Finite-temperature effects on the Gaussian effective potential

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We derive the finite-temperature Gaussian effective potential (FTGEP) from first principles, resolving some confusions arising in earlier treatments. The advantages of the FTGEP approach over the conventional loop expansion, even for high temperatures, are illustrated in a quantum-mechanical example. Results are presented for ϕ^6 theory in two and three dimensions and for the "autonomous" version of $\lambda\phi^4$ theory in four dimensions.

I. INTRODUCTION

Effective potentials are a natural tool for the study of finite-temperature effects in field theories.[1,2] The Gaussian effective potential[3–7] (GEP) offers several important advantages over other approaches, and, for scalar theories, it is known to contain the one-loop and leading-$1/N$ results as limiting cases.[5,8] Several authors have investigated the generalization of the GEP to finite temperatures,[4,9–13] but some confusion has arisen, mainly due to neglect of the crucial distinction between *energy* and *free energy,* as pointed out recently by Bardeen and Moshe.[11] One aim of this paper is to further clarify matters by deriving the finite-temperature Gaussian effective potential (FTGEP) from first principles.

Our other main aim is to study the finite-temperature behavior of "autonomous" ϕ^4 theory.[14,15,8] Recent GEP work indicates that there may be two distinct $(\lambda\phi^4)_{3+1}$ theories.[15,16] The existing FTGEP results relate to "precarious" ϕ^4 theory which has a *negative,* infinitesimal λ_B. While this theory is clearly related to the familiar perturbative and $1/N$-expansion results, it has some strange properties, particularly at finite temperature.[4,9,11] (For a thorough discussion of "precarious" ϕ^4 theory, see Ref. 6.) The "autonomous" ϕ^4 theory has so far shown up only in GEP studies:[14,15] it turns out that it is inaccessible to perturbative, loop-expansion, or $1/N$-expansion approaches.[8] It can exhibit spontaneous symmetry breaking (SSB), but in its unbroken-symmetry phase the particles are always massless. This feature could perhaps explain how this theory has evaded the "triviality" studies.[17–19] Obviously, though, more work is needed to conclusively establish whether autonomous ϕ^4 theory is real or a mirage. (In particular, there is a need for calculations which go beyond the Gaussian approximation, perhaps along the lines suggested in Ref. 5.) In this paper we show that at least the finite-temperature behavior of autonomous ϕ^4 theory is perfectly sensible, which is reassuring.

The plan of the paper is as follows. In Sec. II we summarize the basic GEP formalism and the necessary thermodynamics. The FTGEP is calculated in Sec. III, where we also comment on the recent literature.[9–13] Section IV discusses an example illustrating the advantages of the FTGEP over the conventional loop-expansion approach. Numerical results for ϕ^6 theory in two and three dimensions are briefly summarized in Sec. V. The autonomous ϕ^4 theory is discussed in Sec. VI, with conclusions drawn in Sec. VII.

II. FORMALISM AND THERMODYNAMIC PRELIMINARIES

For definiteness, we shall consider the $\lambda\phi^4$ theory in $\nu+1$ dimensions: generalizations to other scalar theories are straightforward. The theory's Hamiltonian is

$$H=\int d^\nu x\,\mathcal{H}\ ,\quad \mathcal{H}=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B^2\phi^2+\lambda_B\phi^4\ . \tag{2.1}$$

At zero temperature the GEP can be described as a variational approximation to the effective potential constructed with Gaussian trial wave functions. It can be conveniently calculated by writing ϕ as

$$\phi=\phi_0+\hat{\phi}\ , \tag{2.2}$$

where ϕ_0 is a constant classical background field, and

$$\hat{\phi}=\int (dk)_\Omega[a_\Omega(\mathbf{k})e^{-ik\cdot x}+a_\Omega^\dagger(\mathbf{k})e^{ik\cdot x}]\ , \tag{2.3}$$

with

$$(dk)_\Omega=\frac{d^\nu k}{(2\pi)^\nu 2\omega_{\underline{k}}}\ ,\quad \omega_{\underline{k}}=(\mathbf{k}^2+\Omega^2)^{1/2}\ ,$$
$$[a_\Omega(\mathbf{k}),a_\Omega^\dagger(\mathbf{k}')]=2\omega_{\underline{k}}(2\pi)^\nu\delta(\mathbf{k}-\mathbf{k}')\ . \tag{2.4}$$

In this formalism the Gaussian trial vacuum is represented by $|0\rangle_\Omega$, the free-field vacuum annihilated by $a_\Omega(\mathbf{k})$, and one can easily evaluate

$$V_G(\phi_0,\Omega)\equiv{}_\Omega\langle 0|\mathcal{H}|0\rangle_\Omega \tag{2.5}$$

to obtain

$$V_G(\phi_0,\Omega)=I_1+\tfrac{1}{2}(m_B^2-\Omega^2)I_0+\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4+6\lambda_B I_0\phi_0^2+3\lambda_B I_0^2\ , \tag{2.6}$$

where

$$I_N\equiv\int (dk)_\Omega[\omega_{\underline{k}}^2]^N\ . \tag{2.7}$$

The GEP itself, $\bar{V}_G(\phi_0)$, is obtained by minimizing this expression with respect to Ω. Except in cases where $\Omega=0$ gives the global minimum, $\bar{\Omega}$, the optimum value of Ω, is a solution of the equation

$$\bar{\Omega}^2=m_B^2+12\lambda_B[I_0(\bar{\Omega})+\phi_0^2] \ . \tag{2.8}$$

Making use of this equation the GEP can be simplified to

$$\bar{V}_G(\phi_0)\equiv I_1-3\lambda_B I_0^2+\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4 \ . \tag{2.9}$$

