Conditions for nondegeneracy in supersymmetric quantum mechanics

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It is shown that the positive "bosonic" and "fermionic" bound-state spectra in spherically symmetric supersymmetric (SUSY) quantum mechanics are degenerate if and only if the superpotential $W(r)$ satisfies $\delta \equiv \lim_{r\to 0} r|W(r)| \geq 0.5$. Also, if $\delta < 0.5$, then SUSY is broken.

Recently, Roy and Roychoudhury[1] constructed two examples which violate the standard expectation of degeneracy of bound-state energy levels between two supersymmetric partner potentials $V_\pm$.[2,3] Both their examples are equivalent to the consideration of the supersymmetric quantum mechanics of the s-wave reduced radial Schrödinger equation in three spatial dimensions, where all wave functions are required to vanish at the origin. The "bosonic" and "fermionic" potentials $V_\pm(r)$ are generated from the superpotential $W(r)$ in the usual manner: $V_\pm(r) = W^2(r) \mp W'(r)$. For the choices $W(r) = \mu r$ and $W(r) = A \tanh(r)$, it was found[1] that the spectra of V_+ and V_- were nondegenerate. In this Comment, we first give a simple qualitative understanding of these (and similar) phenomena and then determine precise conditions on $W(r)$ under which nondegeneracy occurs.

We denote the reduced radial Hamiltonians for the potentials $V_\pm(r)$ by $H_\pm$. In terms of the operators $Q^\pm = \mp d/dr + W(r)$ we have $H_\pm = Q^\pm Q^\mp = -d^2/dr^2 + V_\pm(r)$ (note that we have taken $\hbar = 2m = 1$). The potentials $V_\pm(r)$ represent the complete effective potentials for the partner Schrödinger equations: i.e.,

$$V_\pm(r) = l_\pm(l_\pm + 1)/r^2 + \hat{V}_\pm(r) \ .$$

The spectra of $H_\pm$ are positive semidefinite, and if either of $H_\pm$ possesses a zero-energy bound state then we say that supersymmetry (SUSY) is unbroken; otherwise SUSY is spontaneously broken.[2,3] The standard "degeneracy theorem" states that $H_\pm$ have identical spectra except for possible zero-energy states. The usual proof goes as follows. If ψ_n^+ is an eigenstate of H_+ with eigenvalue $E_n > 0$, i.e., $H_+\psi_n^+ = E_n\psi_n^+$, then $Q^-\psi_n^+$ is an eigenstate of H_- with the same eigenvalue:

$$H_-(Q^-\psi_n^+) = Q^-Q^+Q^-\psi_n^+ = Q^-H_+\psi_n^+ = E_n(Q^-\psi_n^+)$$

(note that this construction does not work if $E_n = 0$ since $Q^-\psi_n^+ = 0$ for this case). This proof implicitly assumes that if ψ_n^+ satisfies the correct "physical" boundary conditions at $r=0$ (Ref. 4) and $r = \infty$, then so does $Q^-\psi_n^+$. Indeed, if $\psi_n^+ \to 0$ as $r \to \infty$, then $Q^-\psi_n^+$ does also. Moreover, the fact that ψ_n^+ is normalizable implies that $Q^-\psi_n^+$ is normalizable. However, this is not always true for the vanishing of the wave function at the origin. That is, the fact that $\psi_n^+ \to 0$ (in the proper manner[4]) as $r \to 0$, does not in general force $Q^-\psi_n^+$ to do so. In fact, the examples of Ref. 1 are an illustration of this possibility. They can be simply understood in the following manner. Dealing with spherically symmetric partner potentials $V_\pm(r)$ is equivalent to use of only half of symmetric *one-dimensional* SUSY partner potentials defined by $\tilde{V}_\pm(x) = \tilde{V}_\pm(-x) = V_\pm(r = x)$. Any antisymmetric eigenstate of one of these potentials is paired in energy with a symmetric eigenstate of the other. For the potentials $V_\pm(r)$ of Ref. 1, the antisymmetric states of the "extended potentials" $\tilde{V}_\pm(x)$ have $\psi_A(0) = 0$ while the symmetric states obey $\psi_s(0) = C \neq 0$. By considering the spectra of $V_\pm(r)$ we are requiring all acceptable wave functions to vanish at the origin which corresponds to ruling out the symmetric states of $\tilde{V}_\pm(x)$ as unphysical for the spherically symmetric case. Clearly this leads to the nondegeneracy of the energy levels of $V_\pm(r)$.

The above discussion suggests the following picture for general $V_\pm(r)$.

(1) If $V_\pm(r)$ contain terms as $r \to 0$ which are "singular enough" so as to force all normalizable solutions of the extended potentials $\tilde{V}_\pm(x)$ to vanish (fast enough[4]) at the origin, then the "half-potentials" $V_\pm(r)$ will maintain degeneracy.

