Factorization of soft and collinear divergences in QCD in the Feynman gauge via a background field gauge

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We prove Ward-type identities which are needed in the proof of factorization of the Drell-Yan process. Previous treatments prove them only for an axial gauge, and the proofs are diagrammatic in nature. We, on the other hand, establish the identities for the Feynman gauge and through symmetry considerations at the Lagrangian level. Our strategy is to first derive exact results in a background field gauge and then to show that to leading order in the mass divergences the background-field-gauge results can be used in the Feynman gauge.

I. INTRODUCTION

The purpose of this paper is to generalize the following QED result to non-Abelian gauge theories.

Consider a Feynman graph $\mathscr{G}$ with a subgraph $\mathscr{G}'$ such that all the internal propagators of $\mathscr{G}'$ are jet + or hard. By a jet + line (or collinear + propagator) we mean one such that its momentum p satisfies $|p^+|\sim Q$, $|p^-|\sim\delta^2 Q$, $|\mathbf{p}_T|\sim\delta Q$, where $\delta\sim m/Q$ and $p^\pm=(p^0\pm p^3)/\sqrt{2}$. Here m is the mass of the electron, Q is a typical external momentum, and we consider the limit $\delta\to 0$. By a soft (hard) line p we mean one such that $|p_\mu|\sim\delta^2 Q$ ($|p_\mu|\sim Q$) for all μ. Suppose the external lines of $\mathscr{G}'$ are an incoming and an outgoing jet + electron plus N soft photons. Our motivation for studying $\mathscr{G}'$ is that in QCD jets are represented by similar graphs. Indeed the amplitude for a quark to break up into an arbitrary intermediate state times the complex conjugate of this amplitude gives diagrams with the $\mathscr{G}'$ topology. (However, for simplicity we have omitted the cut in this paper. Jets without cuts running through them occur in the graphs for the process $\gamma^*\to q\bar{q}$ considered in the paper companion to this one, Ref. 1.)

The sum of graphs $\mathscr{G}'$ is given by

$$Y=\frac{1}{Z^0[0]}\prod_{i=1}^{N}\int_{y_i'y_i}D_{\mu_i\nu_i}(y_i-y_i')\frac{\delta}{\delta a_{\nu_i}(y_i')}\frac{\delta}{\delta\bar{\eta}(x_2)}\frac{-\delta}{\delta\eta(x_1)}Z^a[J,\eta,\bar{\eta}]\Big|_{\substack{a=0\\ J=\eta=\bar{\eta}=0}}\,, \tag{1.1}$$

where

$$Z^a[J,\eta,\bar{\eta}]=\int\mathscr{D}(A\psi\bar{\psi})\exp\left[\int_x\left[-\frac{i}{4}F_{\mu\nu}{}^2(A)+i\bar{\psi}(i\partial\!\!\!/-m)\psi-\frac{i}{2\xi}(\partial_\mu A^\mu)^2\right]-\int_x ie\bar{\psi}(A\!\!\!/+a\!\!\!/)\psi+\int_x(J_\mu A^\mu+\bar{\psi}\eta+\bar{\eta}\psi)\right]. \tag{1.2}$$

In Z^a, it is understood that the integration is only over those functions $A(x)$, $\psi(x)$, and $\bar{\psi}(x)$ such that their Fourier transforms $A(k)$, $\psi(k)$, and $\bar{\psi}(k)$ vanish in the soft region $|k_\mu|<\delta^2 Q$. This and the fact that the external lines of Y are either jet + or soft implies that the internal lines of Y are either jet + or hard. By power counting (Ref. 1), we know that in $\bar{\psi}(x)a\!\!\!/(x)\psi(x)$ we may replace $a\!\!\!/$ by

$$\gamma^+a^-\simeq\frac{\gamma^+\partial^-v^+a^-}{v^+\partial^-}\simeq\partial\!\!\!/\left[\frac{1}{v\cdot\partial}v\cdot a\right], \tag{1.3}$$

where $v^\mu=\langle v^+,\mathbf{v}_T,v^-\rangle=\langle 1,0,0\rangle$. The operator $1/(v\cdot\partial)$ is defined carefully in Appendix A. We recognize Eq. (1.3) as the Grammer-Yennie approximation.[2,3] Equation (1.3) has two important features that should be noted. First notice that the jet is blind to all components of a^μ except the minus one. The index μ in a^μ labels the different polarizations of the soft photon cloud so the jet is blind to all but one polarization of the cloud. This is a type of spin selection rule. Second, notice that the interaction of the soft cloud and the jet is longitudinal. Since longitudinal interactions are unphysical in the sense that they can be gauged away, we expect to be able to sweep them out of the interior of the jet. We do this next.