The above procedure can be described alternatively[5] as a calculation of the vacuum energy density to first order in the perturbation theory generated by

$$\begin{aligned}
&\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\rm int} \ ,\\
&\mathcal{H}_0=\tfrac{1}{2}[\dot{\phi}^2+(\nabla\phi)^2+\Omega^2\hat{\phi}^2] \ ,\\
&\mathcal{H}_{\rm int}=-\tfrac{1}{2}\Omega^2\hat{\phi}^2+\tfrac{1}{2}m_B^2(\phi_0+\hat{\phi})^2+\lambda_B(\phi_0+\hat{\phi})^4 \ .
\end{aligned} \tag{2.10}$$

Here the parameters ϕ_0 and Ω arise from the split up of the Hamiltonian and the I_N integrals arise as

$$I_1={}_\Omega\langle 0|\mathcal{H}_0|0\rangle_\Omega \ , \quad I_0={}_\Omega\langle 0|\hat{\phi}^2|0\rangle_\Omega \ . \tag{2.11}$$

This quasiperturbative point of view supplies a framework for systematically calculating higher-order corrections to the Gaussian approximation.[5] It also provides a natural basis for discussing the correct finite-temperature generalization of the GEP. We stress that although we borrow some methodology from perturbation theory, the approach is genuinely nonperturbative in the sense that the results are not series expansions in λ, and are not limited to weak coupling.[5]

We now turn to finite-temperature field theory, by which is meant the following situation. A fixed spatial volume V (ultimately taken to be arbitrarily large) is surrounded by a heat bath at a fixed temperature T. A quantum field ϕ lives in the region V and obeys some suitable (e.g., periodic) boundary conditions. The set of modes of the ϕ field constitute the "system" in the thermodynamic sense. Thermal equilibrium is achieved when the system has minimized its (Helmholz) *free energy*

$$F=E-TS \ , \tag{2.12}$$

where E is the (internal) energy, and S is the entropy, of the system. Thus, the physically meaningful effective potential for the system corresponds to the function of ϕ_0 obtained by minimizing the *free energy F*, subject to the constraint that $\langle\phi\rangle=\phi_0$. To use the energy E instead, as in Ref. 10, would correspond to the case where the entropy, rather than the temperature, is somehow held fixed.[20]

The proper finite-temperature generalization of the GEP is therefore obtained as follows. Calculate F to first order in the perturbation theory specified by (2.10); divide out by an overall volume factor V; and minimize with respect to Ω.

Standard thermodynamics[20] gives both F and E in terms of the partition function Z:

$$Z\equiv{\rm Tr}(e^{-\beta H})\equiv\sum_\alpha\langle\alpha|e^{-\beta H}|\alpha\rangle \ , \tag{2.13}$$

$$F=-\frac{1}{\beta}\ln Z \ , \tag{2.14}$$

$$E=-\frac{d}{d\beta}\ln Z=\frac{{\rm Tr}(e^{-\beta H}H)}{{\rm Tr}(e^{-\beta H})} \ , \tag{2.15}$$

where $\beta=1/(kT)$ ($k=$Boltzmann's constant) and $|\alpha\rangle$ denotes an eigenstate of H. We begin by calculating Z. In principle, the trace involves a summation over the unknown eigenstates of the full H. However, in perturbation theory, the correction to each state can be taken to be orthogonal to the unperturbed eigenstate[21] [i.e., $|\alpha\rangle=\mathcal{N}(|\alpha_0\rangle+|\alpha'\rangle)$, with $\langle\alpha'|\alpha_0\rangle=0$, where $|\alpha_0\rangle$ is the corresponding eigenstate of H_0]. It follows that, to first order in $H_{\rm int}$, the trace can be replaced by a summation over the free-particle eigenstates of H_0. Furthermore, we can effectively write

$$e^{-\beta(H_0+H_{\rm int})}\simeq e^{-\beta H_0}(1-\beta H_{\rm int})+O(H_{\rm int}^2) \tag{2.16}$$

because, for products of $H_{\rm int}$ with powers of H_0, the operator ordering does not matter in the *diagonal* matrix elements we need. Thus, dropping terms of order $H_{\rm int}^2$ or higher, we have

$$\begin{aligned}
Z&={\rm Tr}_0[e^{-\beta H_0}(1-\beta H_{\rm int})]\\
&=Z_0(1-\beta\langle H_{\rm int}\rangle_T) \ ,
\end{aligned} \tag{2.17}$$

where

$$Z_0\equiv{\rm Tr}_0(e^{-\beta H_0})\equiv\sum_{\alpha_0}\langle\alpha_0|e^{-\beta H_0}|\alpha_0\rangle \ , \tag{2.18}$$

and the notation $\langle A\rangle_T$ stands for the "thermal average" of the operator A, with respect to H_0,

$$\langle A\rangle_T\equiv\frac{{\rm Tr}_0(e^{-\beta H_0}A)}{{\rm Tr}_0(e^{-\beta H_0})} \ , \tag{2.19}$$

and can be regarded as the finite-temperature generalization of ${}_\Omega\langle 0|A|0\rangle_\Omega$.

