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Lattice formulation of the superstring

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The formulation of closed superstring theory on a discrete world sheet is studied. We work throughout in the light-cone gauge, and attempt to preserve exact discrete global supersymmetry. It is shown that difficulties arise in the attempt to preserve both supersymmetries in the type-II closed superstring, but that truncation of one-half of the bosonic degrees of freedom leads to an exactly supersymmetric action which is just a lattice implementation of the heterotic string. The continuous-time (Hamiltonian) formalism is also studied for this case.

The recent explosion of interest in superstring theory[1] initiated by the discovery of anomaly cancellations for precisely two gauge groups[2] has focused mainly on questions of formal structure (invariances and covariant quantization,[3] second-quantized formalism,[4] finiteness,[5] etc.) and phenomenological viability.[6] In the latter regard, it is particularly important to settle the questions of (a) compactification and (b) supersymmetry breaking, as the theory as formulated with exact ten-dimensional supersymmetry is clearly unsuitable for a description of known (sub-TeV) physics. It is highly unlikely that these questions can be settled by perturbative calculations—indeed, there are indications[7] that superstrings are intrinsically strongly coupled, and semiclassical calculations would consequently be doomed from the outset.

The options for reliable nonperturbative calculations are extremely limited. Large-N techniques, so useful in other contexts,[8] are unlikely to be of much use in superstring theory as consistency of the theory essentially fixes the number of degrees of freedom from the start. Lattice methods seem to offer much greater hope. Of course, lattice calculations with a second-quantized string field theory (on a ten-dimensional lattice) would be out of the question. Fortunately, string theories can equally well be formulated in first quantization as supersymmetric σ models on the two-dimensional world sheet of the string, and two-dimensional lattice theories are readily treated by a variety of techniques, numerical as well as analytic. The ten- (or 26-) dimensional information is relegated in this approach to the "flavor" structure of the two-dimensional field theory. It is important to realize that the world-sheet (i.e., first-quantized) formulation of string theory is a completely adequate framework for nonperturbative studies—we need only recall the importance of random path formulations of $(\phi^4)_d$ theories[9] in the study of triviality questions, for example, in a rigorous constructive framework.

In the formulation of the string theory on the lattice (first discussed in the purely bosonic case by Giles and Thorn[10]) one inevitably loses reparametrization invariance on the world sheet, so it is clearly reasonable to exploit this invariance ahead of time by going to a completely physical gauge *before* discretizing the theory. Thus, we shall work throughout in the light-cone gauge. In this gauge, supersymmetry is manifest, and our primary concern in this paper will be to preserve as much of this symmetry on the lattice as possible.[11] It will turn out that the free *heterotic* superstring action[12] admits a lattice formulation with exact supersymmetry. When interactions are included (in first quantization, by altering the link structure on the lattice) the supersymmetry is violated at the points where strings break and join and local terms have to be added to restore the symmetry. The question of interactions, as also of compactification, will be discussed in a future publication. Here, we work throughout with free closed superstrings in flat ten-dimensional space-time. At the end we shall make some comments on fermionic doubling and on the feasibility of Monte Carlo simulation of a superstring theory.

We shall begin from the superstring action in a light-cone gauge as formulated by Green and Schwarz:[13]

$$I=\frac{1}{2\pi}\int_{\tau_i}^{\tau_f}d\tau\int_0^{\pi}d\sigma\left(-\partial_\alpha X^a\partial_\alpha X^a+\tfrac{1}{2}\psi^T\rho^0\partial\!\!\!/\psi\right) \quad . \quad (1)$$

The notation is summarized in Ref. 14. A Wick rotation to Euclidean space (on the world sheet) has been made so that the action is negative definite in the bosonic string variables. Also, the Majorana, Weyl, and light-cone constraints have been applied in a specific Dirac representation to reduce the original 64-(complex) component spinor S^{Aa} to ψ^{Aa} ($A=1,2$; $a=1,\ldots,8$). Note that there are still twice as many fermion as boson degrees of freedom in (1), so that the global-supersymmetry algebra of (1) will be realized only on shell. We deal throughout with closed strings only, so $\psi^{Aa}(\sigma+\pi)=\psi^{Aa}(\sigma)$, $X^a(\sigma+\pi)=X^a(\sigma)$.