Defining $\beta(x)=[1/(v\cdot\partial)](v\cdot a)$, one can change variables in Eq. (1.2) as follows: $\psi(x)\to\psi'(x)=e^{-ie\beta(x)}\psi(x)$, $\bar{\psi}\to\bar{\psi}'=e^{+ie\beta}\bar{\psi}$, but A is kept fixed. We find that

$$Z^a[J,\eta,\bar{\eta}]=Z^{a=0}[J,\eta',\bar{\eta}']\,, \tag{1.4}$$

where $\eta'=e^{ie\beta}\eta$ and $\bar{\eta}'=e^{-ie\beta}\bar{\eta}$. Thus if

$$\frac{1}{Z^0[0]}\frac{\delta}{\delta\bar{\eta}(x_2)}\frac{-\delta}{\delta\eta(x_1)}Z^a[J,\eta,\bar{\eta}]\Bigg|_{J=\eta=\bar{\eta}=0} = \langle \psi(x_2)\bar{\psi}(x_1)\rangle^a \quad (1.5)$$

then

$$\langle \psi(x_2)\bar{\psi}(x_1)\rangle^a = e^{-ie\beta(x_2)}\langle \psi(x_2)\bar{\psi}(x_1)\rangle^0 e^{ie\beta(x_1)} . \quad (1.6)$$

All the a_μ dependence has been factored into $e^{-ie\beta(x_2)}$ and $e^{ie\beta(x_1)}$ and the functional derivatives with respect to a_μ in Y act only on these exponentials.

The natural generalization of the above QED result to QCD has been presented by Collins and Soper in Ref. 4 and reexamined by the author in Ref. 5. However, both of the latter treatments are for QCD in an axial gauge. It is not clear from these references which results carry over to a Feynman gauge, or even how to derive corresponding results in a Feynman gauge. Furthermore Refs. 4 and 5 prove identities like Eq. (1.6) diagrammatically. One would like to know how to derive them through symmetry considerations at the Lagrangian level (as we did in the QED example above). In this paper we generalize the above example to QCD in a Feynman gauge, and we prove our results at the Lagrangian level.

The generalization of Eq. (1.6) to QCD in a Feynman gauge is a basic ingredient of the proof of factorization of the Drell-Yan process.[6,7] For this process, it is convenient to prove factorization in a Feynman gauge. In fact, nobody has succeeded in proving Drell-Yan factorization in any other gauge. The problem with an axial gauge is that it has extra singularities in the propagator which complicate contour deformation arguments. Furthermore, it is very difficult to calculate Feynman integrals in the axial gauge, even at the one-loop level.

The strategy of this paper is to first derive exact results in a background field gauge and then to show that to leading order in the mass divergences these results can be used in the Feynman gauge. We would like to stress this feature of our paper (which we believe to be new), namely, that we use the background-field-gauge formalism as an aid in proving factorization. These two fields hitherto have existed separately in the literature but we feel that they are intimately connected and consider this paper an illustration of this.

The paper is organized as follows. Section II is a review of QCD in a background field gauge. Section III defines a special gauge transformation which is obtained by performing a sequence of infinitesimal displacements of a jet with respect to its background cloud of soft gluons. Section IV derives the QCD analog of the QED Eq. (1.6). Finally Sec. V clarifies the connection of preceding sections with previous papers. It derives for the Feynman gauge the "tree" Ward identities that have been presented in Refs. 4 and 5 for the axial gauge.

II. QCD IN BACKGROUND FIELD GAUGE

As in the QED example, the soft gluon cloud traversed by the jet will be represented by a background field (BF) $a_\mu(x)$. Whereas in QED it is simple to introduce a BF into the action $[\bar{\psi}A\psi \to \bar{\psi}(A+a)\psi]$, it is a more complicated matter to do so in a symmetric way for QCD. Luckily this is a well-developed subject.[8] In this section we review the BF gauge formalism and introduce our conventions in the process.

Below we will use $[T^a,T^b]=if^{abc}T^c$ and $(\tilde{T}^a)_{bc}=-if^{abc}$ (check that $[\tilde{T}^a,\tilde{T}^b]=if^{abc}\tilde{T}^c$). ψ and $\bar{\psi}^T$ will represent 3×1 matrices; $\mathbf{A}_\mu$, $\mathbf{c}$, and $\bar{\mathbf{c}}$ are 8×1 matrices; $A_\mu \equiv A^a_\mu T^a$, c, and $\bar{c}$ are 3×3 matrices; $\tilde{A}_\mu = A^a_\mu \tilde{T}^a$, $\tilde{c}$, and $\tilde{\bar{c}}$ are 8×8 matrices. For any 8-tuples A^a and B^a, $a=1,\ldots,8$, we define $(\mathbf{A}\times\mathbf{B})^a = \mathrm{i}f^{abc}A^bB^c$. The vector notation is convenient because Jacobi's identity becomes simply $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\times\mathbf{C}+\mathbf{B}\times(\mathbf{A}\times\mathbf{C})$ for non-Grassmann 8-tuples $\mathbf{A},\mathbf{B},\mathbf{C}$. Another useful identity is $\tilde{A}\mathbf{B}=\mathbf{A}\times\mathbf{B}$. We will allow the following exceptions to the above notation:

$$\begin{aligned}
&\mathscr{D}_\mu(A)=\mathbf{e}\partial_\mu+ig\,\mathbf{A}_\mu,\quad \mathbf{e}=(1,1,\ldots,1)\\
&\mathscr{D}_\mu(A)=I_3\partial_\mu+igA_\mu,\quad I_3=3\times 3 \text{ identity matrix} \qquad (2.1)\\
&\tilde{\mathscr{D}}_\mu(A)=I_8\partial_\mu+ig\tilde{A}_\mu .
\end{aligned}$$

We refer to the latter definitions as exceptions because $T^ae^a=\sum_{i=1}^8 T^a \neq I_3$. In fact $\det I_3=1$ but $\det(\sum_{a=1}^8 T^a)=0$. Likewise $\tilde{T}^ae^a\neq I_8$.

