Chiral symmetry at finite temperatures and densities in the Coulomb gauge

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The gap equation at finite temperatures and densities obtained from the Schwinger-Dyson equation in the ladder approximation to the QCD Hamiltonian in the Coulomb gauge is solved numerically for several potential models. Critical temperatures for chiral-symmetry restoration and critical lines in the T-μ plane are obtained showing the conditions for symmetry restoration. The $\langle \bar{\psi}\psi \rangle$ order parameter is evaluated for various potentials and the agreement with experiment is found to be within 15% for the case of a confining plus an attractive Coulomb potential when the string tension $\sqrt{\sigma}$ is taken to be 350 MeV.

I. INTRODUCTION

Chiral-symmetry breaking in QCD and QCD-like theories has been actively studied recently both on the lattice[1] and in field theory.[2] Chiral-symmetry restoration at finite temperatures and quark densities, on the other hand, is only beginning to be explored. Langevin simulation techniques are being used to investigate this phenomenon on the lattice.[3] These studies indicate a first-order phase transition at a temperature $T_c \simeq 180$ MeV. Recently, Kocic[4] has studied, in field theory, symmetry restoration at finite densities for the confining potential. He solves the Schwinger-Dyson equation in the Hartree-Fock approximation for the confining potential. Use of the Coulomb gauge facilitates studying this problem using nonrelativistic potentials for the $q\bar{q}$ kernel.[2] A strictly confining potential exhibits chiral-symmetry breaking at any finite temperature and zero density.[5] Two relevant physical quantities evaluated numerically in Ref. 4 are the order parameter $\langle \bar{\psi}\psi \rangle$ and the pion-decay constant f_π. It is found that the calculated values are consistently smaller than the experimental ones. It seems appropriate to investigate more thoroughly chiral symmetry at finite temperatures and densities for QCD-like theories with an additional short-range attractive potential to describe quark interactions at high densities. Moreover, the study of chiral and deconfinement phase transitions is important for the following reasons, among others: (i) These phenomena are relevant in studying states of matter such as the quark-gluon plasma in the very early Universe; (ii) temperatures of the order of 200 MeV, just in the range where these transitions are estimated to occur, may be attainable for short times in heavy-ion collisions.[1] Both these cases probably involve high quark densities.

In this article we undertake a systematic study of chiral-symmetry restoration in the Coulomb gauge and within the Hartree-Fock approximation, at finite temperature and density. The finite-temperature formalism[6] appropriate to Coulomb-gauge QCD is discussed in Ref. 4, to which the reader is referred for details. We briefly discuss the reasons for choosing to study this problem in the Coulomb gauge. In this gauge, which has been investigated by Finger and Mandula[7] and by Adler and Davis[2] and others,[4,5] the gap equation for the generated quark mass allows use of $q\bar{q}$ potentials similar to those used in potential theory treatments of heavy-quark bound states. Since these potentials are static (nonrelativistic), which amounts to ignoring the effect of transverse gluons on the $q\bar{q}$ Bethe-Salpeter (BS) kernel, the gap equation at finite temperatures and densities can still be reduced to a one-dimensional nonlinear integral equation, which can be solved numerically fairly readily for various potentials. This has the obvious advantage over calculations in a covariant gauge such as the Landau gauge. The Landau-gauge study of chiral-symmetry breaking at zero temperature has been extensive,[8] using the effective-potential formalism for composite operators.[9] The finite-temperature version in the Landau gauge breaks explicit O(4) invariance and the gap equation is complicated to solve even numerically.

As discussed by Adler and Davis,[2] the $q\bar{q}$ BS kernel can be approximated by a static potential in single-gluon exchange and consistent with the Ward identities by

$$V(\mathbf{q})=\frac{\alpha}{|\mathbf{q}|^2} . \tag{1}$$

V depends only on the three-momentum transfer. In higher loops, renormalization-group improvements make α a running function of $|\mathbf{q}|$ at high momentum, thus exhibiting features of asymptotic freedom. Furthermore it is shown[10] that multiple longitudinal gluon exchange produces an additional confining-potential contribution to make V in coordinate space of the form

$$V(r)=\sigma r-\frac{4}{3}\frac{\alpha}{r} . \tag{2}$$

Here σ is the string tension.

