-1)\Gamma(-\frac{1}{2}u-1)}{\Gamma(\frac{1}{2}s+2)\Gamma(\frac{1}{2}t+2)\Gamma(\frac{1}{2}u+2)}\,. \tag{5.16}$$

In the s channel, we can expand the four-point amplitude as

$$S=S_{-1}+S_0+\cdots\,, \tag{5.17}$$

where S_{-1} (S_0) is the contribution from an intermediate tachyon (massless mode). A simple computation yields

$$S_{-1}^{(\mathrm{VS})}=-\frac{2}{s+2},\quad S_0^{(\mathrm{VS})}=-\frac{(t+4)^2}{2s}\,. \tag{5.18}$$

We shall compute these quantities using the Witten action [Eq. (5.13)]. We shall only work out explicitly the contributions of the tachyon and the graviton. However, our method is general and can be used to compute the contributions from all other modes. Unfortunately we have only been able to calculate residues analytically up to proportionality constants. The calculation of these constants seems to require numerical techniques.

The propagators for the tachyon ($|0_{\mathrm{cl}}\rangle$) and the graviton ($\zeta_{\mu\nu}A_{-1}^{\mu}\tilde{A}_{-1}^{\nu}|0_{\mathrm{cl}}\rangle$) are

$$\begin{aligned}\Delta_{-1}(p,p')&\equiv\langle 0_{\mathrm{cl}}|\Delta|0_{\mathrm{cl}}\rangle=\frac{1}{p^2-2}\delta(p+p')\,,\\ \Delta_0(p,p')&\equiv\langle 0_{\mathrm{cl}}|\zeta_{\mu\nu}A_1^{\mu}\tilde{A}_1^{\nu}\Delta\zeta_{\rho\sigma}A_{-1}^{\rho}\tilde{A}_{-1}^{\sigma}|0_{\mathrm{cl}}\rangle\\&=\frac{1}{p^2}\zeta_{\mu\nu}\zeta^{\mu\nu}\delta(p+p')\,,\end{aligned} \tag{5.19}$$

respectively. One also needs the couplings between various states. These are obtained by saturating the three legs of the interaction vertex $\langle V_3|$ with the relevant states. We have found it advantageous to employ the open-string picture; i.e., use Eq. (5.1) for the vertex and Eqs. (4.5) and (4.6) for the tachyon and graviton, respectively. Defining a state $|\lambda\rangle$ by

$$|\lambda\rangle=\exp\left[\frac{1}{2}\sum_{m,n=0}^{\infty}\phi_{mn}\alpha_{-2m-1}^{\mu}\alpha_{-2n-1}^{\mu}+\sum_{m=1}^{\infty}\lambda_m^{\mu}\alpha_{-m}^{\mu}\right]\times|0_{\mathrm{op}}\rangle\,, \tag{5.20}$$

we wish to calculate

$$\bar{V}_{\lambda}(p_1,p_2,p_3)\equiv{}_{123}\langle V_3|0_{\mathrm{cl}},p_1\rangle_1|0_{\mathrm{cl}},p_2\rangle_2|\lambda,p_3\rangle_3\,. \tag{5.21}$$

By taking suitable derivatives with respect to λ as many times as needed, one can thus compute all couplings between two tachyons and an arbitrary closed-string state.

To calculate $\bar{V}_{\lambda}$, we define matrices N and ϕ, and vectors λ and a, by

$$\begin{aligned}(N)_{mn}^{rs,\mu\nu}&=\sqrt{mn}\,N_{mn}^{rs}\delta^{\mu\nu}\,,\\ (\phi)_{2m+1,2n+1}^{rs,\mu\nu}&=\sqrt{(2m+1)(2n+1)}\phi_{mn}\delta^{rs}\delta^{\mu\nu}\end{aligned} \tag{5.22}$$

(with all other components of ϕ vanishing) and

$$\begin{aligned}&(\lambda)_m^{r,\mu}=\sqrt{m}\,\lambda_m^{\mu}\delta^{r3},\quad (a)_m^{r,\mu}=\frac{1}{\sqrt{m}}\alpha_m^{r,\mu}\,,\\ &(\mathcal{P})_m^{r,\mu}=\sum_{s=1}^{3}\sqrt{m}\,N_m^{rs}\alpha_0^{s,\mu}\,.\end{aligned} \tag{5.23}$$