(2) If neither $V_\pm(r)$ contain terms which are sufficiently singular near $r = 0$ to force all normalizable solutions of *either* $\tilde{V}_\pm(x)$ to vanish at the origin, then all even wave functions of $\tilde{V}_\pm(x)$ will go to a nonzero constant as $x \to 0$ and hence are unacceptable as physical states of $V_\pm(r)$ (such as for the potentials of Ref. 1). As a result, the degeneracy theorem will not hold for the spherically symmetric case. Also, since the ground states of $\tilde{V}_\pm(x)$ are even and we have $E \geqq 0$, $V_\pm(r)$ have no physical zero-energy states (i.e., SUSY is broken).

We will find that these two cases do not exhaust all possibilities for partner potentials $V_\pm(r)$. One more situation must be considered.

(3) $V_\pm(r)$ are such that two small-x behaviors are allowed for the normalizable solutions of $\tilde{V}_\pm(x)$, both of which make them vanish as $x \to 0$. However, only one of these is acceptable for physical states of $V_\pm(r)$.[4,5] Here we have no way of knowing which behavior belongs to even states and which to odd states of $\tilde{V}_+(x)$ and $\tilde{V}_-(x)$. Two situations can occur.

(a) The unacceptable solutions of one potential are paired in energy with the acceptable solutions of the other. This is similar to situation (2) above. For $V_\pm(r)$, the degeneracy theorem does not hold and SUSY is broken.[6]

(b) The unacceptable solutions of both potentials are paired in energy. Clearly the degeneracy theorem holds for $V_\pm(r)$.

We now state precise theorems concerning the superpotential $W(r)$ which determine nondegeneracy and SUSY breaking, i.e., which quantify terms used above, such as "singular enough."

Theorem 1. A necessary and sufficient condition for the degeneracy theorem to hold in spherically symmetric SUSY quantum mechanics is

$$\delta = \lim_{r \to 0} r |W(r)| \geq 0.5 \ .$$

Theorem 2. If $\delta < 0.5$, then there is no physical zero-energy state of H_+ or H_- (SUSY is broken).

A trivial corollary of Theorems 1 and 2 follows.

Corollary. If SUSY is unbroken (i.e., there exists a physical $E=0$ state), then the degeneracy theorem holds.

The proof of the above theorems is based on examining the behavior of ψ_n^+ and $Q^- \psi_n^+$ as $r \to 0$ (since, as previously mentioned, nothing goes wrong at large r). Since this proof is rather tedious for general $W(r)$, we illustrate the ideas for the case where $W(r) \sim A r^p (1 + B r^\epsilon + \cdots)$, $\epsilon > 0$, as $r \to 0$.

Proof of Theorem 1.

Part I. Necessity.

$$\delta = \lim_{r \to 0} r |W(r)| = \lim_{r \to 0} |A| r^{p+1} \ .$$

We want to show that if ψ_n^+ is any physical eigenstate of H_+ $(H_+ \psi_n^+ = E_n \psi_n^+)$ and $\delta < 0.5$, then $Q^- \psi_n^+$ is *not* a physical eigenstate of H_-. Note that this actually proves the stronger statement: If $\delta < 0.5$, then the spectra of $H_\pm$ are *completely* nondegenerate.

Case I. $\delta = 0$ $(p > -1)$.

This case contains an infinite class of problems similar to those in Ref. 1 [corresponding to situation (2) above]. For such a problem $\psi_n^+ \sim Cr$ as $r \to 0$. Then $Q^- \psi_n^+ = d\psi_n^+ / dr + W \psi_n^+$ goes to C as $r \to 0$ and is not an acceptable state of H_-.

Case II. $0 < \delta < 0.5$ $(p = -1,\ 0 < |A| < 0.5,\ \delta = |A|)$.

For this case $\psi_n^+ \sim r^{1+A}$ as $r \to 0$ and $Q^- \psi_n^+ \sim r^A$, which is again unacceptable.[4] This case corresponds to situation (3)(a) above.

Part II. Sufficiency.

Here we want to show that if ψ_n^+ is any physical eigenstate of H_+ and $\delta \geq 0.5$, then $Q^- \psi_n^+$ is a physical eigenstate of H_-.

Case I. $0.5 \leq \delta < \infty$ $(p = -1,\ \delta = |A|)$.

Here $\psi_n^+ \sim r^{1+A}$ (r^{-A}) as $r \to 0$ for $A \geq 0.5$ (≤ -0.5) and $Q^- \psi_n^+ \sim r^A$ (r^{1-A}), which is acceptable. [$\delta = 0.5$ and $\delta > 1$ correspond to situation (1), $0.5 < \delta < 1$ yields situation (3)(b).[7]]

Case II. $\delta = \infty$ $(p < -1)$.

Here both ψ_n^+ and $Q^- \psi_n^+$ go to zero as $r \to 0$ faster than any power of r.[8] (Situation 1.)

This completes the proof of Theorem 1.

