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One-loop corrections to solitons in two-dimensional theories

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We have investigated quantum corrections to topological and nontopological solitons in several two-dimensional theories at the one-loop level utilizing a numerical Green's-function method. For the ϕ^4 kink and sine-Gordon soliton, as well as the kink coupled to fermions, we reproduce the masses of static solitons obtained analytically in the cases for which analytic results exist. The advantage of the Green's-function method is that it is suited to dealing with a wide range of theories and with most background-field configurations. This becomes apparent from our study of theories where tractable analytic methods of calculating quantum corrections at the one-loop level do not exist. We have used the method to investigate in detail the derivative and inverse-mass expansions of the one-loop, nonlocal actions of bosons and fermions for some typical nontopological background fields.

I. INTRODUCTION

In the hope of describing hadrons as solitons, the study of solitons in nonlinear field theories has been actively pursued in particle and nuclear physics.[1] In general solitons have been treated at the classical level (i.e., tree level), and little is known about the effects of quantum fluctuations.[2] One can systematically study quantum corrections to the properties of solitons by developing the loop expansion of the effective action. This expansion is expected to converge only when one employs small coupling constants, in which case quantum corrections are expected to be small perturbations. For the present study this assumption need not be made, because we will be primarily concerned with the accurate evaluation of these corrections, not their employment.

In this paper we will calculate the one-loop corrections to the energy of a wide variety of solitons. The soliton background fields will initially be fixed at their classical values, for comparison to previously obtained analytic results, but the methods employed are suited to arbitrary background fields, provided the one-loop corrections do not become imaginary. This means that one can allow the soliton to adjust itself in response to quantum effects. Naturally a quantum soliton will minimize the full quantum energy.

In the process of studying one-loop corrections for four-dimensional (i.e., three spatial dimensions and time) nontopological solitons,[3] we have developed a Green's-function method which is suitable for calculating such corrections for two-dimensional solitons. Because this method utilizes numerical calculations, it does not depend on any special feature of the boson self-interactions or on the form of classical soliton solutions. It is especially suitable for the study of general two-dimensional theories of bosons coupled to fermions through Yukawa interactions. In this paper we demonstrate the power of such methods by reproducing the few analytic one-loop results which exist, and by extending these results to cover arbitrary background fields and several new theories. We also use these numerical calculations to test local derivative and inverse-mass expansions of the one-loop corrections.

In Sec. II we introduce the Green's-function method for two-dimensional theories of bosons as well as bosons coupled to fermions through Yukawa coupling. In Sec. III we apply this formalism to calculate one-loop quantum corrections for several models. In Sec. IV we discuss derivative and inverse-mass expansions of the one-loop corrections to the quantum soliton mass. Our conclusions are given in Sec. V.

II. THE GENERAL FORMALISM

Much of the formalism needed to study quantum corrections of solitons is contained in the excellent papers of Dashen, Hasslacher, and Neveu.[4] In the following we outline only the essential aspects of this formalism with emphasis on the connections to the Green's-function method.

The Lagrangian density of a general renormalizable two-dimensional scalar field coupled to fermions through a Yukawa coupling is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-U(\phi)+\bar{\psi}(i\partial\!\!\!/-g\phi)\psi \ . \tag{2.1}$$

The effective action at the one-loop level can be derived through the loop expansion.[5] The generating function of the Green's functions is

$$Z(J)=\int D\phi\, D\psi\, D\bar{\psi} \exp\left[i\int dx\, dt[\mathcal{L}(\phi,\bar{\psi},\psi)+J\phi]\right] . \tag{2.2}$$

The generating function of the connected Green's functions $\Omega(J)$ is related to the $Z(J)$ by

$$\Omega(J)=-i\ln Z(J) \ . \tag{2.3}$$

The effective action S_{eff} is then given by the Legendre transformation

$$S_{\text{eff}}=\Omega(J)-\int dx\, dt\, \phi_c(x)J(x) \ , \tag{2.4}$$

where

$$\phi_c(x)=\frac{\delta\Omega(J)}{\delta J} . \tag{2.5}$$

Let $\phi(x)=\phi_c(x)+\bar{\phi}(x)$, where $\phi_c(x)$ is the c-number field defined by Eq. (2.5). We make a functional Taylor-series expansion of $\mathcal{L}$ around ϕ_c:

$$\mathcal{L}(\phi,\bar{\psi},\psi)=\mathcal{L}_0(\phi_c)+\tfrac{1}{2}\bar{\phi}D^{-1}(\phi_c)\bar{\phi} -\bar{\psi}S^{-1}(\phi_c)\psi+J\phi_c+\cdots , \tag{2.6}$$

where

$$\mathcal{L}_0(\phi_c)=\tfrac{1}{2}(\partial_\mu\phi_c)^2-U(\phi_c) \tag{2.7}$$

is the classical action for the soliton, and

$$D^{-1}(\phi_c)=-(\partial^2+W) , \tag{2.8}$$

$$S^{-1}(\phi_c)=-(i\partial\!\!\!/-g\phi_c) , \tag{2.9}$$

with

$$W=\frac{\partial^2U(\phi_c)}{\partial\phi_c{}^2} . \tag{2.10}$$

At the one-loop level, we need only keep the terms that are explicitly written above and approximate

$$Z(J)=\exp\left[i\int dx\,dt[\mathcal{L}_0(\phi_c)+J\phi_c]\right] \times\int D\bar{\phi}D\psi D\bar{\psi}\exp\left[i\int(\tfrac{1}{2}\bar{\phi}D^{-1}\bar{\phi}-\bar{\psi}S^{-1}\psi)\right] . \tag{2.11}$$

After performing the functional integrations in $Z(J)$ and using Eqs. (2.3) and (2.4), we obtain the one-loop effective action as

$$S_{\text{eff}}=\int dx\,dt\,\mathcal{L}_0(\phi_c)+\frac{i}{2}\text{Tr}\ln D^{-1}(\phi_c) -i\text{Tr}\ln S^{-1}(\phi_c) . \tag{2.12}$$

Here $\frac{1}{2}i\,\text{Tr}\ln D^{-1}(\phi_c)$ represents the boson-loop contribution, and $-i\,\text{Tr}\ln S^{-1}(\phi_c)$ represents the fermion-loop contribution. Traces are taken in the functional sense for both the boson- and fermion-loop contributions.