It is convenient to define a QCD generating function with an external field $\mathbf{a}_\mu$ by

$$Z^{a,G}[\mathscr{J}]=\int \mathscr{D}\mathscr{F}\exp\left[i\int_x \mathscr{L}_{\text{eff}}+\int_x \mathscr{J}\mathscr{F}\right] \quad (2.2)$$

with

$$\mathscr{L}_{\text{eff}}=\mathscr{L}_{\text{inv}}(A+a)+\tfrac{1}{4}\mathbf{F}_{\mu\nu}{}^2(a)+\mathscr{L}_{\text{gf}}+\mathscr{L}_c .$$

In the above, $\mathscr{J}=(\mathbf{J}_\mu,-\eta,\bar{\eta},-\bar{\mathscr{C}},\mathscr{C})$, $\mathscr{F}=(\mathbf{A}_\mu,\bar{\psi},\psi,\bar{\mathbf{c}},\mathbf{c})$ so $\mathscr{J}\mathscr{F}=\mathbf{J}_\mu\cdot\mathbf{A}^\mu+\bar{\psi}\eta+\bar{\eta}\psi+\bar{\mathbf{c}}\cdot\mathscr{C}+\bar{\mathscr{C}}\cdot\mathbf{c}$. (Note we use purely imaginary sources to avoid factors of i.) Also

$$\begin{aligned}
&\mathscr{L}_{\text{inv}}(A)=-\tfrac{1}{4}\mathbf{F}_{\mu\nu}{}^2(A)+\bar{\psi}[i\mathscr{D}(A)-m]\psi ,\\
&\mathbf{F}_{\mu\nu}(A)=\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+ig\,\mathbf{A}_\mu\times\mathbf{A}_\nu , \qquad (2.3)\\
&\mathscr{L}_{\text{gf}}=-\frac{1}{2\xi}\mathbf{G}^2(A,a),\quad \mathscr{L}_c=\bar{\mathbf{c}}\cdot\mathbf{G}(\tilde{\mathscr{D}}(A+a)\mathbf{c},\mathbf{a}) ,\\
&\mathbf{G}(\mathbf{A},\mathbf{a})=\partial_\mu\mathbf{A}^\mu+ig\mathbf{a}_\mu\times\mathbf{A}^\mu=\tilde{\mathscr{D}}_\mu(a)\mathbf{A}^\mu .
\end{aligned}$$

When $\mathbf{a}_\mu=0$, $Z^{a,G}[\mathscr{J}]$ reduces to the usual generating function for QCD in a covariant gauge. Clearly there is no unique way of introducing $\mathbf{a}_\mu$ into the usual QCD action. One important reason for choosing the above way is that $\mathscr{L}_{\text{eff}}$ is invariant under two types of transformations.

Type I:

$$\begin{aligned}
&\delta\mathbf{a}_\mu=0 ,\\
&\delta\mathbf{A}_\mu=\tilde{\mathscr{D}}_\mu(A+a)\lambda\mathbf{c} ,\\
&\delta\psi=-ig(\lambda c)\psi , \qquad (2.4)\\
&\delta\bar{\psi}=\bar{\psi}ig(\lambda c) ,\\
&\delta\mathbf{c}=\lambda\left[-\frac{ig}{2}\right]\mathbf{c}\times\mathbf{c} ,\\
&\delta\bar{\mathbf{c}}=\frac{\lambda\mathbf{G}}{\xi} .
\end{aligned}$$

($\lambda = x$-independent Grassmann number.) Type-I transformations act only on the quantum fields and they are just a trivial generalization of the Becchi-Rouet-Stora transformations for the usual QCD.

Type II:

$$\begin{aligned}
&\delta \mathbf{a}_\mu = \tilde{\mathscr{D}}_\mu(a)\boldsymbol{\beta}\,,\\
&\delta \mathbf{A}_\mu = -ig\boldsymbol{\beta}\times \mathbf{A}_\mu\,,\\
&\delta\psi = -ig\beta\psi\,,\\
&\delta\bar{\psi} = \bar{\psi}ig\beta\,,\\
&\delta \mathbf{c} = -ig\boldsymbol{\beta}\times\mathbf{c}\,,\\
&\delta\bar{\mathbf{c}} = -ig\boldsymbol{\beta}\times\bar{\mathbf{c}}\,.
\end{aligned} \tag{2.5}$$

($\boldsymbol{\beta}$=arbitrary x-dependent 8-tuple.) Type-II transformations change both the quantum fields and the BF. Proving that $\delta\mathscr{L}_{\text{eff}}=0$ under type-II transformations sheds some light onto why very few other choices of $\mathscr{L}_{\text{eff}}$ would be invariant under gauge transformations of the external field. The proof goes as follows. Since $\delta(\mathbf{a}_\mu+\mathbf{A}_\mu) = \tilde{\mathscr{D}}_\mu(a+A)\boldsymbol{\beta}$, it is clear the $\delta\mathscr{L}_{\text{inv}}(A+a)=0$. $\delta\mathbf{a}_\mu = \tilde{\mathscr{D}}_\mu(a)\boldsymbol{\beta}$ in turn gives $\delta\mathbf{F}_{\mu\nu}{}^2(a)=0$. Finally, the invariance of $\mathscr{L}_{\text{gf}}$ and $\mathscr{L}_c$ is a consequence of the next theorem.