Inserting (2.17) in (2.14), the free energy to first order in $H_{\rm int}$ is

$$F=-\frac{1}{\beta}\ln Z_0+\langle H_{\rm int}\rangle_T \ . \tag{2.20}$$

The explicit evaluation of this expression is discussed in the next section. In passing we note that the corresponding result for the energy E is

$$\begin{aligned}
E&=\langle H_0\rangle_T+\langle H_{\rm int}\rangle_T+\beta(\langle H_0\rangle_T\langle H_{\rm int}\rangle_T\\
&\qquad -\langle H_0H_{\rm int}\rangle_T) \ ,\\
&=\langle H\rangle_T+\beta\frac{d}{d\beta}\langle H_{\rm int}\rangle_T \ ,
\end{aligned} \tag{2.21}$$

where the last term arises from the difference between $e^{-\beta H}$ and $e^{-\beta H_0}$ in the thermal averages appearing in Eqs. (2.15) and (2.19). We also note that the GEP approximation to the entropy is given by

$$S=T^{-1}\left[\langle H_0\rangle_T+\frac{1}{\beta}\ln Z_0+\beta\frac{d}{d\beta}\langle H_{\rm int}\rangle_T\right], \qquad (2.22)$$

by virtue of (2.12).

III. CALCULATION OF THE FTGEP

From the preceding discussion, the FTGEP

$$\bar V_G^{\rm FT}(\phi_0)=\min_{\Omega} V_G^{\rm FT}(\phi_0,\Omega) \qquad (3.1)$$

is obtained from the free energy of Eq. (2.20) as

$$V_G^{\rm FT}(\phi_0,\Omega)\equiv\frac{F}{V}=-\frac{1}{\beta V}\ln Z_0+\langle \mathcal{H}_{\rm int}\rangle_T\ . \qquad (3.2)$$

The first term is a free-field-theory quantity and represents the finite-temperature generalization of the I_1 integral. Although the result is well known, it is crucial to a full understanding of the FTGEP and we shall briefly describe its evaluation from first principles.

A free-field theory in a finite volume V consists of a countably infinite set of harmonic-oscillator modes, labeled by an index i, with frequencies ω_i. In the large-volume limit the discrete index i will go over to a continuous momentum label $\mathbf{k}$, with

$$\begin{aligned}&\sum_i\cdots\to V\int\frac{d^\nu k}{(2\pi)^\nu}\cdots\ ,\\ &\omega_i\to\omega_{\underline{k}}\equiv(\mathbf{k}^2+\Omega^2)^{1/2}\ .\end{aligned} \qquad (3.3)$$

The eigenstates of H_0 are $|n_1,n_2,\ldots\rangle$, corresponding to n_i quanta in mode i, with

$$H_0|n_1,n_2,\ldots\rangle=I_1V+n_1\omega_1+n_2\omega_2+\cdots\ , \qquad (3.4)$$

where I_1V is the vacuum energy, arising from the zero-point energy of each mode:

$$\sum_i\tfrac{1}{2}\omega_i\to V\int\frac{d^\nu k}{(2\pi)^\nu}\frac{1}{2}\omega_{\underline{k}}=VI_1\ . \qquad (3.5)$$

The trace appearing in the definition of Z_0 can now be written explicitly as

$$Z_0=\sum_{\substack{n_1=0,\infty\\ n_2=0,\infty\\ \cdots}}\langle n_1,n_2,\ldots|e^{-\beta H_0}|n_1,n_2,\ldots\rangle\ , \qquad (3.6)$$

where the summation is over all momentum modes i, and for each mode we sum over its occupation number n_i. (Note that for fermions we would sum over $n_i=0,1$ rather than $n_i=0,\ldots,\infty$.) Using (3.4) this becomes

$$\begin{aligned}Z_0&=e^{-\beta I_1V}\left[\sum_{n_1=0}^{\infty}e^{-\beta n_1\omega_1}\right]\left[\sum_{n_2=0}^{\infty}e^{-\beta n_2\omega_2}\right]\cdots\\ &=e^{-\beta I_1 V}\prod_i[(1-e^{-\beta\omega_i})^{-1}]\ .\end{aligned} \qquad (3.7)$$

Hence, defining

$$I_1^{\rm FT}\equiv I_1+I_1^\beta\equiv-\frac{1}{\beta V}\ln Z_0\ , \qquad (3.8)$$

we obtain

$$I_1^\beta\equiv\frac{1}{\beta}\int\frac{d^\nu k}{(2\pi)^\nu}\ln(1-e^{-\beta\omega_{\underline{k}}})\ . \qquad (3.9)$$

This result agrees with that of Roditi,[9] who took the manifestly covariant form of I_1,

$$I_1=-\frac{i}{2}\int\frac{d^{\nu+1}k}{(2\pi)^{\nu+1}}\ln(k^2-\Omega^2)+\text{const}\ , \qquad (3.10)$$

and made the replacements

$$\begin{aligned}&k^0\to 2\pi n/(-i\beta)\ ,\\ &-i\int\frac{dk^0}{(2\pi)}\cdots\to\frac{1}{\beta}\sum_n\cdots\ .\end{aligned} \qquad (3.11)$$

In this Roditi was following the classic work of Dolan and Jackiw[2] who worked in the context of the loop expansion. To us it does not seem immediately obvious that the rules (3.11) must lead to the *free energy,* rather than the energy, generalization of I_1. (We may note that Allès and Tarrach's[10] incorrect result differs from Roditi's only in the form of $I_1^{\rm FT}$; the former authors have an *energy* type of generalization.) Because of this, we feel that a first-principles derivation, such as we have given above, is logically necessary.