In the case that $\phi_c(x)$ is time independent, the energy is related to the effective action by

$$E(\phi_c)=-S_{\text{eff}}\Big/\int dt . \tag{2.13}$$

The $\phi_c(x)$ is required to minimize the energy, that is, to satisfy the following conditions:

$$\frac{\partial E(\phi_c)}{\partial\phi_c}=0 , \tag{2.14}$$

$$\frac{\partial^2E(\phi_c)}{\partial\phi_c{}^2}\geq 0 . \tag{2.15}$$

In the following sections we will restrict ourselves to the study of time-independent solitons. Since $\phi_c(x)$ is time independent, we can define

$$\tilde{D}(\omega)=\int dt\,\exp(-i\omega t)D(\phi_c) , \tag{2.16}$$

$$\tilde{S}(\omega)=\int dt\,\exp(-i\omega t)S(\phi_c) , \tag{2.17}$$

and can express

$$E=\int dx[\tfrac{1}{2}(\nabla\phi_c)^2+U(\phi_c)] -\frac{i}{4\pi}\int d\omega\,\text{tr}\ln\tilde{D}^{-1}(\omega) +\frac{i}{2\pi}\int d\omega\,\text{tr}\ln\tilde{S}^{-1}(\omega) , \tag{2.18}$$

where the tr differs from the Tr by excluding the trace over time. Integrating by parts and throwing away an unimportant constant, we can write

$$E=E_0+\frac{i}{2\pi}\int\omega^2d\omega\,\text{tr}\tilde{D}(\omega) +\frac{i}{2\pi}\int\omega\,d\omega\,\text{tr}[\gamma^0\tilde{S}(\omega)] , \tag{2.19}$$

where

$$E_0=\int dx[\tfrac{1}{2}(\nabla\phi_c)^2+U(\phi_c)] \tag{2.20}$$

represents the classical energy of the soliton.

The ω integration and trace are not well defined until we make appropriate renormalization subtractions and specify boundary conditions for the Green's functions $\tilde{D}(\omega)$ and $\tilde{S}(\omega)$. The necessary subtractions are well known in two dimensions. Because of mild ultraviolet divergences we need only subtract four terms from the energy to make it finite. We need to subtract mass counterterms and terms with free Green's functions replacing the interacting Green's functions in (2.19), corresponding to vacuum-energy subtraction. For the boundary conditions, we require the integral contour in the ω plane to run under negative-energy poles and over positive-energy poles. If one chooses to occupy positive-energy fermion states, their poles should also be included in the contour (see Fig. 1). This allows us to deform the contours to run along the imaginary axis. After making the renormalization subtractions and deforming the contours, the energy becomes

$$E=E_0+\frac{1}{2\pi}\int_{-\infty}^{+\infty}dy\,y^2\text{tr}[\tilde{D}(iy)-\tilde{D}_0(iy)-\tilde{D}_0{}^2(iy)\tilde{W}]+\sum_i n_i\epsilon_i -\frac{i}{2\pi}\int_{-\infty}^{+\infty}dy\,y\,\text{tr}\gamma^0[\tilde{S}(iy)-\tilde{S}_0(iy)-\tilde{S}_0{}^2(iy)g\tilde{\phi}-\tilde{S}_0{}^3(iy)(g\tilde{\phi})^2] , \tag{2.21}$$

where we have separated out the fermion valence states from the rest of the fermion loop. Here ϵ_i is the ith valence fermion eigenvalue and n_i is the corresponding valence fermion occupation number. $\tilde{D}_0$, $\tilde{S}_0$, $\tilde{W}$, and $\tilde{\phi}$

are defined as

$$\tilde{D}_0(iy)=\frac{1}{\nabla^2-y^2-W_0}\ , \tag{2.22}$$

$$\tilde{S}_0(iy)=\frac{1}{-i\gamma^0 y-i\gamma^1\nabla+g\phi_0}\ , \tag{2.23}$$

$$\tilde{W}(x)=W(x)-W_0\ , \tag{2.24}$$

$$\tilde{\phi}(x)=\phi_c(x)-\phi_0\ , \tag{2.25}$$

and $W_0=W(+\infty)$, $\phi_0=\phi_c(+\infty)$ are both x independent. We have separated the free Green's functions in such a way that the traces over the spatial variable in Eq. (2.21) converge. This way of defining the free Green's function is, of course, for convenience of later calculations. Different definitions will simply change the renormalization conditions in Eq. (2.21). We choose the Dirac matrices to be $\gamma^0=\beta=\sigma_y$, $\gamma^1=\beta\alpha=i\sigma_z$.

The general problem of obtaining the soliton solution is somewhat involved. We need to solve Eq. (2.14), which is an integro-differential equation, subject to (2.15). The Green's functions we need to calculate the energy can be constructed by standard methods. Denote $u_y(x)$ and $v_y(x)$ the solutions regular at $x\to\mp\infty$, respectively, of

$$[\nabla^2-y^2-W(x)]\phi=0\ . \tag{2.26}$$

The Green's function $\tilde{D}(iy)$ is given by

$$\tilde{D}(iy)=\frac{1}{j_B(y)}[\theta(x'-x)u_y(x)v_y(x')+\theta(x-x')v_y(x)u_y(x')]\ , \tag{2.27}$$

where $j_B(y)$ is the Wronskian and is independent of x,

$$j_B(y)=u_y(x)v_y'(x)-v_y(x)u_y'(x)\ , \tag{2.28}$$

and $\theta(x)$ is the step function.