Motivated by these possibilities for the $q\bar{q}$ kernel, we investigate in this article the gap equation at finite temperature and density for the following cases.

(a) δ-shell potential: $V(r)=-\lambda\delta(r-a)$.

(b) Purely Coulomb potential; i.e., α is a constant in (1).

(c) $\alpha(q)=+\lambda/\ln(q^2/\Lambda^2)$ in (1). This incorporates renormalization-group corrections to the running coupling constant. $V(q)$ with $\alpha(q)$ as above represents a potential exhibiting asymptotic freedom.

(d) Coulomb plus confining potential as in (2) above.

(e) Potentials inspired by string theories.[11]

(f) Finally we consider the case where the string tension σ is taken as a function of temperature. While a microscopic treatment of $\sigma=\sigma(T)$ does not seem to exist, it is argued in Ref. 12 that the string tension softens with T and may be taken in the form

$$\sigma(T)=\sigma_0\left[1-\left[\frac{T}{T_D}\right]^2\right]^{1/2} \quad \text{for } T<T_D\ , \tag{3}$$

where T_D is of the order of the deconfinement temperature. It is possible that the string tension remains finite near deconfinement temperature[13] and in this case T_D may be taken to be slightly larger than the actual deconfinement temperature. Thus we take the potential in this case to be of the form

$$V(r)=\begin{cases}\sigma(T)r-\dfrac{4}{3}\dfrac{\alpha}{r}, & T<T_D\ ,\\ -\dfrac{4}{3}\dfrac{\alpha}{r}\exp\left\{-\alpha_0^{1/2}\left[\left[\dfrac{T}{T_D}\right]^2-1\right]^{1/4}r\right\}, & T>T_D\ .\end{cases} \tag{4}$$

Potential for $T>T_D$ is taken to be a screened Coulomb potential as the Coulomb potential is expected to be screened in the quark-gluon plasma.

In each case above, the finite T and μ gap equation and the equation for the chemical potential are solved numerically and critical lines in the T-μ plane and the gap as a function of T for different densities are exhibited. We also evaluate the order parameter $\langle\bar\psi\psi\rangle$ for the above cases by taking the string tension to be $\sqrt{\sigma}=350$ MeV from charmonium spectroscopy.[14] The values for $\langle\bar\psi\psi\rangle$ are compared with experimentally accepted value $\langle\bar\psi\psi\rangle_{\rm expt}=(-250\ \text{MeV})^3$.

II. CALCULATIONS

We discuss briefly the finite-temperature formalism employed in discussing chiral-symmetry restoration following Kocic.[4] We start with the renormalized Coulomb-gauge gap equation

$$\Sigma(p)=C_F\int_q \tilde V(p-q)\gamma^0 S(q)\gamma^0\ , \tag{5}$$

where the quark propagator is given by

$$S(q)=\frac{1}{\not q-\Sigma(q)}\ . \tag{6}$$

In Eq. (5), C_F is the quadratic Casimir invariant for color SU(N) in the fundamental representation. $\Sigma(p)$ is the quark self-energy. $\int_q$ includes sum over discrete frequencies

$$\omega_n=(2n+1)\frac{\pi}{\beta}$$

with $q_0=(i\omega_n+\mu)$, where μ is the chemical potential for quarks. $\tilde V(k)$ is the Fourier transform of the $q\bar q$ potential which is taken to be a function of $|\mathbf{k}|$ only. Actually the Fourier transform of the coordinate-space potential $V(r)$ is related to $\tilde V(k)$ by (see Adler and Davis[2])

$$\left[-\frac{3}{16\pi}\right]V(\mathbf{k})=\tilde V(\mathbf{k})\ .$$

Thus the confining potential $V(r)=\sigma r$ $(\sigma>0)$ has a Fourier transform

$$V(k)=-\frac{8\pi\sigma}{(|\mathbf{k}|^2)^2}$$

and $\tilde V(k)$ will be taken as $3\sigma/2(\mathbf{k}^2)^2$.