The proof of the invariance of (1) (up to the usual boundary terms at τ_i and τ_f) under

$$\delta\mathbf{X}=\epsilon\rho^0\boldsymbol{\gamma}\psi \ ,$$
$$\delta\psi=-2\boldsymbol{\gamma}^T\partial\!\!\!/\cdot\mathbf{X}_\epsilon \ , \quad (2)$$

involves (a) integration by parts with respect to σ and τ, (b) commutativity of the derivatives $\partial_\sigma,\partial_\tau$. In order to obtain a discrete version of (1) with an intact global supersymmetry, we must therefore choose lattice derivatives compatible with these manipulations. One is therefore led [if one insists on local differences, but see comment (3) below] to use symmetric derivatives defined as follows (i,j are spatial and temporal site labels on the lattice world sheet):

$$\Delta_\sigma \mathbf{X}_{ij} = \tfrac{1}{2}(\mathbf{X}_{i+1\,j} - \mathbf{X}_{i-1\,j}) \;,$$
$$\Delta_\tau \mathbf{X}_{ij} = \tfrac{1}{2}(\mathbf{X}_{i\,j+1} - \mathbf{X}_{i\,j-1}) \;. \tag{3}$$

After separating the upper ($A=1$) and lower ($A=2$) components of ψ^{Aa} in the manner of Susskind[15] (namely, $\psi^{1a} \to \psi^a_{ij}$ for i odd, $\psi^{2a} \to \psi^a_{ij}$ for i even—note that the asymmetry between i and j follows from the Majorana property of ψ which prevents us from treating $\psi^T \rho_0$ and ψ independently) one is led directly to a discrete action

$$I_{\text{latt}} = \frac{1}{2\pi} \sum_{ij} [-(\Delta_\tau \mathbf{X}_{ij})^2 - (\Delta_\sigma \mathbf{X}_{ij})^2 + \psi_{ij} \Delta_\tau \psi_{ij} - i\psi_{ij} \Delta_\sigma \psi_{ij}] \tag{4}$$

(factors of lattice spacing have been absorbed in field redefinitions). At first sight, the action (4) does indeed appear to have a full $N=2$ supersymmetry, namely, invariance under the upper supersymmetry (corresponding to ϵ^{1a}),

$$\delta \mathbf{X}_{ij} = \begin{cases} -i\epsilon^1 \boldsymbol{\gamma} \psi_{ij}, & i \text{ even} \;, \\ 0, & i \text{ odd} \;, \end{cases}$$
$$\delta \psi_{ij} = \begin{cases} -i\Delta_\tau \mathbf{X}_{ij} \cdot \boldsymbol{\gamma}^T \epsilon^1, & i \text{ even} \;, \\ \Delta_\sigma \mathbf{X}_{ij} \cdot \boldsymbol{\gamma}^T \epsilon^1, & i \text{ odd} \;, \end{cases} \tag{5a}$$

as well as the lower supersymmetry (corresponding to ϵ^{2a})

$$\delta \mathbf{X}_{ij} = \begin{cases} i\epsilon^2 \boldsymbol{\gamma} \psi_{ij}, & i \text{ odd} \;, \\ 0, & i \text{ even} \;, \end{cases}$$
$$\delta \psi_{ij} = \begin{cases} i\Delta_\tau \mathbf{X}_{ij} \cdot \boldsymbol{\gamma}^T \epsilon^2, & i \text{ odd} \;, \\ -\Delta_\sigma \mathbf{X}_{ij} \cdot \boldsymbol{\gamma}^T \epsilon^2, & i \text{ even} \;. \end{cases} \tag{5b}$$

The action (4) is also trivially invariant under a shift of ψ_{ij} by a constant Grassmann spinor, which is the other part of the original ten-dimensional supersymmetry when decomposed in terms of SO(8) spinors.