The Green's function for the fermions can be constructed in a similar fashion. Let us denote here

$$U_y(x)=\begin{bmatrix}U_{y1}\\U_{y2}\end{bmatrix}\ , \tag{2.29}$$

$$V_y(x)=\begin{bmatrix}V_{y1}\\V_{y2}\end{bmatrix}\ , \tag{2.30}$$

and let the U_y,V_y be the two linearly independent solutions of

$$[-i\alpha\nabla-iy+\beta g\phi_c(x)]\psi=0\ , \tag{2.31}$$

which are regular at $x\to\mp\infty$, respectively. Then $\tilde{S}(iy)$ is given by

$$\gamma^0\tilde{S}(iy)=i\gamma^1\frac{1}{j_F(y)}[\theta(x'-x)U_y(x)V_y^T(x')+\theta(x-x')V_y(x)U_y^T(x')]\ , \tag{2.32}$$

where

$$j_F(y)=U_{y2}(x)V_{y1}(x)-V_{y2}(x)U_{y1}(x)\ , \tag{2.33}$$

which is x independent. The free Green's functions $\tilde{D}_0$ and $\tilde{S}_0$ are simple and known exactly:

$$\tilde{D}_0(iy,x,x')=\frac{-1}{2a_y}\exp(-a_y\,|x-x'|)\ , \tag{2.34}$$

$$\gamma^0\tilde{S}_0(iy,x,x')=\frac{1}{2b_y}(+i\alpha\nabla+iy+\beta g\phi_0)\times\exp(-b_y\,|x-x'|)\ , \tag{2.35}$$

where $a_y=(y^2+W_0)^{1/2}$ and $b_y=[y^2+(g\phi)^2]^{1/2}$. Given a background field, we are now in a position to calculate the energy using Eq. (2.21).

We should note that the fermion loop can be calculated in a slightly different fashion. Since we have

$$\mathrm{tr}(\gamma^0\tilde{S})=\mathrm{tr}\frac{iy}{\nabla^2-y^2-(g\phi_c)^2-\sigma_z g\phi_c'}\ , \tag{2.36}$$

we can calculate the fermion-loop contribution in Eq. (2.21) in exactly the same way as the boson loop, because $\gamma^0\tilde{S}$ is diagonal and hence equivalent to two bosonlike contributions. This nice feature is special to two dimensions and cannot be directly generalized to higher dimensions. For example, in three spatial dimensions, after the partial-wave decomposition of the fermion Green's function, the two-dimensional Green's functions occurring for each partial wave cannot in general be diagonalized. Hence we have used the first-order form of $\tilde{S}(iy)$ in our calculations, so that the procedure is easily generalized to higher dimensions.

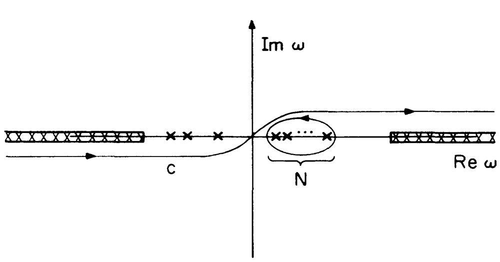

FIG. 1. The ω-plane contour that corresponds to having N valence fermion states occupied.

III. APPLICATIONS

In this section we apply the general formalism to various two-dimensional theories. Throughout this section we will assume that the one-loop corrections are small, allowing us to analyze them as first-order perturbations. What this means is that we will assume that the background scalar field is a solution to the classical equations of motion, and analyze the energy shift due to the loop corrections without allowing the field to respond.

A. The kink (Refs. 1 and 6)

The Lagrangian density is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4}\lambda\phi^4\ . \tag{3.1}$$

After making the scaling $mx \to x$, $my \to y$, and $\phi \to (m/\sqrt{\lambda})\phi$, the static soliton solution to the classical Euler-Lagrange equations of motion is found to be

$$\phi_c = \tanh\left[\frac{x}{\sqrt{2}}\right], \tag{3.2}$$

and the corresponding classical energy is

$$E_0 = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{3.3}$$

From Eq. (2.21) the boson-loop quantum correction is

$$\Delta E_B = \frac{m}{2\pi}\int_{-\infty}^{+\infty} dy\, y^2 \mathrm{tr}\left[\frac{1}{\nabla^2 - y^2 - 2 + 3/\cosh^2(x/\sqrt{2})} - \frac{1}{\nabla^2 - y^2 - 2} + \left[\frac{1}{\nabla^2 - y^2 - 2}\right]^2 \frac{3}{\cosh^2(x/\sqrt{2})}\right]. \tag{3.4}$$

Following the procedure described in Sec. II, we make variable changes $x = \tan(\pi/2\bar{x})$, $y = \tan(\pi/2\bar{y})$, and work on a discrete grid of equally spaced points in the interval $\bar{x}, \bar{y} \in [-1,1]$. We can construct the Green's functions numerically, subtracting the renormalization terms and performing the x and y integrations. By using 100 grid points in $\bar{x}$ and 32 points in $\bar{y}$, and by using repeated Richardson extrapolation, we obtain the result $\Delta E_B = -0.471\,108$ m, which agrees with the exact result[1,6] $\Delta E_B = m[1/(2\sqrt{6}) - 3/(\pi\sqrt{2})]$ to four significant figures. This accuracy can be improved by increasing the grid points.

B. The sine-Gordon soliton (Ref. 1)

The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{m^4}{\lambda}\left[\cos\left[\frac{\sqrt{\lambda}}{m}\phi\right] - 1\right]. \tag{3.5}$$

After making the scaling $mx \to x$, $my \to y$, and $\phi \to (m/\sqrt{\lambda})\phi$, the static soliton solution is

$$\phi_c = 4\arctan\exp(x), \tag{3.6}$$

and the corresponding classical energy is

$$E_0 = \frac{8m^3}{\lambda}. \tag{3.7}$$

From Eq. (2.21) the boson-loop quantum correction is

$$\Delta E_B = \frac{m}{2\pi}\int_{-\infty}^{+\infty} dy\, y^2 \mathrm{tr}\left[\frac{1}{\nabla^2 - y^2 - 1 + 2/\cosh^2 x} - \frac{1}{\nabla^2 - y^2 - 1} + \left[\frac{1}{\nabla^2 - y^2 - 1}\right]^2 \frac{2}{\cosh^2 x}\right]. \tag{3.8}$$

The calculation is identical to that of Sec. III A. By using equal-distance grid points with $n_{\bar{x}} = 100$ and $n_{\bar{y}} = 32$, and by using repeated Richardson extrapolation, we obtain the result $\Delta E_B = -0.318\,28$ m, which agrees with the exact result[1] $\Delta E_B = -m/\pi$ to four significant figures.