Proof of Theorem 2. The zero-energy solutions of $H_\pm$ are

$$\psi_{E=0}^{\pm} \sim \exp\left[\mp \int^r W(r) dr \right] \ .$$

We wish to show that for $\delta < 0.5$, neither of these are physical solutions.

Case I. $0 < \delta < 0.5$ $(p = -1,\ \delta = |A|)$.

Here $\psi_{E=0}^{\pm} \sim r^{\mp A}$ both of which are not acceptable. [Situation (3)(a).]

Case II. $\delta = 0$ $(p > -1)$.

Both $\psi_{E=0}^{\pm}$ go to nonzero constants as $r \to 0$ and hence are unacceptable. (Situation 2.)

This completes the proof of Theorem 2. We can also show that if $\delta \geq 0.5$, then one of $\psi_{E=0}^{\pm}$ is acceptable as $r \to 0$. However, we *cannot* conclude from this that there exists a physical zero-energy state since we do not know about the behavior of $\psi_{E=0}^{\pm}$ at large r.

It should be noted here that for most problems of interest one wishes to generate potentials (which we take to be bosonic) of the form

$$V_+(r) = l(l+1)/r^2 + V(r) \quad (l = 0, 1, 2, \ldots) \ ,$$

with $V(r)$ satisfying $\lim_{r \to 0} r^2 V(r) = 0$, and study their fermionic partners. In such cases, the only choices for the behavior of the superpotential $W(r)$ as $r \to 0$ are $W(r) \sim -(l+1)/r$ or $+l/r$. Theorem 1 tells us that we are assured degeneracy of the bosonic and fermionic spectra regardless of which choice we make except for the case $l = 0$, where in order to get degeneracy we must choose the former behavior.[9,10]

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[1]P. Roy and R. Roychoudhury, Phys. Rev. D **32**, 1597 (1985).

[2]E. Witten, Nucl. Phys. **B188**, 513 (1981).

[3]For a review, see, for example, F. Cooper and D. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).

[4]We require a *physical solution* of the reduced radial Schrödinger equation to vanish as $r \to 0$ at least as fast as $r^{0.5}$ (in addition to vanishing at large r). This guarantees that the *full* Hamiltonian is self-adjoint with respect to the space of its physical solutions (or, equivalently, that such solutions have finite energy). This is in contrast with the case of a *one-dimensional* Hamiltonian with a potential $V(x)$ where imposition of the above finite-energy condition again forces all physical solutions of the Schrödinger equation to vanish as $x \to 0$ at least as fast as $x^{0.5}$, except in the case when $\lim_{x \to 0} x V(x) = 0$, where physical solutions which go to a nonzero constant (with zero slope) at $x = 0$ are also allowed.

[5]Such states are not even physical solutions of the one-dimensional potentials $\tilde{V}_\pm(x)$ (see Ref. 4).

[6]In this case, there is no degeneracy even between the physical spectra of $\tilde{V}_\pm(x)$; i.e., the degeneracy theorem does not hold even in the one-dimensional SUSY system.

[7]The case of $\delta = 1$ is interesting. For this case [which does not correspond to any of situations (1)–(3)] the one-dimensional SUSY system $\tilde{V}_\pm(x)$ in general admits *negative*-energy states, and normalizable solutions of one potential may be "paired" by the degeneracy theorem with non-normalizable solutions of the other. This happens because in this situation the (one-dimensional) operators $Q^\pm$ are not adjoints of each other in the manner which is assumed in the proof of the non-negativity of the energy and the pairing of normalizable states. However, these problems occur in such a way as to yield a sensible SUSY system when one considers the half-potentials $V_\pm(r)$.

[8]The case $\delta = \infty$ corresponds to so-called repulsive "singular" potentials $V_\pm(r)$ [see W. Frank, D. Land, and R. Spector, Rev. Mod. Phys. **43**, 36 (1971)]. It is interesting that one can never generate attractive singular potentials in SUSY quantum mechanics. Such potentials lead to problems of physical interpretation (see Frank, Land, and Spector).

[9]It is also common to use a superpotential constructed from the

physical ground-state wave function ψ_0 of a given Hamiltonian H_+, $W(r)=\psi_0'/\psi_0$. This is often called the "method of factorization" {see, e.g., A. Adrianov, N. Borisov, and M. Ioffe, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 78 (1984) [JETP Lett. **39**, 93 (1984)], and references therein} or the "ground-state wave-function representation" [E. Gozzi, Phys. Lett. **129B**, 432 (1983)]. For such systems, one is assured that the degeneracy theorem holds for $V_\pm(r)$ (i.e., $\delta \gtrsim 0.5$), and that SUSY is unbroken.

[10]Connections between supersymmetry breaking and physically acceptable states have also been discussed by A. Jevicki and J. P. Rodrigues, Phys. Lett. **146B**, 55 (1984) and J. Fuchs, J. Math. Phys. **27**, 349 (1986).