The general form of $\Sigma(\mathbf{p})$ is taken to be

$$\Sigma(\mathbf{p})=a_p+b_p\gamma\cdot\mathbf{p}+C_p\gamma^0\ . \tag{7}$$

The only frequency dependence is in $S(q)$ in Eq. (5). Performing this summation in a standard way and making use of (6) and (7) in (5), one obtains the following system of coupled equations for a_p, b_p, and c_p:

$$a_p=\frac{C_F}{2}\int_{\mathbf{q}}\tilde V(|\mathbf{p}-\mathbf{q}|)F_q\frac{a_q}{\omega_q}\frac{d^3q}{(2\pi)^3}\ , \tag{8a}$$

$$(1+b_p)p=p+\frac{C_F}{2}\int\tilde V(|\mathbf{p}-\mathbf{q}|)F_q\frac{(1+b_q)q}{\omega_q}\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\frac{d^3q}{(2\pi)^3}\ , \tag{8b}$$

$$c_p=-\frac{C_F}{2}\int\tilde V(|\mathbf{p}-\mathbf{q}|)(n_q-\bar n_q)\frac{d^3q}{(2\pi)^3}\ . \tag{8c}$$

In Eqs. (8a)–(8c) we find

$$F_q=1-n_q-\bar n_q\ , \tag{9}$$

where n_q and $\bar n_q$ are occupation numbers for quarks and antiquarks, respectively, defined by

$$n_q=\frac{1}{\exp[\beta(\omega_q-\nu_q)+1]}\ , \tag{10a}$$

$$\bar n_q=\frac{1}{\exp[\beta(\omega_q+\nu_q)+1]}\ , \tag{10b}$$

where

$$\omega_q=[a_q^2+(1+b_q)^2q^2]^{1/2}$$

is the quasiparticle energy and $\nu_q=\mu-c_q$ is the shifted chemical potential.

It is convenient to introduce the quantities

$$a_p=\omega_p\sin\phi_p\ , \tag{11a}$$

$$(1+b_p)p=\omega_p\cos\phi_p\ , \tag{11b}$$

together with the generalized gap functions

$$\Delta(p)=\frac{C_F}{2}\int\frac{d^3q}{(2\pi)^3}\tilde V(|\mathbf{p}-\mathbf{q}|)F_q\sin\phi_q\ , \tag{12a}$$

$$\Gamma(p)=\frac{C_F}{2}\int\frac{d^3q}{(2\pi)^3}\tilde V(|\mathbf{p}-\mathbf{q}|)F_q\cos\phi_q\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\ . \tag{12b}$$

The quasiparticle energy can then be expressed as

$$\omega_p=\{\Delta^2(p)+[\Gamma^2(p)+p]^2\}^{1/2} . \quad (13)$$

Using (11a), (11b), (12a), (12b) and (13) in (8a)–(8c) and performing the angular integration, we arrive at the following system of nonlinear integral equations:

$$\Delta(p)=\frac{C_F}{8\pi^2}\int_0^\infty q^2dq\, K(p,q)F_q\frac{\Delta(q)}{\{\Delta^2(q)+[q+\Gamma(q)]^2\}^{1/2}} , \quad (14a)$$

$$\Gamma(p)=\frac{C_F}{8\pi^2}\int_0^\infty q^2dq\, \tilde{K}(p,q)\frac{q+\Gamma(q)}{\{\Delta(q)^2+[q+\Gamma(q)]^2\}^{1/2}} , \quad (14b)$$

$$c_p=-\frac{C_F}{8\pi^2}\int_0^\infty q^2dq\, K(p,q)(n_q-\bar{n}_q) . \quad (14c)$$