C. The kink coupled to a Fermion field (Ref. 7)

The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}\lambda\phi^4 + \bar{\psi}(i\partial\!\!\!/ - g\phi)\psi. \tag{3.9}$$

After making the same scaling as in Sec. III A, the fermion-loop quantum correction in Eq. (2.21) is given by

$$\Delta E_F = \frac{im}{2\pi}\int_{-\infty}^{+\infty} dy\, y\, \mathrm{tr}\gamma^0 \left[\tilde{S}(iy,x,x) - \tilde{S}_0(iy,x,x) + \tilde{S}_0{}^2(iy,x,x)\left[G\tanh\frac{x}{\sqrt{2}} - G\right] - \tilde{S}_0{}^3(iy,x,x)\left[G\tanh\frac{x}{\sqrt{2}} - G\right]^2\right], \tag{3.10}$$

where

$$\tilde{S}(iy,x,x) = \frac{1}{-i\gamma^0 y - i\gamma^1\nabla + G\tanh x/\sqrt{2}}, \tag{3.11}$$

$$\tilde{S}_0(iy,x,x) = \frac{1}{-i\gamma^0 y - i\gamma^1\nabla + G}, \tag{3.12}$$

and $G = g/\sqrt{\lambda}$. The boson-loop contribution is the same as Sec. III A. The fermion-loop calculation proceeds as described in Sec. II. We present the results in Table I for several values of G. Because boson and fermion loops give contributions of opposite sign, at some value of G they cancel each other, and the quantum soliton has the same energy as the classical soliton. For this model, this value of G is calculated to be $G = 1.366$.

When $\sqrt{2G}$ is an integer, the fermion-loop quantum correction can be calculated analytically[7] and our results are correct to three significant figures. We can achieve higher accuracy by increasing the grid points.

D. The sine-Gordon soliton coupled to a fermion field

The Lagrangian density is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{m^4}{\lambda}\left[\cos\left[\frac{\sqrt{\lambda}}{m}\phi\right]-1\right]+\bar{\psi}\left[i\partial\!\!\!/-g\left[\phi-\frac{m\pi}{\sqrt{\lambda}}\right]\right]\psi\ , \tag{3.13}$$

The calculation proceeds as Sec. III C. The fermion-loop quantum correction is given by

$$\Delta E_F=\frac{mi}{2\pi}\int_{-\infty}^{+\infty}dy\,y\,\mathrm{tr}\gamma^0[\tilde{S}(iy,x,x)-\tilde{S}_0(iy,x,x)+\tilde{S}_0{}^2(iy,x,x)G(4\arctan(e^x)-2\pi)$$
$$-\tilde{S}_0{}^3(iy,x,x)G^2(4\arctan(e^x)-2\pi)^2]\ , \tag{3.14}$$

where

$$\tilde{S}(iy,x,x)=\frac{1}{-i\gamma^0y-i\gamma^1\nabla+G(4\arctan(e^x)-\pi)}\ , \tag{3.15}$$

$$\tilde{S}_0(iy,x,x)=\frac{1}{-i\gamma^0y-i\gamma^1\nabla+G\pi}\ , \tag{3.16}$$

and $G=g/\sqrt{\lambda}$. The results are presented in Table I for several values of G. In this case, the value of G for which the boson and fermion-loop contributions cancel each other is $G=0.306$.

E. The Christ-Lee model (Ref. 8)

The Lagrangian density is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{1}{8}\frac{m^2}{d^2(1+c^2)}(d^2\phi^2+c^2)(1-d^2\phi^2)^2+\bar{\psi}(i\partial\!\!\!/-g\phi)\psi\ . \tag{3.17}$$

This model has a classical static soliton solution

$$\phi_c=\frac{[1+c^{-2}+\sinh^2(1/2mx)]^{-1/2}}{d}\sinh(\tfrac{1}{2}mx)\ , \tag{3.18}$$

which consists of two kinks. In the following, for the purpose of illustration we will consider only the case where $d=c=1$. For different values of d and c the necessary calculations are the same.

In this case, the classical energy of the soliton is given by

$$E_0=\frac{1}{4}m\left[\frac{1}{2}+\frac{5}{2\sqrt{2}}\ln(1+\sqrt{2})\right]\ , \tag{3.19}$$

and the calculations of quantum corrections proceed exactly as in cases in Secs. III A–III D. We present in Table I our results for the fermion-loop correction for several values of G, which is defined as $G=g/m$ in this model. The boson-loop quantum correction for this model is $\Delta E_B=-0.121$. In this model, it is at $G=0.38$ that the boson- and fermion-loop quantum corrections cancel one another.

IV. THE DERIVATIVE AND INVERSE-MASS EXPANSIONS

We have given nonlocal expressions for the boson and fermion one-loop contributions to the effective action, and shown how to analyze these expressions numerically. In this section we present local approximations to the one-loop corrections, and study their validity.[9] The approximations we wish to study can be derived from functional Taylor-series expansions of $(i/2)\mathrm{Tr}\ln\tilde{D}^{-1}(\phi_c)$ and $i\,\mathrm{Tr}\ln\tilde{S}^{-1}(\phi_c)$. In this article we will not reproduce the derivations of these expansions, which have been presented in several articles.[10,11] Throughout this section we will assume that the background scalar field is time independent, and we will absorb coupling constants by suitably redefining the scalar field. It should be noted that one cannot in general absorb both the fermion and boson coupling constants in this fashion.

