Modest step towards a Young-tableau method for E_6

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I present a set of simple prescriptions for calculating the dimensions of the irreducible representations of E_6 and for reducing Kronecker products, using diagrams in parallel to the Young-tableau method for $SU(n)$ groups.

The exceptional group E_6 has been considered by many authors as a candidate for the grand-unifying gauge group.[1] It has many attractive features such as no anomaly, left-right symmetry, and more room for explaining the mass relations between fermions.[2] Recently, it looks as if E_6 is "the grand-unifying group" if one believes in superstring theories,[3] whose attractive features include anomaly cancellation, finiteness, and incorporation of quantum gravity. In any case, E_6 is currently of great interest. But finding out the irreducible representations and carrying out the reduction of the Kronecker products, which is most required for model building, is not easy. Although there are tables[4] which one can refer to, our motivation here is not to depend on the tables, but to be able to do calculations from a given set of simple rules and to go beyond the limit of the tables. There have been discussions in the literature[5] about different methods of calculations, but all of them are difficult and time consuming. In E_6 there are six independent Casimir invariants, and obviously to establish a set of systematic Young-tableau rules like the ones for $SU(n)$ groups is exceptionally difficult. What we would like to present here is a set of simple rules which might make this task easier.

E_6 is a rank-6 group with 78 generators. Its Dynkin diagram is given in Fig. 1. The 36 positive roots are listed in Table I. In the Dynkin basis each irreducible representation (irrep) can be represented by a set of six non-negative integers $\Lambda=(a_1a_2a_3a_4a_5a_6)$ called the highest weight of the irrep, and the dimension of the irrep is given by the relation[6]

$$N(\Lambda)=\prod_{\substack{\text{all the}\\ \text{positive roots}}}\frac{(\Lambda+\delta,\alpha_i)}{(\delta,\alpha_i)}\ , \tag{1}$$

where $\delta=(111111)$, the brackets denote scalar products, and the metric which goes into the scalar product is given in Table II. The actual calculation by hand is cumbersome. Even for simple irreps like $\Lambda_{27}=(100\,000)$ one has to first calculate 36 factors and then multiply them together.

What we propose here is to replace each irrep by a diagram similar to a Young tableau; namely, we will put an a_i number of i-box columns with numbers of boxes in a column decreasing to the right. We will then find out the number of boxes in row $j=l_j'$. Add $(7-j)$ to the number of boxes in the row j. Let us call that quantity l_j. Then in terms of l_j the expression for the dimension of irrep Λ is given by

$$N(\Lambda)=\left(\frac{\prod_{i\neq 4,5} l_i\prod_{i<j}(l_i-l_j)\prod_{\substack{i<j=1,2,3,6\\ k=4,5}}(l_i+l_j-l_k)}{(\text{the same factors with } l_j'=0)}\right)\left(\frac{\prod_{i<j<k=1,2,3,6}(l_i+l_j+l_k-l_4-l_5)\times(l_1+l_2+l_3+l_6-l_4-l_5)}{(\text{the same factors with } l_j'=0)}\right)\ . \tag{2}$$

A critical reader would immediately ask: What good is such a formula? After all, it is the same equation written in a different form. But a little thought will convince him that once one does some practice with Eq. (2) one will not need to calculate all the 36 factors. Equation (2) is much nicer to handle than Eq. (1). We will give an example here to clarify our point.

Consider the evaluation of N for the irrep $(100\,000)$. For this irrep l_i''s and l_i's are given as $(100\,000)$ and $(754\,321)$, respectively. To calculate the dimension of this irrep one has to find out only the factors which contain l_1 because other factors will cancel both in the numerator and denominator. And one notices that for the factors containing l_1, the numerator is one more than the denominator. So one

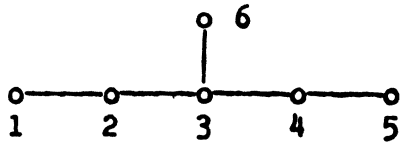


FIG. 1. Dynkin diagram for E_6.

TABLE I. Positive roots of E_6 in dynamic basis.