The kernels in the integral equations above are defined by

$$K(p,q)=\int \sin\theta\, d\theta\, V(\,|\mathbf{p}-\mathbf{q}|\,) , \quad (15a)$$

$$\tilde{K}(p,q)=\int \sin\theta\, d\theta \cos\theta\, V(\,|\mathbf{p}-\mathbf{q}|\,) . \quad (15b)$$

As remarked in the Introduction, we can study the generalized gap equation for various phenomenological $q\bar{q}$ potentials. For all but the simplest potential studied here, analytic solutions seem difficult to obtain. We have therefore resorted to a numerical solution using Newton's iterative method with 200 point mesh for p and q. The kernels $K(p,q)$ and $\tilde{K}(p,q)$ are expressible in closed form in all the cases we have studied below.

III. NUMERICAL RESULTS

The system of coupled nonlinear integral equations (14a)–(14c) must be solved simultaneously. We are interested in determining the gap $\Delta(p)$ at $p=0$ as a function of T and μ (or n) as well as the critical line in the (T,μ) plane. We have, however, used the following observations to simplify our calculations. It can be seen from (8b) that $\Gamma(p)\to p$ as $p\to 0$ and as p^{-n} $(n>1)$ as $p\to\infty$ for the potentials considered here. The latter statement is modified in the case of δ-shell potential. $\Gamma(p)$ occurs in (10a) in F_q and in the square-root factor. It can be seen qualitatively that the effect of $\Gamma(p)$ in determining $\Delta(p)$ is negligible. Hence, we have set $\Gamma(p)=0$ in (14a) and discarded Eq. (14b). Furthermore, c_p determined by Eq. (14c) enters (14a) in the Fermi factors. c_p is strictly zero only for zero density, i.e., the number of quarks equals the number of antiquarks. But for nonzero but small densities, we may ignore the (p,μ,T) dependence of c_p. The validity of these approximations was tested numerically and it was found that the behavior of $\Delta(p=0)$ as a function of T and μ did not change significantly from the results reported below when all three equations were solved simultaneously.

A. δ-shell potential

The relevant quantities are

$$V(r)=-\lambda\delta(r-a) , \quad (16a)$$

$$K(p,q)=\lambda j_0(pa)j_0(qa) . \quad (16b)$$

$j_n(pa)$ are the spherical Bessel functions. Since the kernel is separable, the problem can be reduced to quadratures. The zero-temperature term is ultraviolet divergent. A cutoff momentum can, however, be traded for the critical coupling λ_c defined by

$$1=\frac{\lambda_c C_F}{8\pi^2 a^2}\int_0^\Lambda q\, dq\, j_0^2(qa) . \quad (17)$$

In Fig. 1 the gap $\Delta(p=0)$ is plotted as a function of T for various values of μ and the critical line is shown in Fig. 2. The phase separation line is similar to that recently reported in a lattice calculation.[15] Since the gap goes to zero continuously at T_c, the phase transition is second order. In fact, in all the cases we have studied, chiral-symmetry restoration, if it occurs, is second order.

B. Coulomb potential

We find

$$\tilde{V}(k)=\frac{\lambda}{k^2} ,$$
$$K(p,q)=\frac{\lambda}{pq}\ln\left|\frac{p+q}{p-q}\right| . \quad (18)$$

In Figs. 2 and 3, the critical line and the gap as a function of T are, respectively, plotted. It is interesting to observe that the gap for small μ is a linear function of T for large temperatures. Thus, the critical exponent for the gap is different from the previous case. Although the Coulomb potential model is not adequate for QCD, the features observed here will be useful in our analysis of the models with Coulomb plus confining potentials. In the Coulomb case, it is known that there is a critical coupling λ_c (Ref.