Theorem 2.1. Suppose $\mathbf{V}_\mu$ and $\mathbf{W}$ are arbitrary 8-vectors in color space such that $\delta\mathbf{V}_\mu = \tilde{\mathscr{D}}_\mu(V)\boldsymbol{\beta}$ and $\delta\mathbf{W} = -ig\boldsymbol{\beta}\times\mathbf{W}$. Then $\delta(\tilde{\mathscr{D}}_\mu(V)\mathbf{W}) = -ig\boldsymbol{\beta}\times\tilde{\mathscr{D}}_\mu(V)\mathbf{W}$. The proof is left to the reader.

According to Theorem 2.1, $\delta\mathbf{G}(A,a) = -i\boldsymbol{\beta}\times\mathbf{G}$. Since $\mathbf{G}$ undergoes a simple rotation under the gauge transformation, $\delta(\mathbf{G}\cdot\mathbf{G})=0$;

$$\bar{\mathbf{c}}\cdot\mathbf{G}(\tilde{\mathscr{D}}_\mu(A+a)\mathbf{c},a) = \bar{\mathbf{c}}\cdot\tilde{\mathscr{D}}_\mu(a)\tilde{\mathscr{D}}^\mu(A+a)\mathbf{c}\,. \tag{2.6}$$

By the theorem, $\delta(\tilde{\mathscr{D}}^\mu(A+a)\mathbf{c}) = -ig\boldsymbol{\beta}\times\tilde{\mathscr{D}}^\mu(A+a)\mathbf{c}$. Again, according to the theorem $\tilde{\mathscr{D}}_\mu(a)\tilde{\mathscr{D}}^\mu(A+a)\mathbf{c}$ is rotated. Since $\bar{\mathbf{c}}$ is rotated by the same amount, their dot product is invariant.

In order to find out what kind of vertices $i\mathscr{L}_{\text{eff}}$ implies, we next expand $i\mathscr{L}_{\text{eff}}$. It is convenient to express $i\mathscr{L}_{\text{eff}}$ in terms of the following functions:

$$Y^{abc}_{\alpha\beta\gamma}(k_1,k_2,k_3) = gf^{abc}[(k_1-k_3)_\beta g_{\alpha\gamma} + (k_2-k_1)_\gamma g_{\beta\alpha} + (k_3-k_2)_\alpha g_{\gamma\beta}]\,, \tag{2.7}$$

$$Y'^{abc}_{\alpha\beta\gamma}(k_1,k_2) = -\frac{g}{\xi}f^{abc}(g_{\beta\gamma}k_{1\alpha} - g_{\alpha\gamma}k_{2\beta})\,, \tag{2.8}$$

$$H^{abcd}_{\alpha\beta\gamma\delta} = -ig^2\left[f^{rab}f^{rcd}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} + f^{rac}f^{rbd}\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} + f^{rad}f^{rbc}\begin{bmatrix}\alpha & \delta\\ \beta & \gamma\end{bmatrix}\right]\,, \tag{2.9}$$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} = g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}\,. \tag{2.10}$$

We call $(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix})$ a toroid. It is symmetric under exchanges of two rows or two columns and antisymmetric under exchanges within a row. We also use the functions

$$H'^{abcd}_{\alpha\beta\gamma\delta} = -\frac{ig^2}{\xi}(f^{rac}f^{rbd}g_{\alpha\gamma}g_{\beta\delta} + f^{rbc}f^{rad}g_{\beta\gamma}g_{\alpha\delta})\,, \tag{2.11}$$

$$E^{abc}_\gamma(k_1) = -gf^{abc}k_{1\gamma}\,, \tag{2.12}$$

$$E'^{abc}_\gamma(k_2) = gf^{abc}k_{2\gamma}\,, \tag{2.13}$$

$$K^{(ac)(bd)}_{\gamma\delta} = -ig^2g_{\gamma\delta}f^{rac}f^{rbd}\,. \tag{2.14}$$

In terms of the above functions

$$\begin{aligned}
i\mathscr{L}_{\text{eff}} = &\left[-\bar{\psi}\left[\frac{i\partial-m}{i}\right]\psi - ig\bar{\psi}a\psi\right] + \left[-\bar{\mathbf{c}}\left[\frac{\Box}{i}\right]\mathbf{c} + (E+E')_{123}\bar{c}^{\,1}c^2a^3 + (K_{(14)(23)}+K_{(13)(24)})\bar{c}^{\,1}c^2\frac{a^3a^4}{2}\right] + [-ig\bar{\psi}A\psi]\\
&+[E_{123}\bar{c}^{\,1}c^2A^3 + K_{(14)(23)}\bar{c}^{\,1}c^2A^3a^4] + \left[-\frac{i}{4}(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu)^2 - \frac{i}{2\xi}(\partial_\mu\mathbf{A}^\mu)^2 + (Y+Y')_{123}\frac{A^1A^2a^3}{2} + (H+H')_{1234}\frac{A^1A^2a^3a^4}{4}\right]\\
&+\left[Y_{123}\frac{A^1A^2A^3}{3!} + H_{1234}\frac{A^1A^2A^3a^4}{3!}\right] + \left[H_{1234}\frac{A^1A^2A^3A^4}{4!}\right]\,.
\end{aligned} \tag{2.15}$$