The evaluation of $\langle H_{\rm int}\rangle_T$, the other term in $V_G^{\rm FT}$, involves the calculation of $\langle\hat\phi^2\rangle_T$ and $\langle\hat\phi^4\rangle_T$. At zero temperature these would give I_0 and $3I_0^2$, respectively. We have carefully performed a first-principles calculation to check that the finite-temperature generalization is simply to replace I_0 by $I_0^{\rm FT}$, where

$$I_0^{\rm FT}=I_0+I_0^\beta\equiv\langle\hat\phi^2\rangle_T\ , \qquad (3.12)$$

with

$$I_0^\beta=2\int(dk)_\Omega\frac{1}{e^{\beta\omega_{\underline{k}}}-1}\ , \qquad (3.13)$$

in accord with the rules (3.11). In particular, we verified that, up to a volume-suppressed term, $\langle\hat\phi^4\rangle_T$ equals $3(\langle\hat\phi^2\rangle_T)^2$, as at zero temperature. A similar result holds for higher powers. Since all authors are in agreement on these points, we shall omit the details of our straightforward, but lengthy, calculations.[22]

The final result can be summarized thus: *the FTGEP is obtained from the GEP by the replacements*

$$I_1\to I_1^{\rm FT}\ ,\quad I_0\to I_0^{\rm FT}\ . \qquad (3.14)$$

Furthermore, the relation

$$\frac{dI_1^{\rm FT}}{d\Omega}=\Omega I_0^{\rm FT} \qquad (3.15)$$

holds, so that the minimization with respect to Ω parallels the zero-temperature case. Thus, the rule $I_0\to I_0^{\rm FT}$ also applies to the $\bar\Omega$ equation, (2.8).

Our conclusions therefore agree with Roditi's[9] (and with Okopinska's[13]), although we do not agree with his starting point [his Eq. (4)]. Allowing for the large-N limit, our results also agree with Bardeen and Moshe,[4] who obtained the correct $I_1^{\rm FT}$ by assuming the relation

(3.15). The result of Allès and Tarach,[10] who defined their FTGEP as $\langle \mathcal{H} \rangle_T$, differs from the correct result only in the form of the I_1^β integral.[23] This seemingly small difference has a huge effect on the physics, however, and their initial claims for "temperature-induced interaction" in precarious ϕ^4 theory should be discounted. The energy/free energy distinction is the crucial point:[11,24] As in many systems, as the temperature increases, the energy goes up, but the *free energy* goes *down*. Finally, we note that $\langle H \rangle_T$, calculated by Allès and Tarrach is not quite the energy in the GEP approximation: there should be an extra term—see Eq. (2.21).

IV. A QUANTUM-MECHANICAL EXAMPLE

In this section we present a quantum-mechanical example where we are able to compare the FTGEP and one-loop results with the exact high-temperature asymptotic behavior. We consider the anharmonic oscillator, whose Hamiltonian is

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}m^2\phi^2+\lambda\phi^4 , \tag{4.1}$$

where $[p,\phi]=-i\hbar$. In quantum-field-theory terms this is the $\lambda\phi^4$ theory reduced to $0+1$ dimensions, while in quantum-mechanics language it corresponds to a particle of unit mass oscillating in a one-dimensional, anharmonic potential well.

The high-temperature behavior can be obtained by a semiclassical analysis, because at high temperatures quantum fluctuations become negligible compared to the classical, thermal fluctuations. We shall specialize immediately to the $m^2=0$ case, since it is intuitively clear that the m^2 term is unimportant at high temperatures. (One can check that m^2 corrections to Z are in fact suppressed by $\sqrt{\beta}$.) From the Bohr-Sommerfeld quantization rule, which results from the WKB approximation,[21] one obtains the energy levels as[25]

$$E_n=\kappa\lambda^{1/3}(n+\tfrac{1}{2})^{4/3}\hbar^{4/3} , \tag{4.2}$$

where

$$\kappa=\pi^2[\sqrt{3}/\Gamma(\tfrac{1}{4})]^{8/3}\simeq 1.3765 . \tag{4.3}$$

This formula becomes exact as $n\to\infty$ (and is not too bad even for the lowest-lying states). The partition function can then be calculated directly:

$$\begin{aligned} Z&=\sum_{n=0}^{\infty} e^{-\beta E_n}\\ &\sim\int_0^\infty \exp(-\beta\kappa\lambda^{1/3}\hbar^{4/3}x^{4/3})dx\\ &=\tilde{\kappa}\lambda^{-1/4}\beta^{-3/4}\hbar^{-1} , \end{aligned} \tag{4.4}$$

where

$$\tilde{\kappa}=\Gamma(\tfrac{1}{4})(8\pi)^{-1/2} . \tag{4.5}$$

Note that, in the high-temperature ($\beta\to 0$) limit the sum is dominated by large values of n, justifying both the replacement of the sum by an integral, and the use of the semiclassical formula for E_n. Thus, Eq. (4.4) gives the exact high-temperature asymptotic behavior of Z. Hence, the high-temperature form of the free energy is

$$F\sim\tfrac{3}{4}\beta^{-1}\ln(\beta\lambda^{1/3}\hbar^{4/3})+\beta^{-1}\ln(1/\tilde{\kappa}) . \tag{4.6}$$