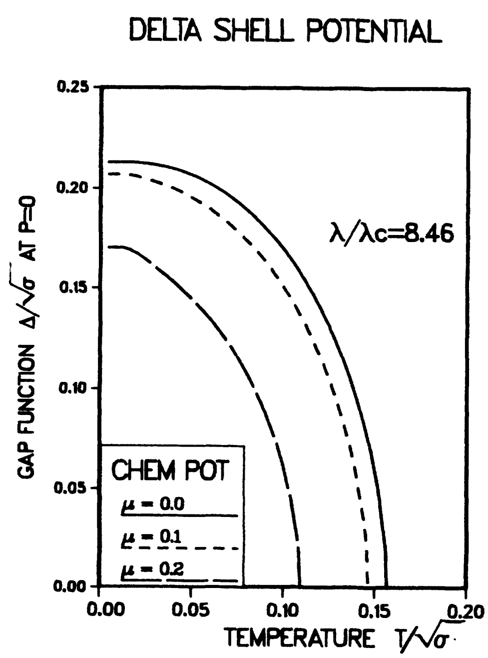

FIG. 1. Gap function at $p=0$ for δ-shell potential as a function of temperature for several chemical potentials.

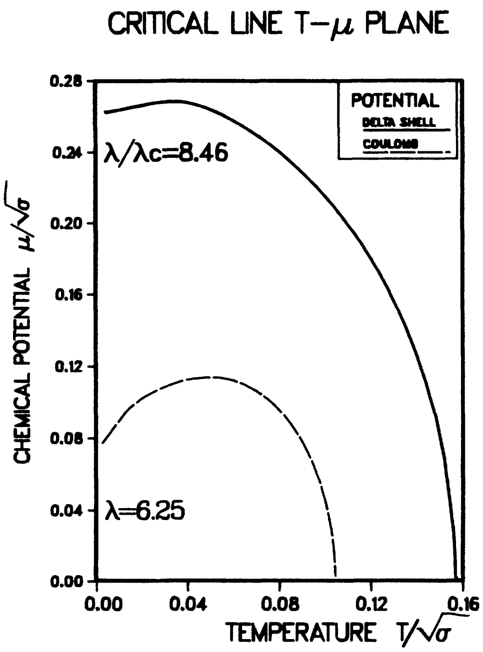

FIG. 2. Critical lines in the T-μ plane for Coulomb and δ-shell cases.

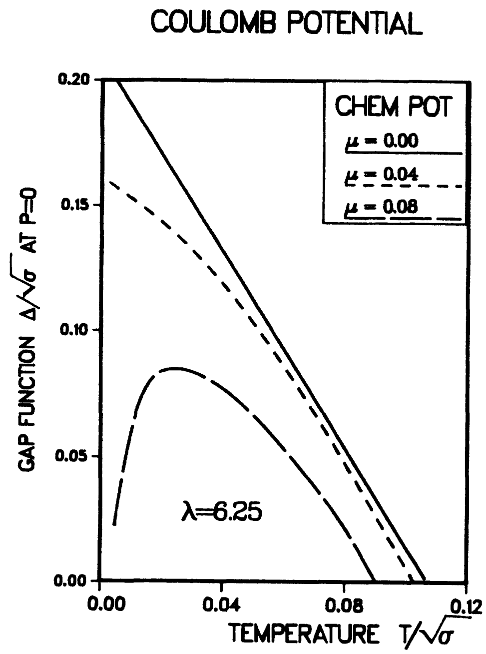

FIG. 3. Gap function at $p=0$ as a function of temperature for Coulomb potential for several chemical potentials.

16) above which chiral symmetry is broken. In our analysis therefore the coupling λ is chosen well above λ_c.