The first expansion we wish to consider is in powers of derivatives of the background field. If one renormalizes the boson loop as in Eq. (2.21), one obtains[10]

$$\frac{i}{2}\mathrm{Tr}\ln D^{-1}(\phi_c)=\frac{i}{2}\mathrm{Tr}\ln(-\partial^2-W)$$
$$=\frac{1}{8\pi}\int dx\,dt\left[W_0-W+W\ln\frac{W}{W_0}-\frac{1}{12}W^{-2}\left[\frac{\partial W}{\partial x}\right]^2+\frac{1}{12}W^{-5}\left[\frac{\partial W}{\partial x}\right]^4+\frac{1}{60}W^{-3}\left[\frac{\partial^2 W}{\partial x^2}\right]^2\right.$$
$$\left.-\frac{1}{12}W^{-4}\left[\frac{\partial^2 W}{\partial x^2}\right]\left[\frac{\partial W}{\partial x}\right]^2+O(\partial^6)\right]\ , \tag{4.1}$$

The first term in this expansion contains no derivatives of the background field, and is the one-boson-loop contribution to the effective potential.[5] The derivative expansion for the fermion loop is most easily derived by employing the identity

$$i\mathrm{Tr}\ln S^{-1}(\phi_c)=i\mathrm{Tr}\ln(i\partial\!\!\!/-\phi_c)=\frac{i}{2}\mathrm{Tr}\ln(-\partial^2-\phi_c{}^2-\sigma_z\phi_c') \ . \tag{4.2}$$

Equation (4.2) makes it obvious that the fermion-loop contribution is equivalent to two boson-loop contributions. Since a derivative of the background field occurs inside the logarithm for the fermion loop, some rearrangement of the terms found in Eq. (4.1) is necessary. The result is

$$-i\mathrm{Tr}\ln S^{-1}(\phi_c)=\frac{1}{4\pi}\int dx\,dt\left[\phi_c{}^2-\phi_0{}^2-\phi_c{}^2\ln\frac{\phi_c{}^2}{\phi_0{}^2}-\frac{1}{6}\phi_c{}^{-2}\left[\frac{\partial\phi_c}{\partial x}\right]^2+\frac{11}{60}\phi_c{}^{-6}\left[\frac{\partial\phi_c}{\partial x}\right]^4\right.$$
$$\left.-\frac{2}{15}\phi_c{}^{-5}\left[\frac{\partial^2\phi_c}{\partial x^2}\right]\left[\frac{\partial\phi_c}{\partial x}\right]^2+\frac{1}{60}\phi_c{}^{-4}\left[\frac{\partial^2\phi_c}{\partial x^2}\right]^2+O(\partial^6)\right] \ . \tag{4.3}$$

The second expansion we consider is in inverse powers of the boson or fermion effective mass. The boson effective mass is given by the square root of W, while the fermion effective mass is ϕ_c. When W becomes negative the boson effective mass becomes imaginary, with results which will be discussed below. The boson and fermion inverse mass expansions can be derived from Eqs. (4.1) and (4.2), respectively. Each term in these expansions has a fixed dimension, so that derivatives and the masses upon which they act must be balanced by powers of some inverse mass multiplying each term. For the simple problems that we consider here, only one mass occurs in each one-loop contribution, so that the relevant mass is the boson (fermion) effective mass for the boson (fermion) loop. These considerations are sufficient to prove that all terms containing four or fewer powers of the relevant inverse mass can be found in Eqs. (4.1) and (4.3). If we eliminate any terms in Eq. (4.1) which contain more than four inverse powers of the effective boson mass, we have for the boson loop

$$\frac{i}{2}\mathrm{Tr}\ln D^{-1}(\phi_c)=\frac{1}{8\pi}\int dx\,dt\left[M_0{}^2-M^2+M^2\ln\frac{M^2}{M_0{}^2}-\frac{1}{3}M^{-2}\left[\frac{\partial M}{\partial x}\right]^2+\frac{1}{15}M^{-4}\left[\frac{\partial^2 M}{\partial x^2}\right]^2+O(M^{-6})\right] \ . \tag{4.4}$$

Similarly, for the fermion loop we find

$$-i\mathrm{Tr}\ln S^{-1}(\phi_c)=\frac{1}{4\pi}\int dx\,dt\left[\phi_c{}^2-\phi_0{}^2-\phi_c{}^2\ln\frac{\phi_c{}^2}{\phi_0{}^2}-\frac{1}{6}\phi_c{}^{-2}\left[\frac{\partial\phi_c}{\partial x}\right]^2+\frac{1}{60}\phi_c{}^{-4}\left[\frac{\partial^2\phi_c}{\partial x^2}\right]^2+O(\phi_c{}^{-6})\right] \ . \tag{4.5}$$

Note that the derivative and inverse mass expansions agree through terms containing two derivatives.

The expansions are now in place, but we have no results which can be used to test them. All of the solitons that have been considered to this point are topological. In a simple scalar theory topological solitons are possible because the scalar potential possesses degenerate minima which are not connected. The topological soliton interpolates between these minima from $x=-\infty$ to $x=+\infty$, and one can easily convince oneself that such a soliton must pass through regions of the scalar potential which have negative curvature. This means that at some point in the soliton, the boson effective mass will vanish and then become imaginary. For such solitons all terms in Eqs. (4.1) and (4.4) which contain inverse powers of this effective mass will diverge, and the expansions will fail. If $\phi_c=0$ lies between the two minima between which the soliton interpolates, the fermion effective mass will vanish at some point, and the expansions given for the fermion loop will also diverge. This happens for all of the solitons considered in the last section.

TABLE I. The fermion-loop quantum correction for models (C)–(E) for several values of G. All quantities are in proper units of the boson mass m.

Model (C)		Model (D)		Model (E)	
$\sqrt{2}G$	ΔE_F	G	ΔE_F	G	ΔE_F
1	0.225	0.05	0.058	0.05	0.022
1.25	0.284	0.30	0.310	0.30	0.096
1.50	0.347	0.55	0.682	0.55	0.179
1.75	0.417	0.80	1.21	0.80	0.282
2	0.492	1.05	1.90	1.05	0.409

When should we expect the expansions given above to converge? The obvious answer is when the effective mass is large, and the field derivatives are small. In order to clarify these ill-defined criteria, let us consider background scalar fields for which the derivative and inverse mass expansions might succeed. For this purpose we do not need fields which satisfy any equation of motion; rather, we want to consider fields whose depth and surface thickness can be freely adjusted. A convenient parametrized choice, which for convenience we will refer to as a soliton, is

$$\phi_c(x)=\phi_0-\frac{\phi_b}{1+\exp[(x^2-R^2)/T^2]} \ . \tag{4.6}$$

Naturally, ϕ_0 should be a minimum of the scalar-field effective potential, to make the one-loop corrections finite.