Let us now compute the FTGEP result, for $\phi_0=0$, at high temperatures, for comparison. From our earlier results, setting the spatial dimension $\nu=0$, we have

$$\begin{aligned} \hbar I_1^{\rm FT}&=\hbar\left[\frac{\Omega}{2}+\frac{1}{\hbar\beta}\ln(1-e^{-\beta\hbar\Omega})\right]\\ &\sim\beta^{-1}\ln(\beta\hbar\Omega)+\tfrac{1}{24}\beta\hbar^2\Omega^2+O(\beta^2\Omega^3) , \end{aligned} \tag{4.7}$$

$$\begin{aligned} \hbar I_0^{\rm FT}&=\hbar\left[\frac{1}{2\Omega}+\frac{1}{\Omega}\frac{1}{e^{\beta\hbar\Omega}-1}\right] ,\\ &\sim\frac{1}{\beta\Omega^2}+\frac{1}{12}\hbar^2\beta+O(\beta^2\Omega) . \end{aligned} \tag{4.8}$$

We have reinstated the factors of $\hbar$ which naturally accompany $I_1^{\rm FT}$ and $I_0^{\rm FT}$ in the FTGEP. At zero temperature these integrals represent purely quantum fluctuations, but at finite temperature they also contain thermal fluctuations, and hence include classical, $O(\hbar^0)$ contributions. The $\bar{\Omega}$ equation for $\phi_0=0$ reduces to

$$\bar{\Omega}^2=m^2+12\lambda\hbar I_0^{\rm FT} \tag{4.9}$$

and hence

$$\bar{\Omega}\sim(12\lambda/\beta)^{1/4} . \tag{4.10}$$

Thus, the effective mass grows as the fourth root of the temperature [and so the expansion parameters in (4.7) and (4.8), is $\beta\bar{\Omega}\sim\beta^{3/4}$]. From (2.9) the FTGEP result for the free energy is then

$$I_1^{\rm FT}-3\lambda(I_0^{\rm FT})^2\sim\tfrac{3}{4}\beta^{-1}\ln(\beta\lambda^{1/3}\hbar^{4/3})+\tfrac{1}{4}(\ln 12-1)\beta^{-1} . \tag{4.11}$$

Comparing to (4.6) we see that the $\beta^{-1}\ln\beta$ term is exact, while the β^{-1} term is approximately correct, the coefficient being 0.37 instead of the exact result's $\ln(1/\tilde{\kappa})=0.32$. Crucial to this success is the fact that $\bar{\Omega}$ grows as $\beta^{-1/4}$, producing the 3/4 coefficient for the logarithm.

By contrast, the conventional one-loop result,[2] for $\phi_0=0$, would be

$$\hbar I_1^{\rm FT}(m)\sim\beta^{-1}\ln(\beta\hbar m) , \tag{4.12}$$

which does not have the 3/4 coefficient, and also becomes singular at $m=0$, spuriously. It might seem surprising that the one-loop method, supposedly a semiclassical approximation, does not yield the right high-temperature behavior. The point is, though, that *at finite temperature the loop expansion ceases to be a semiclassical expansion:* the correspondence between number of loops and powers of $\hbar$ breaks down at nonzero temperatures. Thus, the one-loop result does not account for all the classical fluctuations, and has the wrong high-temperature behavior. Moreover, the true result is nonanalytic in $\hbar$.

The drawbacks of the loop-expansion technique at finite temperatures were well recognized by Dolan and Jackiw[2] who went on to improve the result by summing "daisy" and "superdaisy" graphs, leading them to the $1/N$ expansion. In diagrammatic terms[3] the FTGEP can be viewed as a similar kind of improvement, but—better still—it does not require large N, and its formulation does not have recourse to a power-series expansion in λ, even as an intermediate step. Finally, we note that the "mean-field" result (the leading $1/N$ result with N set equal to one) is similar to the FTGEP's above, but with an effective mass $\sim(4\lambda/\beta)^{1/4}$, and with ln4 replacing ln12 in (4.11). Thus, it is not quite such a good approximation to the exact high-temperature form (4.6).