C. Asymtotically free potential

The asymptotically free potential is taken in the form

$$\tilde{V}(k)=\frac{\lambda}{k^2\ln(k^2/\Lambda^2)}\,, \tag{19}$$

where Λ is the QCD parameter

$$K(p,q)=\frac{\lambda}{pq}\ln\left[\frac{\ln(p+q)}{\ln|p-q|}\right]. \tag{20}$$

All momenta have been scaled by Λ and the solutions are plotted in Figs. 4 and 5.

D. Confining and confining plus Coulomb potentials

We have investigated the standard confining potential in which $V(r)=+\sigma r$ as well as the string-theory-inspired potential in which $V(r)=\sigma(r^2-R^2)^{1/2}$ $(r\geq R)$ and $V(r)=0$ $(r\leq R)$. In the latter case the potential for large r goes like

$$V(r)\simeq\sigma r-\frac{\sigma R^2}{2r}\,,$$

which is just the Coulomb plus confining potential. Thus the string-inspired potential should interpolate between the standard confining potential and confining plus Coulomb potential. Numerical results are, however, exhibited only for the confining and Coulomb plus confining cases. It is well known that chiral symmetry is broken for the confining case for any value of the string tension and if the string tension is taken to be independent of temperature, chiral symmetry, is restored only asymptotically as $T\to\infty$. Kocic[4] has studied for finite densities the confining potential model and finds that symmetry is restored for $T=0$ K at a critical density. To

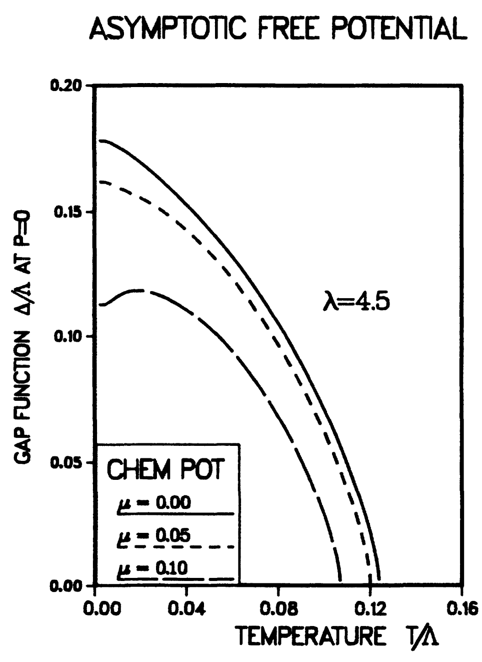

FIG. 4. Gap function at $p=0$ as a function of temperature for asymptotic free potential for several values of μ.

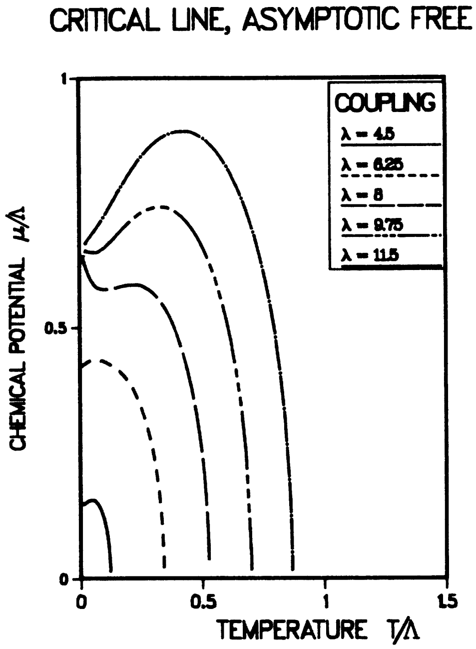

FIG. 5. Critical lines in the T-μ plane for asymptotic free potential for different values of coupling λ.