Then Eq. (5.21) can be written as

$$\begin{aligned}\bar{V}_{\lambda}(p_1,p_2,p_3)=&\left[\frac{4}{3\sqrt{3}}\right]^{p_1^2+p_2^2+p_3^2}\\&\times\langle e^{(1/2)aNa+\mathcal{P}a}e^{(1/2)a^{\dagger}\phi a^{\dagger}+\lambda a^{\dagger}}\rangle\\&\times\delta(p_1+p_2+p_3)\,,\end{aligned} \tag{5.24}$$

where we used $N_{00}^{rs}=\frac{1}{2}\ln(\frac{16}{27})\delta^{rs}$. This vacuum expectation value of exponentials of quadratic oscillator forms can be calculated in a standard way with the result

$$\begin{aligned}\bar{V}_{\lambda}(p_1,p_2,p_3)=&\left[\frac{4}{3\sqrt{3}}\right]^{p_1^2+p_2^2+p_3^2}\delta(p_1+p_2+p_3)\\&\times\det M^{1/2}e^{\lambda M\mathcal{P}+(1/2)\mathcal{P}\phi M\mathcal{P}+(1/2)\lambda MN\lambda}\,,\end{aligned} \tag{5.25}$$

where $M=(1-N\phi)^{-1}$.

Thus, the coupling of three tachyons is (viewing N and ϕ as 3×3 matrices in the indices r and s and making use of the explicit form of the Neumann coefficients N_{mn}^{rs}

(Ref. 15) and ϕ [Eq. (5.22)], it can be shown that $\mathcal{P}\phi M\mathcal{P} \propto p_1^2+p_2^2+p_3^2$)

$$V_{-1}(p_1,p_2,p_3)=\bar{V}_\lambda(p_1,p_2,p_3)|_{\lambda=0} \propto C^{p_1^2+p_2^2+p_3^2}\delta(p_1+p_2+p_3) \ . \tag{5.26}$$

The coupling of two tachyons and a graviton is [cf. Eq. (4.6)]

$$V_0=\tfrac{1}{4}\zeta_{\mu\nu}\left[\frac{\partial}{\partial\lambda_2^\mu}+e^{(\mu)}\cdot\frac{\partial}{\partial\lambda}\right]\left[\frac{\partial}{\partial\lambda_2^\nu}-e^{(\nu)}\cdot\frac{\partial}{\partial\lambda}\right]\bar{V}_\lambda\Bigg|_{\lambda=0} \ . \tag{5.27}$$

where $(e^{(\mu)})^\rho_{2k+1}=2\sqrt{2k+1}\delta^{\mu\rho}\psi_{k1}$ and $(e^{(\mu)})_{2k}=0$. After some algebra, we obtain

$$V_0(p_1,p_2,p_3)\propto C^{p_1^2+p_2^2+p_3^2+2}\zeta_{\mu\nu}(p_1-p_2)^\mu \times(p_1-p_2)^\nu\delta(p_1+p_2+p_3) \ . \tag{5.28}$$

Assembling all ingredients, we obtain the contributions to the S matrix from the first two poles (for the pole at $s=0$, we should also include the dilaton and the antisymmetric tensor contributions; it can be seen that they are of the same form as the graviton):

$$S_{-1}\equiv V_{-1}(p_1,p_2,p)\Delta_1(p)V_{-1}(p_3,p_4,p) \propto \frac{1}{s+2} \ , \tag{5.29}$$

$$S_0\equiv V_0(p_1,p_2,p)\Delta_0(p)V_0(p_3,p_4,p) \propto \frac{(2t+s+8)^2}{s} \ , \tag{5.30}$$

where in Eq. (5.30) we integrated over all directions of $\zeta_{\mu\nu}$.

Therefore, the residues of the poles of S at $s=0$ and -2 have the same functional dependence on the variable t as the corresponding poles in the Virasoro-Shapiro amplitude [cf. Eqs. (5.16) and (5.18)]. This provides an indication that our results agree with dual theory at the tree level.

At this point, some remarks on associativity anomalies are in order. Closed-string states have already been constructed in terms of open-string modes,[12] and shown to suffer from associativity anomalies. These anomalies are essentially due to double infinite sums that are not interchangeable. Thus, for two general states outside of the open-string Fock space, $|A\rangle$ and $|B\rangle$, say, it is not necessarily true that

$$({}_{123}\langle V_3|A\rangle_1)|B\rangle_2=(-)^{AB}({}_{123}\langle V_3|B\rangle_2)|A\rangle_1 \ , \tag{5.31}$$

where the parentheses indicate the order of summation. An example is furnished by the states[11]

$$|A\rangle=x^\mu(\pi/2)|I\rangle, \quad |B\rangle=\int_0^{\pi/2}\partial_\tau x^\mu(\sigma)d\sigma|I\rangle \ . \tag{5.32}$$

For such states, the cubic action[6] is not equivalent to Witten's action. Thus, such a state may obey the equation of motion derived from the cubic action level by level, yet it will not be annihilated by the BRST operator Q. Evidence that our closed-string states are free from associativity anomalies is therefore provided by the fact that, by construction, they are annihilated by Q. This fact is a direct consequence of the equality of open- and closed-string BRST operators [Eq. (3.17)].