The explicit computation of the free closed superstring propagator by path integration over the action (4) leads, however, to an unpleasant surprise—the boson and fermion self-energies do not cancel (the former giving precisely twice the latter) so the theory contains a tachyon. The complete calculation will not be presented here, as the bosonic part is already treated in great detail in the work of Giles and Thorn.[10] Here we outline only the calculation of the fermionic part. Recall from Ref. 10 that the zero-point energy is related to the determinant of the quadratic form coupling the fields in (4) in the limit $N \to \infty$, where we work on a lattice with $i=1,\ldots,M$ and $j=1,\ldots,N$. For the fermionic degrees of freedom this determinant may be computed as follows. After Fourier transforming

$$\psi_{ij} = \frac{1}{\sqrt{M}}[\phi_{j0} + (-)^i \phi_{j\,M/2}] + \left(\frac{2}{M}\right)^{1/2} \sum_{m=1}^{M/2-1} \left[\phi^c_{jm} \cos\left(\frac{2m\pi}{M}(i-\tfrac{1}{2})\right) + \phi^s_{jm} \sin\left(\frac{2m\pi}{M}(i-\tfrac{1}{2})\right)\right] \;, \tag{6}$$

the fermionic action becomes

$$I_{\text{ferm}} = \frac{1}{4\pi} \sum_j \sum_{m=1}^{M/2-1} \left[\phi^c_{jm}(\phi^c_{j+1\,m} - \phi^c_{j-1\,m}) + (c \to s) - 4i \sin\left(\frac{2m\pi}{M}\right) \phi^c_{jm} \phi^s_{jm}\right] \;, \tag{7}$$

where the $\phi_{j0}, \phi_{jM/2}$ variables, which give no contribution to the zero-point energy, have been dropped. We may therefore calculate the path integral over ϕ^c_{jm}, ϕ^s_{jm} for each m, $1 \le m \le M/2-1$, separately. Defining $\alpha_m = -2i\sin(2m\pi/M)$ and dropping the m (and internal a) index temporarily, it is easily shown that

$$\int \prod_{j=1}^{N} d\phi^c_j d\phi^s_j e^{I_{\text{ferm}}} \propto \det(A + i\alpha_m) \;, \tag{8}$$

where A is the $N \times N$ matrix, $A_{jk} = \delta_{j,k-1} - \delta_{j,k+1}$, with eigenvalues $2i\sin[j\pi/(N+1)]$, $j = -(N-1)/2, \ldots, +(N-1)/2$. After some algebra, one finds

$$\det(A + i\alpha_m) = \prod_{j=1/2}^{N-1/2} \left[4\sin^2\left(\frac{j\pi}{N+1}\right) - \alpha_m{}^2\right] = \frac{\cosh\{(N+1)\operatorname{arcsinh}[\sin(2m\pi/M)]\}}{\cosh\{\operatorname{arcsinh}[\sin(2m\pi/M)]\}} \tag{9}$$

so

$$\prod_m \det(A + i\alpha_m) \underset{N\to\infty}{\sim} \exp\left\{N \sum_m \operatorname{arcsinh}\left[\sin\left(\frac{2m\pi}{M}\right)\right]\right\} \;.$$

One may read off directly the contribution of the bosonic degrees of freedom from the results of Ref. 10. One finds

$$\int \prod dX_{ij} e^{I_{\text{bos}}} \underset{N\to\infty}{\sim} \exp\left\{-2N \sum_m \operatorname{arcsinh}\left[\sin\left(\frac{2m\pi}{M}\right)\right]\right\} \tag{10}$$

so the bosonic zero-point energy overpowers (by exactly a factor of 2) the fermionic, despite (5a) and (5b).

The resolution of this apparent paradox lies in the observation that there exist (different) decoupled Bose degrees of freedom with respect to each of the supersymmetries (5a) and (5b). For example, the bosonic fields on the odd sites are invisible to the upper supersymmetry (5a), and similarly for the even sites with respect to (5b). Taken together, the two supersymmetries (5a) and (5b) of course act on all the fields, but the cancellation of Bose and Fermi energies would require a single supersymmetry acting freely.