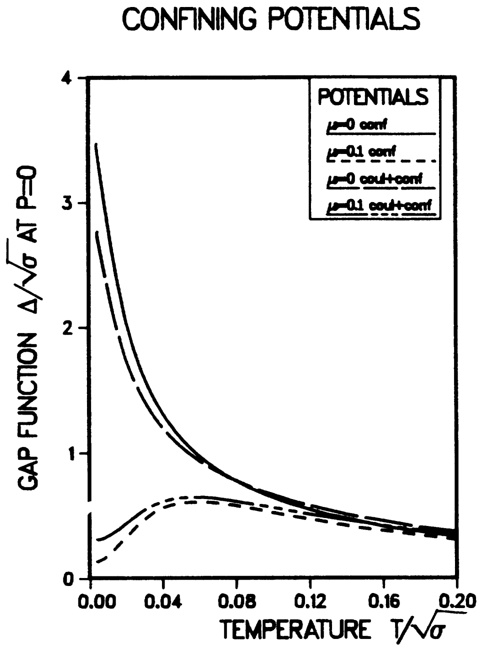

FIG. 6. Gap function at $p=0$ as a function of temperature for confining and confining plus Coulomb potentials are shown for two different values of μ.

understand this from a physical point of view[4] let us recall that there are two length scales in this problem. One of these measures the average distance between $q\bar{q}$ condensate pairs and is proportional to $\rho^{-1/3}$ where ρ is the pair density. The other length scale is determined by the average interparticle distance and is proportional to $n^{-1/3}$. As the density of quarks increases, ρ decreases and hence $1/\rho^{1/3}$ increases. When these two length scales become comparable, one expects symmetry restoration to occur.

In Figs. 6 and 7 the relevant quantities are plotted for these cases. These plots confirm earlier speculation that as long as we have a confining potential with temperature-independent string tension, chiral symmetry remains broken at finite temperature and densities below the critical density. This fact certainly indicates the need for a temperature-dependent string tension that vanishes at deconfinement temperature which allows symmetry restoration at finite temperature with densities below the critical density. Thus, in order to incorporate the temperature dependence of string tension, we used the potential model as outlined in the Introduction where the string tension varies as $\sigma_0[1-(T/T_D)^2]^{1/2}$ for $T \le T_D$. In this case it is easy to see what the properties of the gap equation would be for $T \le T_D$; the gap equation always has a nontrivial solution which goes to zero at $T=T_D$ if the coefficient λ of the Coulomb term is less than the value at which the Coulomb gap equation at T_D ceases to have a solution. Taking λ for strong interactions at about 200 MeV to be of the order of unity, it is clear that chiral symmetry will be restored at $T=T_D$. If this is taken as the deconfinement temperature, then within the approximation scheme employed here, chiral symmetry will be restored at the deconfinement temperature. It is likely that[13] string tension remains finite but small at the deconfinement temperature and chiral-symmetry restoration will then occur at a temperature larger than the deconfinement temperature. These conclusions seem to be in agreement with the commonly accepted notions about these two phase transitions.

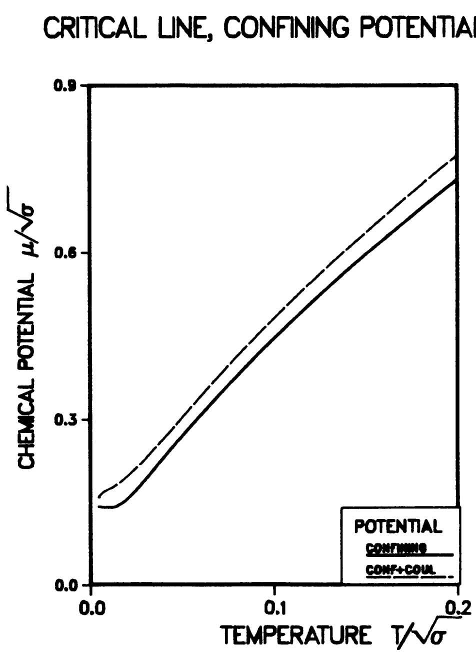

FIG. 7. Critical lines in the T-μ plane for confining and confining plus Coulomb potentials.