(000001)
(00100−1)
(01−1100)
(1−10100) (010−110)
(−100100) (1−11−110) (0100−10)
(−101−110) (10−1011) (1−110−10)
(−11−1011) (−1010−10) (10−11−11) (10001−1)
(0−10011) (−11−11−11) (−11001−1) (100−101) (1001−1−1)
(0−101−11) (0−1101−1) (−110−101) (−1101−1−1) (101−10−1)
(0−11−101) (0−111−1−1) (00−1110) (−111−10−1) (11−1000)
(00−1002) (0−12−10−1) (00−12−10) (000−120) (−12−1000) (2−10000)

TABLE II. The metric tensor for the weight space of E_6.

$$\frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

can immediately write down

$$N(100\,000)=\frac{7}{6}\times\frac{2\times3\times4\times5\times6}{1\times2\times3\times4\times5}\times\frac{9}{8}\times\frac{10}{9}\times\frac{8}{7}\times\frac{9}{8}\times\frac{5}{4}\times\frac{6}{5}\times\frac{11}{10}\times\frac{8}{7}\times\frac{7}{6}\times\frac{12}{11}=27 \ .$$

We hope that the reader will appreciate the simplicity of this method over the other ones.

Before we proceed to do the decomposition of the Kronecker products, let us discuss two quantities: namely, the triality and the Dynkin index. The triality of E_6 has to do with the congruency classes. All irreps in the product of two irreps have the same triality which is the sum of the trialities of the initial irreps modulo 3. For E_6 the triality is given by

$$\text{triality}=(a_1-a_2+a_4-a_5)\text{modulo } 3 \ . \tag{3}$$

The index is the quantity related to the second Casimir invariant. The second Casimir invariant $C(\Lambda)$ and the index $l(\Lambda)$ are defined as

$$C(\Lambda)=(\Lambda,\Lambda+2\delta)$$

and

$$l(\Lambda)=\frac{N(\Lambda)}{N(\text{adjoint})}C(\Lambda) \ . \tag{4}$$

Because of the reflection symmetry of the Dynkin diagram about the 3-6 root direction, the reality and complexity can be seen very easily. If the highest weight $(a_1a_2a_3a_4a_5a_6)$ is invariant under $a_1 \leftrightarrow a_5$ and $a_2 \leftrightarrow a_4$ then it is a real representation; if not, it is complex and its conjugate is given by $a_1 \leftrightarrow a_5$ and $a_2 \leftrightarrow a_4$ and is denoted with a bar on the top of it.

Now we turn our attention to Kronecker products. No attempt is made here to give a rigorous derivation of the rules starting from the Casimir invariants. What I would like to do is to give a set of prescriptions which work for low dimensions and that is what we need in model building anyway. Let us consider the following set of rules to multiply different irreducible representations.

(1) Adding one box: Attach a single box onto the end of any row (including a row of length zero) in all possible ways so as to leave an allowed tableau (no column having more than six boxes and the number of boxes decreasing to the right).

(2) Adding more than one box: Label the boxes in the top row a, the next row b, etc. Add each box one by one, always in a one-box-permissible way, the top row first, then the second, etc., such that reading from right to left and then up to down, the numbers of a boxes encountered $\geqslant$ the number of b boxes $\geqslant \cdots$, etc. Two representations are distinct if $a,b,c,\ldots$ labelings differ. No two or more a's, b's, etc. may appear in the same column.

Because we have not explored all the independent Casimir invariants, it is obvious that we will not get the full story by these two rules. We further look at the general rule $N_1\times N_2=\sum N_i$. If we fall short we look for other irreps as the outcome of the product. For this we should first try three ways to find out the candidates for missing irreps: (i) by taking off a 6-box column altogether from any of the irreps in the product, (ii) by adding a 6-box column to the left of any irrep in the product, and (iii) by taking off two boxes by a box being added to any irrep to give allowed tableau. If ambiguity arises, and to find out whether we need N_i or $\overline{N}_i$, we take advantage of the following rules.

(1) The representation with highest weight $\Lambda=(a_1+b_1, a_2+b_2,\ldots,a_6+b_6)$ must be in the product.