The simplest solution to this difficulty is just to abandon one of the supersymmetries [say, the lower one (5b)] by setting $X_{ij}=0$ for i odd. The upper supersymmetry is then exact, while the exponent in (10) is reduced by precisely the factor of 2 needed to ensure a zero squared-mass ground state. This is, of course, not a standard type-I superstring,[1] as the upper and lower supersymmetries are not treated symmetrically. Remarkably, as we now show, it is precisely a lattice implementation of the ten-dimensional part of the *herterotic* superstring.

The heterotic string action[12] (our normalization agrees

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with Schwarz[1]) is ($a=1,\ldots,8$; $I=9,\ldots,24$)

$$I_{\rm het}=-\frac{1}{2\pi}\int d\sigma\, d\tau\left[\partial_\alpha X^a\partial^\alpha X^a+\partial_\alpha X^I\partial^\alpha X^I - \frac{i}{2}\psi(\partial_\tau-\partial_\sigma)\psi\right] , \quad (11)$$

where the X^I satisfy the constraint $(\partial_\tau+\partial_\sigma)X^I=0$, $I=9,\ldots,24$. To establish the connection with (4), we have adopted a different convention for the world-sheet Dirac matrices: thus $\rho^0=\sigma_2$, $\rho^1=-i\sigma_1$. Now, the X^I variables are only relevant to the propagation amplitude at interaction points, as the insertion of the constraint $(\partial_\tau+\partial_\sigma)X^I=0$ in (11) causes the second term to vanish (the relevant overlap integral at an interaction time is readily computed analytically and may be inserted "by hand" in a Monte Carlo simulation of the system, for example). Thus we shall henceforth drop the second term in (11), which is in any case totally unaffected by the ten-dimensional supersymmetry of the right-moving sector. Introducing lattice spacings a_0, a_1 in the τ and σ directions, respectively, one finds the following discrete version of (11), after rotation to Euclidean space:

$$I_{\rm het,\,latt}=-\frac{a_0a_1}{\pi}\sum_{ij}\left[\frac{1}{a_0^2}(\Delta_\tau\mathbf{X}_{ij})^2+\frac{1}{a_1^2}(\Delta_\sigma\mathbf{X}_{ij})^2 - \tfrac{1}{4}\psi_{ij}\left(\frac{1}{a_0}\Delta_\tau-\frac{i}{a_1}\Delta_\sigma\right)\psi_{ij}\right] , \quad (12)$$

which is equivalent to (4) up to a change in normalization. In (12) the X_{ij} variables are only defined for even i. This action has the exact (on-shell) discrete supersymmetry

$$\begin{aligned}\delta_\epsilon\mathbf{X}_{ij}&=i\epsilon\boldsymbol{\gamma}\psi_{ij},\quad i\ \text{even} ,\\ \delta_\epsilon\psi_{ij}&=4i\left(\frac{1}{a_0}\Delta_\tau+\frac{i}{a_1}\Delta_\sigma\right)\mathbf{X}_{ij}\boldsymbol{\gamma}^\tau\epsilon .\end{aligned}\quad (13)$$

The verification of the commutator algebra (inserting the lattice equations of motion) will be left to the reader: We discuss it explicitly below in the Hamiltonian limit ($a_0\to 0$) where the Noether charges Q^{Aa} are given explicitly in operator form.

Note that the action (12) contains two decoupled bosonic fields: The use of the symmetric lattice derivative Δ_τ means that X_{ij} for j even and odd decouple. This is really a symptom of fermion doubling, now transferred to the bosonic sector via supersymmetry (more on this below).