We have also calculated the order parameter $\langle \bar{\psi}\psi \rangle$ for various potentials studied above at zero temperature and density. It is defined by

$$\langle \bar{\psi}\psi \rangle = -\frac{N_c}{\pi^2}\int_0^\infty q^2 dq \sin\phi_q \ . \tag{21}$$

Equivalently, asymptotically for large q it can be defined as

$$\sin\phi_q \underset{q\to\infty}{\sim} \frac{C_F}{4N_c} \, |\langle \bar{\psi}\psi \rangle| \, \frac{\tilde{V}(q)}{q} \ . \tag{22}$$

From its definition,

$$\sin\phi_q \simeq \frac{\Delta}{(\Delta^2+q^2)^{1/2}} \ .$$

We have extracted the value for $\langle \bar{\psi}\psi \rangle$ for $N_c = 3$ (color) from an asymptotic fitting of calculated value of $\sin\phi_q$ with the right-hand side of Eq. (22). We find that (assuming a value $\sqrt{\sigma} = 350$ MeV)

$$\langle \bar{\psi}\psi \rangle = \begin{cases} (-100\ \text{MeV})^3 & \text{confining potential ,} \\ (-330\ \text{MeV})^3 & \text{pure Coulomb potential ,} \\ (-286\ \text{MeV})^3 & \text{Coulomb plus confining .} \end{cases} \tag{23}$$

The estimate for confining potential agrees with that of Kocic while the Coulomb value agrees with the result of Govaerts, Mandula, and Weyers.[17] As expected, the addition of an attractive Coulomb piece to the confining potential increases $\langle \bar{\psi}\psi \rangle$ to an acceptable value $(-286\ \text{MeV})^3$. This agrees to within 15% of the experimentally accepted value $(-250\ \text{MeV})^3$.

In summary, the gap equation at finite T and μ in the Coulomb gauge has been analyzed with a view to understanding chiral-symmetry restoration. It is gratifying to find that in the case of the potential model with temperature-dependent string tension, symmetry restoration occurs at a temperature T_D which is very close to the deconfinement temperature. The chiral-symmetry restoration is second order, which, however, does not seem to be in agreement with recent lattice calculations. The estimate for the order parameter was in good agreement with the experimental value for the confining plus Coulomb case. It seems then that the combination of Coulomb (or asymptotic free) plus confining potential with temperature-dependent string tension $\sigma(T)$ should be the most realistic potential of all potentials studied here.

[1] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, England, 1983).

[2] S. Adler and A. Davies, Nucl. Phys. **B244**, 469 (1984).

[3] M. Fukugita, S. Ohta, and A. Ukawa, Phys. Rev. Lett. **60**, 178 (1988).

[4] A. Kocic, Phys. Rev. D **33**, 1785 (1986).

[5] A. Davis and A. Matheson, Nucl. Phys. **B246**, 203 (1984).

[6] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).

[7] J. Finger and J. Mandula, Nucl. Phys. **B199**, 168 (1982).

[8] A. Barducci, R. Casalbuoni, S. DeCurtis, D. Dominici, and R. Gatto, Phys. Rev. D **38**, 238 (1988).

[9] J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[10] A. Le Yaouanc, L. Oliver, O. Pène, and J. Raynal, Phys. Rev. D **29**, 1233 (1984).

[11] D. Gross and R. Pisarski, Rev. Mod. Phys. **53**, 43 (1981).

[12] R. Pisarski and O. Alvarez, Phys. Rev. D **26**, 3735 (1982).

[13] R. Pisarski, Phys. Rev. D **29**, 1222 (1984).

[14] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D **21**, 203 (1980).

[15] A. Moreo, Phys. Rev. D **35**, 1442 (1987).

[16] A. Amer, A. Le Yaouanc, L. Oliver, O. Pène, and J. Raynal, Phys. Rev. D **28**, 1530 (1983).

[17] J. Govaerts, J. Mandula, and J. Weyers, Nucl. Phys. **B237**, 59 (1984).