(2) The dimension sum rule

$$N(R_1\times R_2)=N(R_1)N(R_2)=\sum_i N(R_i) \ .$$

(3) The index sum rule

$$l(R_1\times R_2)=l(R_1)N(R_2)+N(R_1)l(R_2)=\sum_i l(R_i) \ .$$

At this stage one can do the calculations in two steps. Because calculation of the index is the most difficult thing in our scheme, one might like to try satisfying the index rule approximately to get an approximate idea about what the representations could be and then do the exact calculation of the indices to verify. To do the rough calculation one might use the following approximate empirical relation:

$$l(\Lambda)\sim\left[\frac{N(\Lambda)}{27}\right]^{\alpha} ,$$

where α decreases from 1.306 for **78** to 1.176 for **112 320** almost monotonically.

(4) Complexity. If two representations are real then all the products on the right-hand side must be real or will occur in conjugate pairs. If two are complex then all the outcomes are complex unless they are conjugates of each other in which case we get all scalar or conjugate pairs. Complex and real will always give complex.

(5) Triality. All the irreps in the product must have triality equal to the sum of the trialities of the initial irreps modulo 3.

(6) If $R_1\times R_2$ contains R_3 exactly p times, then $R_1\times\overline{R}_3$ contains $\overline{R}_2$ exactly p times.

We will, in what follows, give two examples to explain some of the things.

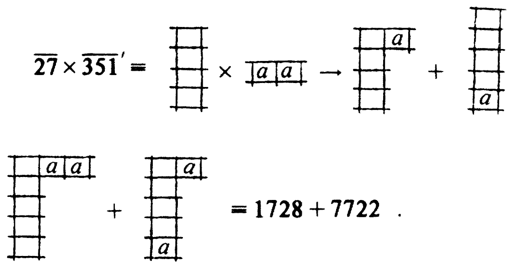
(i) Adding one box. Let us take, for example, $\mathbf{27}\times\mathbf{27}$:

$$\mathbf{27}\times\mathbf{27}=\square\times\square\rightarrow\square\square+\begin{smallmatrix}\square\\ \square\end{smallmatrix}=\overline{\mathbf{351}}'+\overline{\mathbf{351}} \ .$$

The dimension does not add up to $27\times27=729$. We fall short of 27. Now the question is whether it is **27** or $\overline{\mathbf{27}}$. Since triality of **27** is 1 we have to have triality 2 or -1 on the right-hand side so the missing one is $\overline{\mathbf{27}}$.

(ii) Adding more than one box: We will look for example at the two-box cases, first for the symmetric case and then for the antisymmetric case.

$$\overline{27}\times\overline{351}'=$$



$$=1728+7722\ .$$

We see that $1728+7722$ does not equal $27\times 351=9477$. We fall short of 27. Here again the triality consideration will solve the ambiguity between **27** and $\overline{\mathbf{27}}$. Triality of the left-hand side $=-1+2=1$, which implies that the missing irrep in the Kronecker product is, in fact, **27**. $\mathbf{78}\times\overline{\mathbf{351}}$ will yield $\overline{\mathbf{17550}}$ by our prescription. We differ by 9828. If we take off a 6-box column from $\overline{\mathbf{17550}}$ we will get $\overline{\mathbf{351}}$ and it goes with all the requirements.

If we take off two boxes from the 6-box column by the first box being added that will result in a 4-box column. The second box being added can go either to the right, giving $\overline{\mathbf{7371}}$, or to the bottom, giving $\overline{\mathbf{27}}$. These two satisfy all the requirements.

A little struggle with all the combinations satisfying the triality, complexity, and index rules will give, uniquely,

$$\mathbf{78}\times\overline{\mathbf{351}}=\overline{\mathbf{27}}+\overline{\mathbf{351}}+\overline{\mathbf{351}}'+\overline{\mathbf{1728}}+\overline{\mathbf{7371}}+\mathbf{17550}\ .$$

In summary, we have presented in this paper a set of simple rules, easy to remember and to work with, to do calculations in E_6 which might save a lot of time for people working in the field.

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[6]Here we follow the notations of Ref. 4.