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Perturbation theory for an anharmonic oscillator

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We show that a Rayleigh-Schrödinger-type perturbation theory for an anharmonic oscillator can be developed by suitable modification of the results of Hsue and Chern. Energy eigenvalues up to third order are calculated. The results show a distinct improvement over the results of standard perturbation theory.

Recently, Hsue and Chern[1] have developed an elegant method to determine accurate energy eigenvalues for an anharmonic oscillator. Unfortunately, their final results can only be obtained numerically. Here we show that with suitable manipulation their results can be used to develop a Rayleigh-Schrödinger-type perturbation theory.

Hsue and Chern start with the Hamiltonian given by

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}x^2+\lambda x^4 \quad , \tag{1}$$

which, in terms of creation and annihilation operators $a^\dagger$ and a, becomes

$$H=a^\dagger a+\tfrac{1}{2}+\tfrac{1}{4}\lambda(a+a^\dagger)^4 \quad . \tag{2a}$$

By defining a new ground state

$$|\psi_0\rangle=\exp(\tfrac{1}{2}ta^{\dagger 2})|0\rangle \quad , \tag{2b}$$

and a new set of creation and annihilation operators $b^\dagger$ and b given by

$$b^\dagger=\frac{(a^\dagger-ta)}{(1-t^2)^{1/2}} \quad , \tag{3}$$

$$b=\frac{(a-ta^\dagger)}{(1-t^2)^{1/2}} \quad , \tag{4}$$

with

$$b|\psi_0\rangle=0 \tag{5}$$

and

$$[b,b^\dagger]=1 \quad , \tag{6}$$

the Hamiltonian was rewritten in terms of b and $b^\dagger$ and the ground-state energy was obtained by taking the variation of t. The final results are

$$E_0=\frac{1+3w^2}{8w} \quad , \tag{7}$$

$$H=E_0+wb^\dagger b+\frac{\lambda}{4w^2}:(b+b^\dagger)^4: \quad , \tag{8}$$

where : : stands for normal ordering and w is given by the equation

$$w-w^3+6\lambda=0 \quad . \tag{9}$$

TABLE I. Energy levels of an anharmonic oscillator. The first line of entry is our results. The other two lines are reproduced from Table I of Ref. 1 and represent the results from the diagonalization of a 19×19 Hamiltonian matrix and the exact values, respectively.

W_0	W_1	W_2	W_3	W_4
0.559 200 3	1.770 249 0	3.141 394 0	4.650 353 3	6.277 670 0
0.559 146 33	1.769 502 6	3.138 624	4.628 83	6.220 30
0.559 146 33	1.769 502 6	3.138 624	4.628 833	6.220 30

TABLE II. Contributions to energy levels from different orders of perturbation theory. The last line of entry is the results of a normal perturbation calculation for the ground-state energy.

	Zeroth order	First order	Second order	Third order	Total
W_0	0.560 307 5	0.0	−0.001 107 1	0.0	0.559 200 3
W_1	1.781 504 1	0.0	−0.013 746 9	0.002 491 8	1.770 249
W_2	3.203 864 6	0.0	−0.075 726 7	0.013 256 1	3.141 394 0
W_3	4.827 388 9	0.0	−0.230 404 1	0.053 369 8	4.650 353 3
W_4	6.652 077 0	0.0	−0.494 486 2	0.120 079 2	6.277 670
W_0 (Ref. 2)	0.50	0.075	−0.026 25	0.020 81	0.569 56

Hsue and Chern constructed excited states as

$$|\psi_n\rangle=\frac{b^{\dagger n}}{n!}|\psi_0\rangle \tag{10}$$

and constructed H_{nm}, the matrix elements of the Hamiltonian. By truncating the matrix, $n,m \leq N$ for different N values and diagonalizing by numerical methods they obtained the energy eigenvalues.

To develop a Rayleigh-Schrödinger-type perturbation theory we rewrite Eq. (8) as

$$H=H_0+H' , \tag{11}$$

with

$$H_0=E_0+wb^\dagger b+\frac{3\lambda}{2w^2}b^{\dagger 2}b^2 , \tag{12}$$

$$H'=\frac{\lambda}{4w^2}(b^{\dagger 4}+b^4+4b^{\dagger 3}b+4b^\dagger b^3) . \tag{13}$$

It happens that $|\psi_n\rangle$ are exact eigenfunctions of H_0:

$$H_0|\psi_n\rangle=E_n|\psi_n\rangle , \tag{14}$$

$$E_n=E_0+nw+\frac{3\lambda}{2w^2}n(n-1) . \tag{15}$$

The matrix elements of H' are

$$\langle n-2|H'|n\rangle=\langle n|H'|n-2\rangle=\frac{\lambda}{w^2}[n(n-1)]^{1/2}(n-2) , \tag{16}$$

$$\langle n-4|H'|n\rangle=\langle n|H'|n-4\rangle =\frac{\lambda}{4w^2}[n(n-1)(n-2)(n-3)]^{1/2} , \tag{17}$$

with all other matrix elements being zero. With E_n and $|\psi_n\rangle$ as unperturbed energy eigenvalues and eigenfunctions, respectively, we carry out the perturbation calculation. Results for $\lambda=0.1$ up to third order are given in Table I and are compared with the results of Hsue and Chern. Although the results are not as good as those of Hsue and Chern we have the advantage that the results are obtained in analytic form and can be used in actual cases of physical interest.

In Table II the contributions to the energy levels are given separately up to third order. The convergence seems to be good. The last line of entry is the results of a normal perturbation calculation[2] and is given for comparison.

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[1]C. S. Hsue and J. L. Chern, Phys. Rev. D **29**, 643 (1984).

[2]C. M. Bender and T. T. Wu, Phys. Rev. **184**, 1231 (1969).