The Hamiltonian limit is obtained by taking $a_0\to 0$ in (12) (rotated back to real τ) and canonically quantizing the resulting action. Namely,

$$I_{a_0=0}=-\frac{a_1}{\pi}\int d\tau\left[\sum_{i\ \rm odd}\frac{1}{a_1^2}(\Delta_\sigma\mathbf{X}_i)^2-\sum_{i\ \rm even}\dot{\mathbf{X}}_i^2 - \frac{i}{4}\sum_i\psi_i\left(\frac{\partial}{\partial\tau}-\frac{1}{a_1}\Delta_\sigma\right)\psi_i\right] \quad (14)$$

leads to the Hamiltonian

$$H=\frac{1}{\pi}\sum_i\left(\frac{1}{a_1}(\Delta_\sigma\mathbf{X}_i)^2+\frac{\pi^2}{4a_1}\mathbf{p}_i^2+\frac{i}{4}\psi_i\Delta_\sigma\psi_i\right) \quad (15)$$

with the canonical commutation relations (CCR's)

$$\begin{aligned}[p_i^a,X_j^b]&=-i\delta^{ab}\delta_{ij} ,\\ \{\psi_i^a,\psi_j^b\}&=\frac{2\pi}{a_1}\delta^{ab}\delta_{ij} .\end{aligned}\quad (16)$$

The action (14) is invariant under

$$\begin{aligned}\delta\mathbf{X}_i&=\frac{i}{\sqrt{p^+}}\epsilon^{(+)}\boldsymbol{\gamma}\psi_i ,\\ \delta\psi_i&=4\sqrt{2}i\sqrt{p^+}\epsilon^{(-)}\delta_{i,\,\rm even}-\frac{4}{\sqrt{p^+}}\left(\dot{\mathbf{X}}_i+\frac{1}{a_1}\Delta_\sigma\mathbf{X}_i\right)\cdot\boldsymbol{\gamma}^\tau\epsilon^{(+)} ,\end{aligned}\quad (17)$$

where we have restored the second SO(8) spinor supersymmetry parameter $\epsilon^{(-)}$ coming from the original SO(9,1) supersymmetry. The transformations (17) are generated (up to an unpleasant overall numerical factor) by the operator charge

$$\bar\epsilon Q=\frac{i2^{1/4}a_1\sqrt{p^+}}{\pi}\sum_{i\ \rm even}\epsilon^{(-)}\psi_i - \frac{2^{1/4}}{\pi\sqrt{p^+}}\sum_i\epsilon^{(+)}\boldsymbol{\gamma}\psi_i\left(\Delta_\sigma\mathbf{X}_i+\frac{\pi}{2}\mathbf{p}_i\right) . \quad (18)$$

A straightforward computation, using the CCR's in (16), gives the desired supersymmetry algebra

$$\begin{aligned}[\bar\epsilon Q,\bar\epsilon' Q]&=\sqrt{2}p^+\epsilon^{(-)}\epsilon^{(-)'}+\sqrt{2}\frac{H}{p^+}\epsilon^{(+)}\epsilon^{(+)'} + i\,\mathbb{P}\cdot(\epsilon^{(+)}\boldsymbol{\gamma}\epsilon^{(-)'}+\epsilon^{(-)}\boldsymbol{\gamma}\epsilon^{(+)'})\\ &=2\bar\epsilon\gamma^\mu\mathbb{P}_\mu\rho^0\epsilon' .\end{aligned}\quad (19)$$

In the last line of (19), we return to SO(9,1) spinor notation, with $\epsilon^{1a}=0$, $\epsilon^{2a}=(\epsilon^{(+)},\epsilon^{(-)},0,0)$. $\mathbb{P}_\mu$ is the generator of translations in D- ($=10$) dimensional space-time, so $\mathbb{P}^-=H/p^+$ (since H generates translations in τ, and $X^+=x^++p^+\tau$ in the light-cone gauge), and $\mathbb{P}^a=\sum_i p_i^a$ translates the string transversally. Thus, the entire supersymmetry algebra is exactly reproduced at the discrete level. Moreover, we now manifestly have only a single propagating bosonic string, in contrast with the situation discussed previously for discrete time, so that boundary conditions are specified only on the initial and final time slice. As a consequence of the usual difficulties with chiral lattice fermions, however, the model is still doubled [see comment (3) below].