TABLE II. Boson-loop effective action and approximations. The first three columns show the field parameters. The fourth column presents the value for the effective action divided by an integral over time; while the remaining columns give the various contributions to the derivative and inverse mass expansions. Here U is the one-loop correction to the effective potential, which involves no derivatives.

ϕ_b	R	T	$-E$	$-U$	$O(\partial^2)$	$O(\partial^4)$	$O(M^{-4})$
0.2	10	5	0.240 958	0.241 453	−0.000 504	0.000 009	0.000 010
0.2	7.5	3.75	0.180 438	0.181 090	−0.000 671	0.000 022	0.000 023
0.2	5	2.5	0.119 78	0.120 727	−0.001 007	0.000 074	0.000 079
0.2	2.5	1.25	0.058 685	0.060 363	−0.002 014	0.000 592 0	0.000 634
0.2	1	0.5	0.021 265	0.024 145	−0.005 036	0.009 256	0.009 910
0.2	10	4	0.261 962	0.262 736	−0.000 804	0.000 036	0.000 038
0.2	10	3	0.275 17	0.276 477	−0.001 450	0.000 205	0.000 221
0.2	10	2	0.282 46	0.284 819	−0.003 288	0.002 405	0.002 583
0.2	10	1	0.285	0.289 442	−0.013 211	0.156 968	0.168 558
0.3	10	5	0.555 332	0.556 945	−0.001 648	0.000 038	0.000 047
0.4	10	5	1.052 09	1.058 10	−0.006 24	0.000 26	0.000 50
0.42	10	5	1.192 2	1.201 89	−0.010 43	0.000 87	0.001 91
0.43	10	5	1.274 6	1.290 74	−0.030 52	0.984 02	1.277 77
0.430 39	10	5	1.278 1	1.294 96	−0.415 99	4.6×10^6	5.8×10^6

There is a free mass scale in the problem, which we choose so that $\phi_0=1$, where we assume that a nonzero minimum of the effective potential exists. Infinitely far from the center the soliton will sit at the potential minimum, slowly moving away from this minimum as one moves toward the origin. Since ϕ_0 is assumed to be a minimum, both effective masses will be positive in the asymptotic region of the soliton. By adjusting ϕ_b we can control the depth of the soliton at its center. If the potential possesses an inflection point, as do all of the potentials considered in the last section, ϕ_b can be adjusted so that the soliton approaches this point, causing the effective boson mass to approach zero. The parameter R controls the size of the soliton, while T controls its surface thickness, which is proportional to T^2/R. If we fix R and ϕ_b, T will affect the magnitude of the scalar-field derivatives without necessarily having a drastic effect on the magnitude of the relevant effective mass.

The only remaining choice we need to make is the form of the boson potential, which determines the dependence of the boson effective mass on the background scalar-field strength. A convenient choice is the potential from the case in Sec. III A. This choice leads to

$$U_0=\tfrac{1}{4}(\phi_c{}^2-\phi_0{}^2)^2 , \tag{4.7}$$

$$W=M^2=3\phi_c{}^2-\phi_0{}^2 . \tag{4.8}$$

Note that the effective boson mass will vanish when $\phi_c{}^2=\frac{1}{3}\phi_0{}^2$.

Let us first examine the boson loop. The magnitude of the effective boson mass obviously affects the convergence of both the derivative and the inverse mass expansions. This magnitude can be freely adjusted by scaling ϕ_0 and ϕ_b, leaving R and T unchanged. On dimensional grounds this is equivalent to scaling R and T, leaving ϕ_0 and ϕ_b unchanged. We have chosen $\phi_b=0.2$ for convenience. Table II presents all of our boson calculations, including the results of such a scaling. Figure 2 shows the magnitude of the relative error for various approximations of the full one-boson-loop correction to the soliton mass. As the effective mass is decreased (i.e., as R and T are decreased) all approximations get worse. For large values of the effective mass, the effective potential provides the worst approximation, and even it is accurate to 0.2% initially. Over the entire range the two-derivative term im-

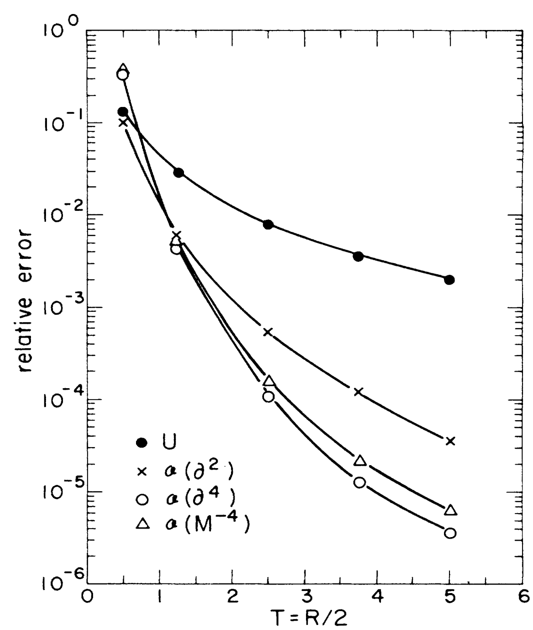

FIG. 2. The magnitude of the relative error for the various approximations to the one boson-loop contribution to the effective action discussed in the text. Here the effect of simultaneously varying both T and R is shown. Lines connecting the calculated points are drawn to roughly indicate the continuous variation of the error ($\phi_0=1$, $\phi_b=0.2$).

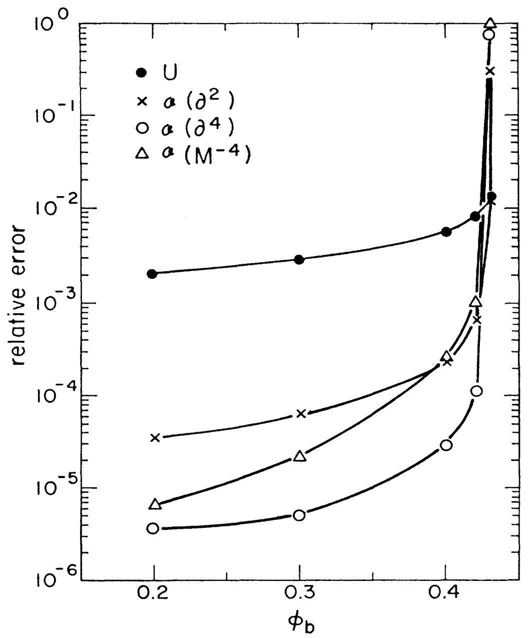

FIG. 3. The same as Fig. 2, but with ϕ_b being varied ($\phi_0=1$, $R=10$, $T=5$).