In the above, 1 is shorthand for $x_1\alpha a$, 2 for $x_2\beta b$, and so forth. In Eqs. (2.7) to (2.14) wherever a momentum k_r appears, it should be replaced by $i\partial/\partial x_r$ before substituting into Eq. (2.15). Also, all x_r should be set equal to x but only after the derivatives $i\partial/\partial x_r$ have acted on the fields. For example,

$$E_{123}\bar{c}^{\,1}c^2A^3 = [E^{abc}_\gamma(i\partial_{x_1})\bar{c}^{\,a}(x_1)c^b(x_2)A^{c\gamma}(x_3)]_{x_r\to x}\,. \tag{2.16}$$

A final comment on Eq. (2.15) is that any vertices with one $\mathbf{A}_\mu$, any number of $\mathbf{a}_\mu$, and no other fields have been ignored. This is justified in our particular application because $\mathbf{a}_\mu(x)$ is restricted to be soft and $\mathbf{A}_\mu(x)$ to be jet + or hard so such vertices are zero by momentum conservation.

The Feynman rules corresponding to $i\mathscr{L}_{\text{eff}}$ are given in Fig. 1. One sees from this figure that most vertices with a's have the same analytical expressions assigned to them

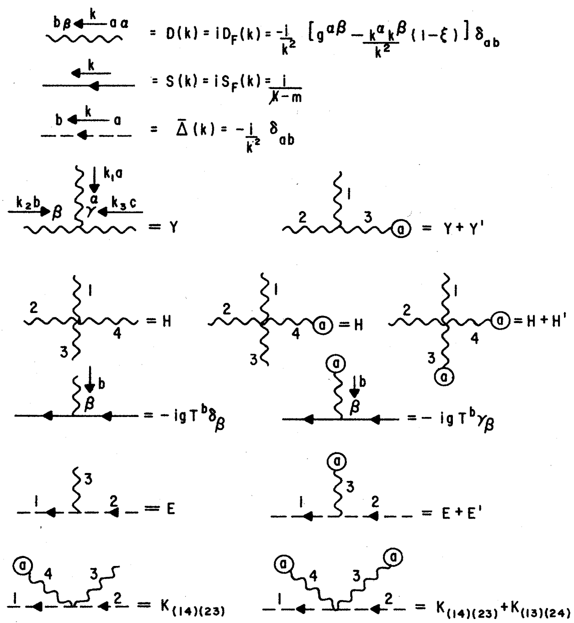

FIG. 1. Feynman rules in the background field gauge.

as those with the **a**'s replaced by **A**'s. The exceptions have been labeled by a primed letter or by a K. From here on, these contributions to $i\mathscr{L}_{\text{eff}}$ will be referred to as auxiliary vertices.

III. A SEQUENCE OF GAUGE TRANSFORMATIONS

Recall that in the QED example we changed variables to

$$\psi'(x)=e^{-ie\beta(x)}\psi(x)$$

with

$$\beta(x)=-\int_0^\infty d\eta\, v\cdot a(x+\eta v)\ .$$

This prompts us to investigate for the non-Abelian case a sequence of infinitesimal gauge transformations parametrized by $\eta\in[0,\infty]$ of the form

$$\delta\mathbf{a}_\mu(x,\eta)=\partial_\mu\delta\boldsymbol{\beta}(x,\eta)+ig\mathbf{a}_\mu(x,\eta)\times\delta\boldsymbol{\beta}(x,\eta)\ . \tag{3.1}$$

We choose to investigate $\delta\boldsymbol{\beta}(x,\eta)$ of the special type

$$\delta\boldsymbol{\beta}(x,\eta)=d\eta\,\mathbf{f}(x+\eta v)\equiv d\eta\,\boldsymbol{\omega}(\eta) \tag{3.2}$$

for an arbitrary function $\mathbf{f}$. Equations (3.1) and (3.2) imply the differential equation

$$\frac{d\,\mathbf{a}_\mu(\eta)}{d\eta}=\partial_\mu\boldsymbol{\omega}(\eta)-ig\widetilde{\omega}(\eta)\mathbf{a}_\mu(\eta) \tag{3.3}$$