V. ϕ^6 THEORIES IN TWO AND THREE DIMENSIONS

Employing the rule (3.14) it is straightforward to generalize the GEP results for ϕ^6 theories in $\nu+1$ dimensions[7] ($\nu=1$ or 2) to finite temperatures.[9] The Hamiltonian is given by (2.1) supplemented by a $\xi\phi^6$ term. Following Ref. 7 the renormalized parameters are defined as[26]

$$\alpha_4=(3\lambda_r/2\pi)m_R^{\nu-3}\,,$$
$$\alpha_6=(45\xi/8\pi^2)m_R^{2\nu-4}\,, \tag{5.1}$$

where

$$\lambda_r=\lambda_B-15\xi I_0(m_R) \tag{5.2}$$

and m_R is the solution to the $\overline{\Omega}$ equation at $\phi_0=0$ (and $T=0$). Various subtleties, including restrictions on the α_4,α_6 parameter space, are discussed in Ref. 7. These properties are unaffected by temperature effects because, of course, the renormalization is temperature independent.

Notice that, unlike the discussion in Ref. 9, our renormalized parameters are related to the bare parameters in a temperature-independent fashion. Hence, when we vary the temperature we do so within one particular theory; i.e., one with fixed, "God-given," bare parameters. We do not mean that "effective" masses and couplings[9] are not useful concepts, but only that temperature changes should be considered at fixed values of the bare parameters, not at fixed values of the "effective" parameters.

Numerical results for these models were obtained, showing that, as expected, the finite-temperature effects tend to restore the symmetry if it was broken at zero temperature.[1,2] Figures 1 and 2 provide a summary of our results,[22] showing how the region of theories with "unbroken symmetry" in the α_4,α_6 parameter space expands as the temperature is raised. Here we are speaking of "unbroken" or "broken" symmetry according to whether the global minimum of the FTGEP is, or is not, located at $\phi_0=0$. This terminology is fine in 2 + 1 dimensions but, as discussed in Refs. 6–8, it is rather sloppy in 1 + 1 dimensions where, just as in quantum mechanics, the symmetry can never truly be said to be broken.

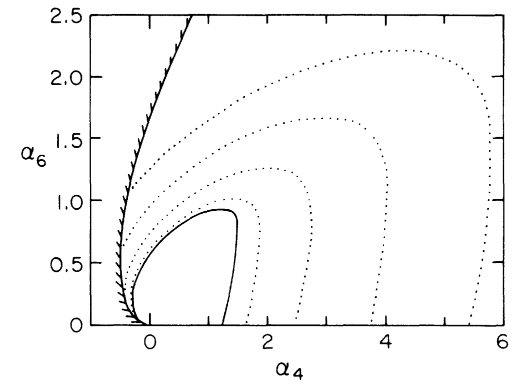

FIG. 1. The parameter space for ϕ^6 theories in 1+1 dimensions, showing the growth of the "unbroken-symmetry" region with increasing temperature. At $T=0$ those theories with "unbroken symmetry" lie inside the region enclosed by the solid line. As T increases this region expands, as shown by the dotted lines for $T=1$, 2, 3.3, 5 (in units of m_R). The parameters α_4 and α_6 are renormalized quartic and sextic couplings, respectively, and the left-hand boundary on their parameter space is explained in Ref. 7.

VI. $\lambda\phi^4$ THEORY IN FOUR DIMENSIONS

As mentioned in the Introduction, there are two quite distinct ways of renormalizing the GEP for $(\lambda\phi^4)_4$. For the "precarious" version the finite-temperature corrections have been correctly computed in Refs. 9 and 13. (Corresponding results for the large-N case were derived earlier by Bardeen and Moshe.[4]) The calculated behavior corresponds to the intuitive expectation of Ref. 6, which discusses the question of interpretation.

We can therefore turn immediately to the "autonomous" case.[14,15,8] Here the renormalization is given by[15]

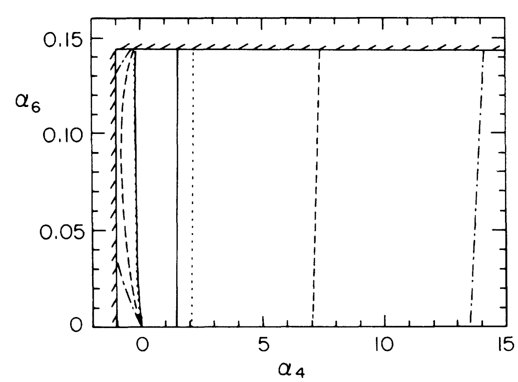

FIG. 2. As Fig. 1, but for (2 + 1)-dimensional ϕ^6 theories. For reasons explained in Ref. 7, the parameter space is restricted to $\alpha_4>-1$ and $0<\alpha_6<0.145$. At $T=0$ the unbroken-symmetry region is contained between the pair of solid lines. Its expansion as T increases is shown by the dotted lines ($T=2$), the dashed lines ($T=10$), and the dot-dashed lines ($T=20$). Units: $m_R=1$.