There are several important issues which have only been discussed in passing, or not at all. As regards the free superstring, the following points deserve further study (and will be treated in detail in a forthcoming publication).

(1) The analysis presented above is manifestly applicable as long as the X^a variables describe a flat manifold (e.g., toroidal compactifications are allowed). Phenomenologically interesting compactifications[6] will involve the full, generally coordinate-invariant form of the supersymmetric model,[16] containing, for example, a quartic fermionic term, and this action needs to be formulated on the lattice in a supersymmetric way.

(2) The reader may find the absence of bosonic variables

on one-half of the lattice sites aesthetically displeasing. We emphasize that this is due to the on-shell formalism in which there are twice as many fermionic as bosonic fields. Clearly, the auxiliary field could be reintroduced, presumably in such a way that bosonic degrees of freedom appear on all lattice sites.

(3) The theory at the Hamiltonian level[15] is, in fact, doubled and contains both left and right movers: Thus, as follows directly from the equations of motion, the fields $(1/a_1)\Delta_\sigma X_i+\dot{X}_i$ and ψ_i are right moving, while $(1/a_1)\Delta_\sigma X_i-\dot{X}_i$ and $(-1)^i\psi_i$ are left moving. So, in fact, we have a duplication[17] of the ten-dimensional part of the heterotic superstring, in accordance with the well-known veto of chiral fermions with local lattice derivatives. However, string interactions inevitably introduce nonlocality on the world sheet, and doubling is known to be avoided if one used nonlocal "SLAC" derivatives,[18] which, in fact, satisfy the contraints discussed following (2). The formulation of the theory in terms of such derivatives will be discussed in a future publication.

In relation to the interacting superstring, the following issues arise.

(4) The joining and breaking of closed strings will lead to violations of the supersymmetry of (4) at the interaction points. For example, if we define Δ_σ always within a single closed string (to maintain the integration-by-parts property) it is easy to see that Δ_σ and Δ_τ then fail to commute at interaction points [recall comments following (2)]. It may be possible to maintain the supersymmetry at the interacting level by adding contact terms to absorb the variation, or it may even be that the violation is "soft" (unimportant in the continuum limit).

(5) A Monte Carlo simulation of the system with interactions may be done by introducing a link field $l_{ss'}$ (s,s' are site variables) taking values 0,1 and satisfying certain topological constraints.[10] In our view, the prospect of studying superstring dynamics nonperturbatively in terms of two-dimensional lattice theories which may be simulated to high precision with present computing facilities is an extremely exciting one.

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$$\gamma^0=i\begin{pmatrix}0 & -\mathbb{1}_{16}\\ \mathbb{1}_{16} & 0\end{pmatrix},$$

$$\gamma^a=i\begin{pmatrix}0 & 0 & \mathbb{1}_8 & 0\\ 0 & 0 & 0 & -\mathbb{1}_8\\ \mathbb{1}_8 & 0 & 0 & 0\\ 0 & -\mathbb{1}_8 & 0 & 0\end{pmatrix},$$

$$\gamma^a=\begin{pmatrix}0 & \Gamma^a\\ \Gamma^a & 0\end{pmatrix}\quad(a=1,8)$$

with

$$\Gamma^a=\begin{pmatrix}0 & \gamma_8^a\\ \gamma_8^{aT} & 0\end{pmatrix}.$$

The 8×8 matrices γ_8^a are the only ones appearing in the text, and we drop throughout the subscript. They satisfy the Fierz identity $(\gamma^a)^{AB}(\gamma^a)^{CD}+(\gamma^a)^{AD}(\gamma^a)^{CB}=-2\delta^{AC}\delta^{BD}$. For the Dirac matrices on the world sheet, we use initially $\rho^0=\sigma_2$, $\rho^1=i\sigma_3$, but switch later to $\rho^0=\sigma_2$, $\rho^1=-i\sigma_1$, when the model is reinterpreted as heterotic.

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