(x dependence suppressed until further notice) which we propose to solve subject to the condition $\mathbf{a}_\mu(x,\infty)=0$. Define

$$I(0,\eta)=T^*\left[\exp\left[\int_0^\eta ig\widetilde{\omega}(\eta_1)d\eta_1\right]\right]\quad\text{for } \eta>0\ , \tag{3.4}$$

where T^* represents anti-time ordering. Letting

$$I(\eta,0)=I^{-1}(0,\eta)=I^\dagger(0,\eta) \tag{3.5}$$

one finds

$$\frac{d}{d\eta}[I(0,\eta)\mathbf{a}_\mu(\eta)]=I(0,\eta)\partial_\mu\boldsymbol{\omega}(\eta) \tag{3.6}$$

so

$$\mathbf{a}_\mu(x,\eta)=-\int_\eta^\infty d\eta' I(\eta,\eta')\partial_\mu\mathbf{f}(x+\eta' v) \tag{3.7}$$

with

$$I(\eta,\eta')=T^*\left[\exp\left[\int_\eta^{\eta'} d\eta_1 ig\widetilde{f}(x+\eta_1 v)\right]\right]\ . \tag{3.8}$$

Theorem 3.1. $\mathbf{a}_\mu(x,\eta)=\mathbf{a}_\mu(x+\eta v,0)$.

Proof. Write right- and left-hand sides in terms of Eq. (3.7).

Theorem 3.2. $v\cdot\mathbf{a}(x,\eta)=\mathbf{f}(x+\eta v)$.

Proof:

$$v\cdot\mathbf{a}(x,\eta)=-\int_\eta^\infty d\eta' I(\eta,\eta')\frac{d}{d\eta'}\mathbf{f}(x+\eta' v)\ . \tag{3.9}$$

Now integrate by parts, assuming $f(x+v\infty)=0$, and use the fact that $\widetilde{f}\mathbf{f}=\mathbf{f}\times\mathbf{f}=0$.

It is clear from Theorem 3.2 that

$$\mathbf{f}(x+\eta v)=v\cdot\mathbf{a}(x+\eta v,0)\ . \tag{3.10}$$

If we introduce the notation $\mathbf{a}_\mu(x)$ for $\mathbf{a}_\mu(x,0)$ then

$$\mathbf{a}_\mu(x)=-\int_0^\infty d\eta' I(0,\eta')\partial_\mu v\cdot\mathbf{a}(x+\eta' v)\ . \tag{3.11}$$

Equation (3.11) has many important consequences. To begin it implies that $\mathbf{a}^+(x)$ and $\mathbf{a}^i(x)$, $i=1,2$ are determined once $\mathbf{a}^-(y)$ is known for all y. Furthermore, if we choose $\mathbf{a}^-(y)$ real then $\mathbf{a}^+(x)$ and $\mathbf{a}^i(x)$ will be too since $(iv\cdot\widetilde{a})^{bc}=f^{dbc}v\cdot a^d$ and f^{dbc} is real. (Indeed $[T^a,T^b]=if^{abc}T^c$ and the T^a are Hermitian.) From here on we will use $\mathbf{a}_\mu(x)$ to refer to an arbitrary BF whereas $\mathbf{V}_\mu(x)$ will represent those special BF's which are pure gauge (produced by gauge transformations starting from 0), i.e., those which satisfy Eq. (3.11). While $\mathbf{a}_\mu$ has four arbitrary Lorentz components, $\mathbf{V}_\mu$ only has one. Feynman rules for $\mathbf{V}_\mu$ are given in Appendix C.

In QED Eqs. (3.3), (3.7), and (3.11) reduce, respectively, to

$$\begin{aligned}
\frac{dV_\mu(x,\eta)}{d\eta}&=\partial_\mu[v\cdot V(x+\eta v)]\ ,\\
V_\mu(x,\eta)&=-\int_\eta^\infty d\eta'\partial_\mu[v\cdot V(x+\eta' v)]\ ,\\
V_\mu(x)&=-\int_0^\infty d\eta'\partial_\mu[v\cdot V(x+\eta' v)]\\
&=\partial_\mu\left[\frac{1}{v\cdot\partial-\epsilon}v\cdot V(x)\right]\ .
\end{aligned} \tag{3.12}$$

We recognize the Grammer-Yennie (GY) substitution. Hence $\mathbf{V}_\mu$ is a non-Abelian generalization of the GY approximation for the coupling of a soft gluon to a jet. It is more complicated than the GY expression because it takes into account soft gluon radiation by the gluon in question before it couples to the jet. In general $\mathbf{V}_\mu$ is pure gauge but it is only longitudinal in the Abelian case. However, in Appendix D we show that even in the non-Abelian case only the longitudinal part of $\mathbf{V}_\mu$ is seen by the jet.

According to Theorem 3.1 and Eq. (3.10), $\mathbf{V}_\mu(x,\eta)=\mathbf{V}_\mu(x+\eta v)$ and $\delta\boldsymbol{\beta}(x,\eta)=d\eta v\cdot\mathbf{V}(x+\eta v)$. The physical picture suggested by these equations[9] is that the differential gauge transformations $\delta\boldsymbol{\beta}$ we are considering represent translations of the jet by an amount $d\eta$ in the direction v relative to the cloud of soft gluons. Figure 2 illustrates this point. In this figure, the x frame is attached to the jet, the y frame to the soft gluon cloud $(y=x+\eta v)$ and the shaded region represents the support of $\mathbf{V}_\mu(y)$.