$$\lambda_B=\frac{1}{12I_{-1}(\mu)}\ ,\quad \phi_0^2=I_{-1}(\mu)\Phi_0^2\ , \tag{6.1}$$

$$m_B^2+12\lambda_B I_0(0)=\tfrac{3}{2}m_0^2/I_{-1}(\mu)\ , \tag{6.2}$$

where μ and m_0 are finite parameters, each with dimensions of mass. I_{-1} is a logarithmically divergent integral, see (2.7). Substituting into the $\overline{\Omega}$ equation, which is Eq. (2.8) with I_0 replaced by $I_0^{\rm FT}=I_0+I_0^\beta$, and using the formulas[6]

$$I_0(0)-I_0(\Omega)=\tfrac{1}{2}\Omega^2[I_{-1}(\Omega)+1/(8\pi^2)]\ , \tag{6.3}$$

$$I_{-1}(\Omega)-I_{-1}(\mu)=-\ln(\Omega^2/\mu^2)/(8\pi^2)\ , \tag{6.4}$$

one obtains

$$\overline{\Omega}^2=\tfrac{2}{3}\Phi_0^2+\frac{1}{I_{-1}(\mu)}\left[m_0^2+\frac{\overline{\Omega}^2}{24\pi^2}(\ln\overline{\Omega}/\mu^2-1)+\frac{2}{3}I_0^\beta(\overline{\Omega})\right]. \tag{6.5}$$

To evaluate the renormalized form of the FTGEP it is easiest to proceed via its first derivative.[8] From (2.6) and (2.8) it follows directly that

$$\frac{d\overline{V}_G^{\rm FT}}{d(\phi_0^2)}=\tfrac{1}{2}(\overline{\Omega}^2-8\lambda_B\phi_0^2)\ . \tag{6.6}$$

Inserting the renormalization (6.1) this becomes

$$\frac{d\overline{V}_G^{\rm FT}}{d(\Phi_0^2)}=\tfrac{1}{2}I_{-1}(\mu)(\overline{\Omega}^2-\tfrac{2}{3}\Phi_0^2)\ . \tag{6.7}$$

Substituting from (6.5), one observes that the divergent terms cancel, leaving

$$\frac{d\overline{V}_G^{\rm FT}}{d(\Phi_0^2)}=\frac{1}{2}m_0^2+\frac{\overline{\Omega}^2}{48\pi^2}(\ln\overline{\Omega}^2/\mu^2-1)+\tfrac{1}{3}I_0^\beta(\overline{\Omega})\ . \tag{6.8}$$

One may now use $\overline{\Omega}^2\simeq\frac{2}{3}\Phi_0^2$, since terms of order $1/I_{-1}$ will vanish when the (implicit) regularization is removed. Integration with respect to Φ_0^2 then gives

$$\overline{V}_G^{\rm FT}(\Phi_0)-D=\tfrac{1}{2}m_0^2\Phi_0^2+\frac{1}{144\pi^2}\Phi_0^4[\ln(\Phi_0^2/\tfrac{3}{2}\mu^2)-\tfrac{3}{2}]+I_1^\beta(\overline{\Omega}^2=\tfrac{2}{3}\Phi_0^2)\ . \tag{6.9}$$

The constant of integration D in (6.9) is, in fact, temperature independent, and is the usual divergent vacuum-energy constant. [To show this requires a separate calculation. Since $\overline{\Omega}=O(1/I_{-1})$ at $\Phi_0=0$ one can establish that $\overline{V}_G^{\rm FT}(\Phi_0=0)=V_G^{\rm FT}(\Phi_0=0,\Omega=0)$, up to vanishingly small terms. The latter, minus its zero-temperature value (D), is just $I_1^\beta(0)$, plus infinitesimal terms.]

It is straightforward to check that the $\overline{\Omega}$ equation is essentially always valid. The end point $\Omega=0$ is only relevant in an infinitesimal region $\Phi_0^2\lesssim 1/I_{-1}$ around the origin where, in cases when $m_0^2<-1/(18\beta^2)$, the $\overline{\Omega}$ equation ceases to have a solution. This situation never arises when the origin is a minimum of the FTGEP, so it has essentially no bearing on the physics.[27]

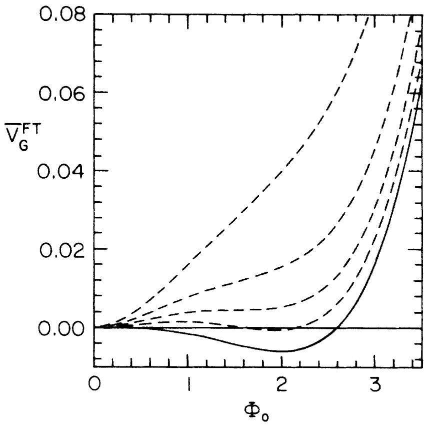

FIG. 3. The shape of the FTGEP for autonomous $(\lambda\phi^4)_{3+1}$ theory with $m_0^2=0$. At $T=0$ there is spontaneous symmetry breaking (solid curve), but the symmetry is restored at high temperatures. The dashed lines correspond to $T=0.5$, 0.625, 0.769, 1.0. A temperature-dependent constant has been added to each curve so that they coincide at the origin. Units: $\mu=1$.

The evolution of the FTGEP with temperature is illustrated, for the case $m_0^2=0$, in Fig. 3. As expected, the symmetry, if broken at $T=0$, is restored at sufficiently high temperatures. For $m_0^2=0$ the critical temperature T_c at which the phase transition occurs is $kT_c\simeq 0.51\ \mu$. For other values of m_0^2/μ^2 the critical temperature T_c can be read from Fig. 4, which is the phase diagram of the autonomous theory.