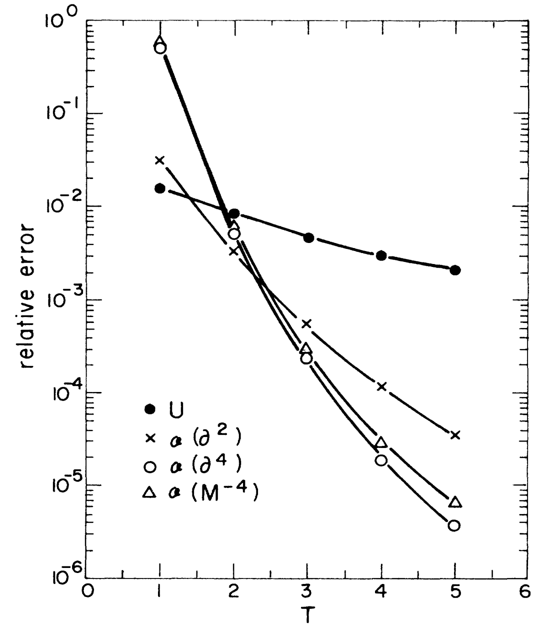

FIG. 4. The same as Fig. 2, but with T being varied alone ($\phi_0=1$, $\phi_b=0.2$, $R=10$).

proves this approximation; although as the magnitude of this correction increases, it starts to overshoot the correct value by increasing amounts. The $O(\partial^4)$ and $O(M^{-4})$ terms improve the approximation for large effective mass. These approximations start out with less than 0.0015% error, but this error rapidly climbs as the boson mass is decreased. It should be pointed out that relative errors less than 10^{-5} may be affected by numerical errors in our estimate of the full one-loop correction to the effective action.

We mentioned above that when the effective mass approaches zero, both expansions will diverge. This behavior is illustrated in Fig. 3, where relative errors are shown as a function of ϕ_b. As ϕ_b is increased, the effective mass in the center of the soliton approaches zero, and one can clearly see that all the approximations diverge away from the correct value as this occurs. Finally, one expects the magnitude of the field derivatives in the surface of the soliton to play an important role in the convergence of both expansions. Ignoring subtleties introduced by trying to consider the volume of the soliton surface, these derivatives are controlled by the magnitude of T. In Fig. 4 one sees each approximation breaking down as T is decreased, with the same pattern as found in Figs. 2 and 3.

The results can be simply summarized by stating that both the derivative and inverse mass expansions seem to be asymptotic in character. Succeeding terms in each approximation oscillate about the correct value, with the last term considered providing an upper bound on the error. By varying the shape of the solitons one can observe each series converge towards and then away from the correct value, with the order of the term which provides the best approximation determined by the shape of the soliton. This means that the sign of the relative error of each approximation remains constant, and this sign changes in successive approximations.

We turn now to the fermion loop. Table III gives all of our fermion results, while Figs. 5–7 show the relative error of various approximations as one varies the same quantities discussed above. Again, both expansions seem to be asymptotic for all field configurations studied. The qualitative behavior one finds for both expansions in this case is quite similar to that seen for the boson loop.

Although we have made no attempt to derive any rigorous criteria with which one can determine when the derivative or inverse mass expansions are converging towards a correct value, we have hopefully clarified the intuitive features of these convergence criteria. It should be clear that precise numerical calculations are an excellent, perhaps necessary, tool for the study of the validity of such approximations.

V. CONCLUSIONS

Solitons are widely studied as interesting structures which can arise from the nonlinearities in field theory, and are widely employed to model physical systems. Most work to date has focused on the classical equations of motion for solitons. The importance of the effective potential for the study of the vacuum is recognized, but the importance of corrections to finite solitons has been studied little. The nonlocalities implied by such corrections usually render analytic investigation impossible, forcing one to turn to numerical study.

One would like to know when a soliton will survive quantum corrections, and how these corrections affect the

TABLE III. Fermion-loop effective action and approximations. The first three columns show the field parameters. The fourth column presents the value for the effective action divided by an integral over time; while the remaining columns give the various contributions to the derivative and inverse mass expansions. Here U is the one-loop correction to the effective potential, which contains no derivatives.

ϕ_b	R	T	E	U	$O(\partial^2)$	$O(\partial^4)$	$O(M^{-4})$
0.2	10	5	0.099 424	0.099 255	0.000 171	-2.8×10^{-6}	-2.9×10^{-6}
0.2	7.5	3.75	0.074 664	0.074 441	0.000 228	-6.7×10^{-6}	-6.9×10^{-6}
0.2	5	2.5	0.049 952	0.049 627	0.000 343	-2.3×10^{-5}	-2.3×10^{-5}
0.2	2.5	1.25	0.025 400	0.024 814	0.000 686	-1.8×10^{-4}	-1.9×10^{-4}
0.2	1	0.5	0.010 996	0.009 925	0.001 714	$-0.002\,839$	$-0.002\,905$
0.2	10	4	0.107 752	0.107 490	0.000 271	-1.1×10^{-5}	-1.1×10^{-5}
0.2	10	3	0.113 242	0.112 798	0.000 486	-6.0×10^{-5}	-6.1×10^{-5}
0.2	10	2	0.116 827	0.116 018	0.001 097	$-0.000\,685$	$-0.000\,702$
0.3	10	5	0.215 572	0.215 146	0.000 433	-8.0×10^{-6}	-8.5×10^{-6}
0.5	10	5	0.549 438	0.547 885	0.001 588	-3.9×10^{-5}	-4.8×10^{-5}
0.7	10	5	0.965 608	0.961 227	0.004 534	$-0.000\,184$	$-0.000\,294$
0.9	10	5	1.375 70	1.362 78	0.014 141	$-0.001\,894$	$-0.004\,728$
1.0	10	5	1.536 13	1.508 96	0.040 675	$-0.152\,852$	$-0.390\,358$

observable properties of the soliton. We have shown a method for computing the one-loop corrections to the energy of a soliton, which can be generalized to include other observables and higher loop corrections. This method relied primarily on the ability to compute a propagator dressed by interactions with a background soliton field, and many observables can be expressed in terms of such Green's functions. The potential accuracy of such numerical methods was demonstrated by comparison to the few analytic results which exist, and in all cases accuracy was primarily limited by computing time.