Our use for the sequence of gauge transformations which we have just defined is the following. Recall

$$\delta\boldsymbol{\beta}(x,\eta)=d\eta v\cdot\mathbf{V}(x+\eta v)\,, \qquad \boldsymbol{\beta}(x)=-\int_{\eta=0}^{\infty}\delta\boldsymbol{\beta}(x,\eta)\,. \tag{3.13}$$

From

$$\mathscr{L}_{\rm eff}(\psi(x,\eta),\mathbf{V}_\mu(x,\eta)) =\mathscr{L}_{\rm eff}(e^{-ig\delta\boldsymbol{\beta}(x,\eta)}\psi(x,\eta),\mathbf{V}_\mu(x,\eta+d\eta)) \tag{3.14}$$

(the dependence on quantum fields other than ψ is analogous and has been suppressed) one gets

$$\mathscr{L}_{\rm eff}(\psi(x,0),\mathbf{V}_\mu(x,0))=\mathscr{L}_{\rm eff}(T[e^{+ig\boldsymbol{\beta}(x)}]\psi(x,0),0)\,. \tag{3.15}$$

We define the special color rotation matrices

$$U_F^\dagger(x)=T^*[e^{-ig\boldsymbol{\beta}(x)}]\,, \tag{3.16}$$

$$U_G^\dagger(x)=T^*[e^{-ig\tilde{\boldsymbol{\beta}}(x)}]\,. \tag{3.17}$$

(Feynman rules for U_F and U_G are given in Appendix C.) Setting $\psi'=U_F^\dagger\psi$, $\bar\psi'=\bar\psi U_F$, $\mathbf{A}'_\mu=U_G^\dagger\mathbf{A}_\mu$, $\mathbf{c}'=U_G^\dagger\mathbf{c}$ and $\bar{\mathbf{c}}'=U_G^\dagger\bar{\mathbf{c}}$, we have shown

$$\mathscr{L}_{\rm eff}(\mathscr{F}',V_\mu)=\mathscr{L}_{\rm eff}(\mathscr{F},0)\,. \tag{3.18}$$

It is instructive to show Eq. (3.18) directly from the definitions (3.16), (3.17), and (3.11) of U_F, U_G, and $\mathbf{V}_\mu$. We do so in Appendix B.

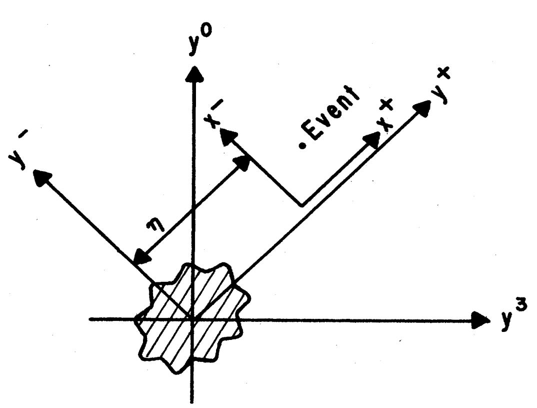

FIG. 2. Interpretation of $\mathbf{V}_\mu(x,\eta)=\mathbf{V}_\mu(x+\eta v)$. The x frame is attached to the jet and the y frame to the soft gluon cloud. The shaded region represents the support of $\mathbf{V}_\mu(y)$.

IV. FACTORIZATION OF SOFT AND COLLINEAR DIVERGENCES

Consider a subgraph $\mathscr{G}'$ of a graph $\mathscr{G}$ such that all internal propagators of $\mathscr{G}'$ are jet + or hard. For definiteness suppose $\mathscr{G}'$ has as external propagators incoming and outgoing jet + quarks and N soft gluons. The generalization of the QED example to QCD in the Feynman gauge can be done in three steps.

Step 1: Show $X=Y^*$. The sum of $\mathscr{G}'$ is given by

$$X=\frac{1}{Z^0[0]}\left[\prod_{i=1}^{N}\frac{\delta}{\delta\mathbf{J}_{\mu_i}(y_i)}\right]\frac{\delta}{\delta\bar\eta(x_2)}\frac{-\delta}{\delta\eta(x_1)} \times Z^{0,G(A,0)}[\mathscr{J}]\,|_{\mathscr{J}=0}\,, \tag{4.1}$$

where, as before, we integrate only over fields which vanish in the soft region. We *would like* to show that $X=Y$, where

$$Y=\frac{1}{Z^0[0]}\left[\prod_{i=1}^{N}\int_{y_i'}D_{\mu_i\nu_i}(y_i,y_i')\frac{\delta}{\delta\mathbf{a}_{\nu_i}(y_i')}\right. \times\frac{\delta}{\delta\bar\eta(x_2)}\frac{-\delta}{\delta\eta(x_1)}Z^{a,G(A,a)}[\mathscr{J}]\Bigg|_{\mathscr{J}=a=0}\,. \tag{4.2}$$

The statement $X=Y$ is clearly true in QED since the diagrams in X and Y are in 1-1 correspondence. However, in QCD Y contains all the diagrams of X plus new diagrams possible because of the auxiliary vertices. In the remaining portion of this step we will show that the auxiliary vertices can be pulled from the interior of a diagram out to its external fermion legs.