Another quantity of interest is the effective particle mass. In a broken-symmetry vacuum, $\Phi_0=\pm v$, the effective mass $\overline{\Omega}_v$, is determined by equating (6.8) to zero. Since $\overline{\Omega}_v^2=\frac{2}{3}v^2$, the effective mass decreases as the temperature increases and the minima of the FTGEP move in toward to the origin. This is a rather small effect: in the $m_0^2=0$ case, for example, $\overline{\Omega}_v$ decreases by only 5% between $T=0$ and T_c. After the phase transition to the symmetric phase the particles become massless, since $\overline{\Omega}$ vanishes at the origin, for all temperatures.

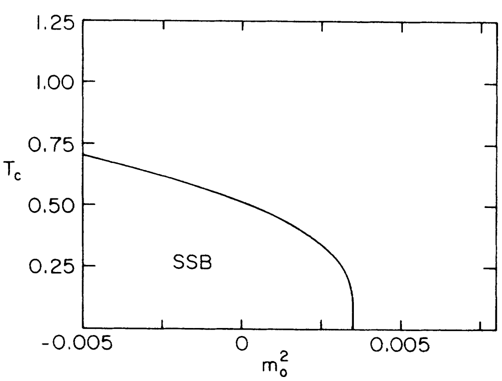

FIG. 4. The phase diagram for autonomous $(\lambda\phi^4)_{3+1}$ theory, according to the FTGEP. The curve represents the critical temperature T_c as a function of m_0^2. The region below this curve corresponds to spontaneous symmetry breaking. Units: $\mu=1$.

The physics in the symmetric vacuum is obviously complicated due to the infrared divergences associated with massless particles. At zero temperature, the fourth and higher derivatives of $\bar{V}_G^{\rm FT}$ are singular at the origin, naively indicating infinite effective couplings. However, since the cross section for $2\to n$ scattering, for any definite number of particles n, is not an observable quantity in a massless theory, this is not necessarily an unphysical result. At finite temperatures a remarkable cancellation takes place. From the high-temperature expansion for I_1^β in four dimensions,[2]

$$I_1^\beta=\frac{1}{\beta^4}\left[-\frac{\pi^2}{90}+\frac{(\beta\Omega)^2}{24}-\frac{(\beta\Omega)^3}{12\pi}-\frac{(\beta\Omega)^4}{64\pi^2}(\ln\beta^2\Omega^2-c)+O((\beta\Omega)^6)\right] \tag{6.10}$$

[where $c=\frac{3}{2}+2(\ln 4\pi-\gamma)\approx 5.41$], one can make a small-$\Phi_0$ expansion of the FTGEP, since $\bar{\Omega}=(\frac{2}{3})^{1/2}\,|\Phi_0|$. This gives

$$\bar{V}_G^{\rm FT}-D=-\frac{\pi^2}{90\beta^4}+\frac{1}{2}\left[m_0^2+\frac{1}{18\beta^2}\right]\Phi_0^2-\left[\frac{2}{3}\right]^{1/2}\frac{1}{18\pi\beta}\,|\Phi_0|^3-\frac{1}{144\pi^2}(\ln\mu^2\beta^2+\tfrac{3}{2}-c)\Phi_0^4+O(\Phi_0^6)\ . \tag{6.11}$$

The fourth derivative thus becomes finite, due to the cancellation of $\ln\Phi_0^2$ terms. However, a finite discontinuity in the third derivative has now appeared, which grows larger as the temperature rises.

Finally, we mention that our result (6.9) is easily extended to the O(N)-symmetric case, by appropriately generalizing the analysis in Ref. 8. The result is simply

$$\bar{V}_G^{\rm FT}=\bar{V}_G+I_1^\beta(\Omega)+(N-1)I_1^\beta(\omega)\ , \tag{6.12}$$

where Ω and ω, both proportional to $|\Phi_0|$, are the variational mass parameters for the radial and transverse fields, respectively.[8] A small-Φ_0 expansion of the result shows the same cancellation noted above.

VII. SUMMARY AND CONCLUSIONS

We have derived the FTGEP from first principles, clarifying and better justifying earlier treatments.[9–13] The result is simply that the I_1 and I_0 integrals appearing in the GEP should be replaced by specific finite-temperature generalizations: see Eqs. (3.8)–(3.15).

At finite temperature the conventional one-loop approach does not correspond to a semiclassical approximation and does not in general give the correct high-temperature behavior. This point was illustrated by a quantum-mechanical example in Sec. IV, which also indicated that the FTGEP approach can be much more successful.

Results for ϕ^6 theories in two and three dimensions, extending those of Roditi[9] are summarized in Figs. 1 and 2.

For autonomous $(\lambda\phi^4)_4$ theory the finite-temperature corrections emerge in a particularly simple form, and give rise to the expected symmetry-restoration phenomenon (Figs. 3 and 4). An intriguing cancellation between the zero- and finite-temperature terms suggests some interesting physics in the massless, symmetric vacuum, which deserves further investigation.

To conclude, we reemphasize the need for "higher-order" Gaussian calculations, to check that the corrections do not upset the picture implied by the leading-order results. Work in this direction is in progress. The finite-temperature calculations here, because they involve excited states, are, to a small extent, indicative of higher-order effects. The fact that we find physically sensible finite-temperature behavior for autonomous ϕ^4 theory is therefore encouraging. However, it is no substitute for a higher-order calculation.

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