We have addressed the question of how to accurately estimate the magnitude of one-loop corrections to the energy of a soliton, but we have said little about the reliability of the one-loop approximation itself. In general one has no reason to believe that a loop expansion will converge unless it is governed by a small coupling constant; and even then, the expansion may not converge. If one-loop corrections are to be employed only when they are sufficiently small that their effects on background fields can be ignored, it is necessary only to estimate their magnitude accurately, as we have done in this paper. On the

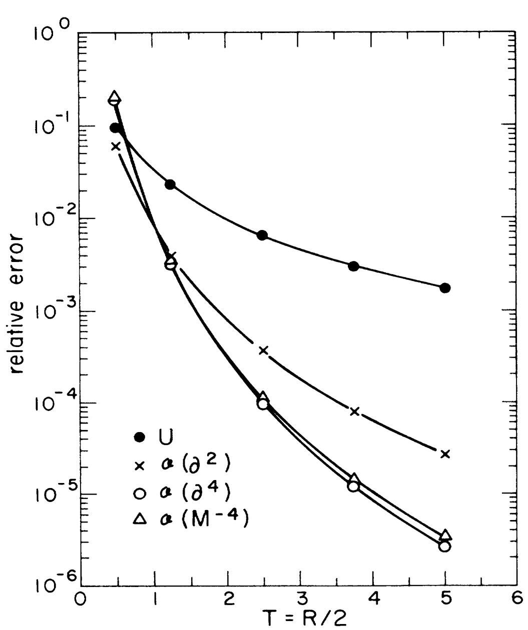

FIG. 5. The magnitude of the relative error for the various approximations to the one fermion-loop contribution to the effective action discussed in the text. Here the effect of simultaneously varying both T and R is shown ($\phi_0=1$, $\phi_b=0.2$).

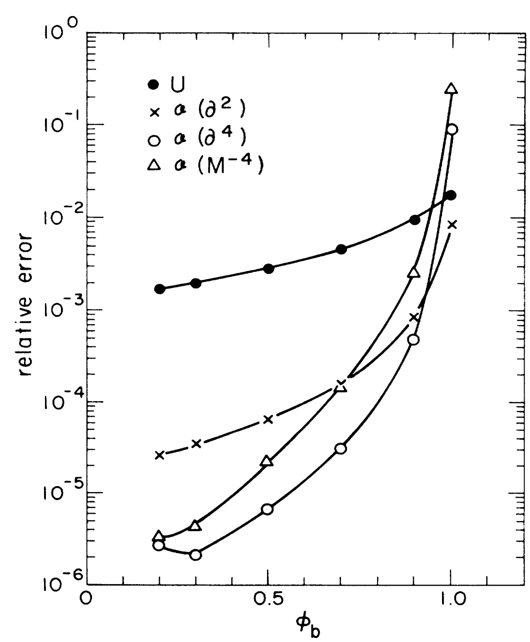

FIG. 6. The same as Fig. 5, but with ϕ_b being varied ($\phi_0=1$, $R=10$, $T=5$).

other hand, there are situations in which a small parameter governs the convergence of the loop expansion, but one-loop effects are important. A relevant example which is similar to cases we have considered here is that of fermions with an unbroken-O(N) symmetry coupled to a scalar boson. In this case the leading terms in an expansion of the effective action in powers of $1/N$ are the classical boson action and the fermion one-loop contribution. When finding the background scalar field to this order, one must minimize the effective action to this order. Such a minimization leads to coupled integro-differential equations, unless a local approximation to the fermion loop is accurate near the minimum. The only sure method for testing this accuracy is the one we employ here: calculate the nonlocal loop correction numerically and use the result to test the approximation.

It is possible that the loop expansion converges asymptotically even when strong coupling constants are employed, in which case one will discover such convergence by calculating quantum corrections. Since such convergence must be questionable, it is necessary to include the effects of quantum corrections in calculating the background fields. On the other hand, the loop expansion may not converge, but other expansions which do not rely on perturbation series in powers of strong coupling constants may be found. This is the case for strong interactions in nonrelativistic many-body theory. Whatever approximation one finds, one will in general be faced with a nonlocal effective action, and issues analogous to those we have considered here will arise.

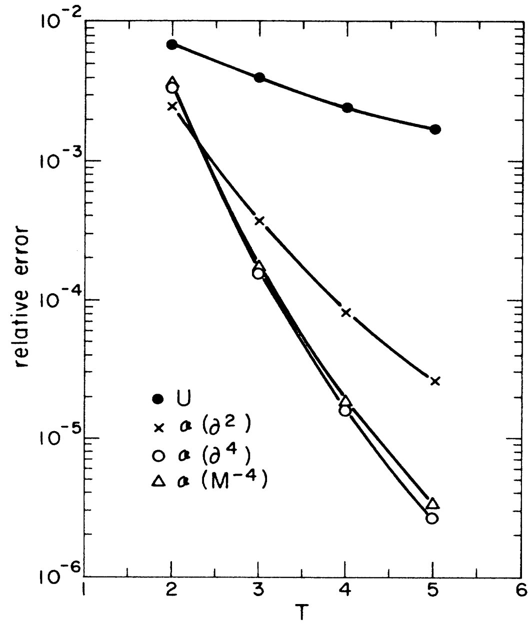

FIG. 7. The same as Fig. 5, but with T being varied alone ($\phi_0 = 1$, $\phi_b = 0.2$, $R = 10$).

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