We shall now discuss in more detail how our construction avoids associativity anomalies. It is convenient to replace $\langle V_3|$ by $\langle V_4|$, the four-point interaction vertex. This does not weaken the argument, since ${}_{123}\langle V_3| = {}_{1234}\langle V_4|I\rangle_4$. The relevant part of $\langle V_4|$ for two states $|A\rangle_1$ and $|B\rangle_2$ is

$${}_{12}\langle 0|e^{\alpha_m^1 N_{mn}\alpha_n^2} \ . \tag{5.33}$$

The Neumann coefficients N_{mn} are given in terms of the binomial coefficients $C_r^{-1/2}$ and can be found in Ref. 15. (In fact, $N_{2n+1,2m+1}=\phi_{nm}$ [Eq. (2.20)].) They should not be confused with those of the three-point vertex $\langle V_3|$. To compute ${}_{1234}\langle V_4|A\rangle_1|B\rangle_2$, one in general has to evaluate double (infinite) sums of the form

$$\sum_{m=0}^\infty\sum_{n=0}^\infty N_{mn}A_mB_n \ , \tag{5.34}$$

which are not necessarily interchangeable. The question then arises: under what circumstances do we encounter such a problem? It is clear that, for problems to arise, the two sums have to involve two different (infinite) sets of indices. This is the case with the example considered above [Eq. (5.32)]: A_m (B_n) contains only even (odd) indices. One can now see how our states avoid this problem. Their coefficients contain infinite sets of only odd indices, whereas only isolated even indices appear [cf. Eqs. (2.6) and (2.7)]. This ensures the interchangeability of double infinite sums like in Eq. (5.34) and therefore the absence of associativity anomalies. We have checked this explicitly for low-lying states. The algebra involves a straightforward manipulation of binomial coefficients [cf. Eq. (2.22)].

Even though associativity anomalies are absent, it may still be the case that our theory does not cover moduli space completely. In this respect, there is a compelling argument by Giddings and Martinec,[14] showing that a naive extension of Witten's open-string field theory cannot lead to a single cover of moduli space. Unfortunately, we cannot provide conclusive evidence one way or another, because calculating amplitudes in the oscillator formalism is a formidable task at present. Thus in the present formulation of the theory it seems impossible to even calculate the four-point tree-level diagram. A better geometrical understanding of the vertex is needed before explicit expressions for any amplitudes can be found. Nevertheless, it appears that one has to modify this theory in order to obtain agreement with dual theory, even at the tree level.

VI. CONCLUSIONS

Following a suggestion by Strominger,[13] we wrote the closed-string oscillators as linear combinations of open-string modes and vice versa. We then constructed a Fock space that accommodated both open- and closed-string states. The resulting string theory could be described solely in terms of open- or closed-string modes and which picture one chooses is a matter of convenience.

We also built the enlarged Fock space of the ghost fields and discussed the BRST operator Q that took on its standard forms in the two pictures. Thus, in effect $Q_{\text{open}} = Q_{\text{closed}}$, therefore, the condition for physical states was the same for open- and closed-string states. This disagrees with recent results on the construction of closed-string states in open-string field theory.[8,12] Therefore it is not clear whether there exists a connection between our formulation and the results of Refs. 8 and 12.

We extended our approach to Witten's string field theory. We were able to write down a closed-string field theory simply by expressing all oscillators in Witten's action in terms of closed-string modes. We showed that the closed-string states thus constructed do not suffer from associativity anomalies. Thus we obtained an interaction vertex that was manifestly BRST invariant. We made contact with dual theory by calculating the residues of poles in the four-tachyon tree amplitude.

Unfortunately, it is very hard to calculate scattering amplitudes in the oscillator formalism. Therefore, in order to proceed any further, a geometrical understanding of the notion of integration and the interaction vertex is imperative. If one is allowed to speculate, it seems that the vertex should look like a Witten vertex where three strings join together at their midpoints. However, because we are dealing with closed strings, the end points must all be identified. The interaction, thus, becomes bilocal and one has to check whether or not the resulting theory is local.[5,9] Because of the ghost-number assignments discussed in Sec. V, it is hard to construct a surface, similar to the hexagon for open strings,[5] whose integrated curvature would lead to these assignments.[9] The resolution might be the insertion of an operator that will absorb any excess ghost number in the vertex, in analogy with open superstrings.[16] At any rate, due to a general argument by Giddings and Martinec,[14] the closed-string field theory we discussed probably does not lead to a single cover of moduli space, even at the tree level. The main reason lies in the fact that the interaction we have considered is essentially planar in nature. One has to modify this theory in a way that will account for the nonplanar character of the interaction of closed strings. The connection we established between open and closed strings should provide the clue to the appropriate modification of the theory. This issue is currently under investigation.

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