It is convenient to define $\mathscr{L}_{\rm eff}^{\mathscr{G}}$ and the corresponding moment generator $Z^{a,\mathscr{G}}$ where $\mathscr{G}$ is an arbitrary gauge-fixing function not necessarily equal to $\mathbf{G}$. We let $\mathscr{G}=\mathscr{N}_\mu\mathbf{A}^\mu$ for some $\mathscr{N}_\mu$ independent of A. To obtain $\mathscr{L}_{\rm eff}^{\mathscr{G}}$ we replace the gauge-fixing and ghost Lagrangians of $\mathscr{L}_{\rm eff}$ by $\mathscr{L}_{\rm gf}^{\mathscr{G}}=-(1/2\xi)\mathscr{G}^2$ and $\mathscr{L}_c^{\mathscr{G}}=\bar{\mathbf{c}}\cdot\mathscr{N}_\mu\mathscr{D}^\mu(A)\mathbf{c}$. In the BF gauge case $\mathscr{G}=\mathbf{G}=\mathscr{D}_\mu(a)\mathbf{A}^\mu$ and $\mathscr{N}_\mu=N_\mu=\mathscr{D}_\mu(a)$.

One observes that auxiliary vertices arise only in $\mathscr{L}_{\rm gf}$ and $\mathscr{L}_c$. In fact

$$(\mathscr{L}_{\rm gf}+\mathscr{L}_c)^G=-\frac{1}{2\xi}(N_\mu\mathbf{A}^\mu)^2+\bar{\mathbf{c}}\cdot N_\mu\mathscr{D}^\mu(A+a)\mathbf{c} =-\frac{1}{2\xi}(\partial_\mu\mathbf{A}^\mu)^2+\bar{\mathbf{c}}\cdot\partial_\mu\mathscr{D}^\mu(A+a)\mathbf{c}+\mathscr{L}_{\rm aux}\,. \tag{4.3}$$

Therefore if we define $\Delta\mathbf{G}=-ig\mathbf{a}_\mu\times\mathbf{A}^\mu=\Delta N_\mu\mathbf{A}^\mu$ then

$$(\mathscr{L}_{\rm gf}+\mathscr{L}_c)^G=(\mathscr{L}_{\rm gf}+\mathscr{L}_c)^{G+\Delta G}+\mathscr{L}_{\rm aux}\,, \qquad \mathscr{L}_{\rm eff}^{G+\Delta G}=\mathscr{L}_{\rm eff}^{G}-\mathscr{L}_{\rm aux}\,. \tag{4.4}$$

We conclude that

$$X=\frac{1}{Z^0[0]}\left[\prod_{i=1}^{N}\int_{y_i'}D_{\mu_i\nu_i}(y_i,y_i')\frac{\delta}{\delta\mathbf{a}_{\nu_i}(y_i')}\right] \times\frac{\delta}{\delta\bar\eta(x_2)}\frac{-\delta}{\delta\eta(x_1)}Z^{a,G+\Delta G}[\mathscr{J}]\Bigg|_{\mathscr{J}=a=0}\,. \tag{4.5}$$

Next we shall try to relate $Z^{a,G+\Delta G}$ to $Z^{a,G}$ (Ref. 10). A consequence of type-I symmetry for an arbitrary gauge-fixing function $\mathscr{G}$ is that

$$\left\langle \int_x \mathscr{J}\delta_I\mathscr{F} \right\rangle = 0 , \tag{4.6}$$

where by $\langle Q \rangle$ we mean

$$\frac{1}{Z^0[0]}\int \mathscr{D}\mathscr{F}\, Q\exp\left[i\int_x \mathscr{L}^{\mathscr{G}}_{\text{eff}} + \int_x \mathscr{J}\mathscr{F} \right] .$$

It follows that

$$\frac{-\delta}{\delta\mathscr{C}_y}\left\langle \int_x \mathscr{J}\delta\mathscr{F} \right\rangle = \left\langle \bar{c}_y \int_x \mathscr{J}\delta\mathscr{F} + \delta\bar{c}_y \right\rangle . \tag{4.7}$$

Thus

$$\left\langle \frac{\mathscr{G}_y}{\xi} \right\rangle = \left\langle \bar{c}_y \frac{d}{d\lambda}\int_x \mathscr{J}\delta\mathscr{F} \right\rangle . \tag{4.8}$$

We want to relate Z at $\mathscr{G}=\mathbf{G}+\Delta\mathbf{G}$ to Z at $\mathscr{G}=\mathbf{G}$ where $\Delta\mathbf{G}$ is finite. We start by relating Z at $\mathscr{G}$ and $\mathscr{G}+d\mathscr{G}$ for infinitesimal $d\mathscr{G}$. Since

$$Z^{a,\mathscr{G}+d\mathscr{G}}[\mathscr{J}] = \int \mathscr{D}\mathscr{F}\exp\left[i\int_x \mathscr{L}_{\text{inv}}(A+a) + \int_x \frac{i}{4}\mathbf{F}_{\mu\nu}{}^2(a) + i\int_x \bar{\mathbf{c}}\cdot(\mathscr{N}_\mu + d\mathscr{N}_\mu)\tilde{\mathscr{D}}^{\mu}(A+a)\mathbf{c} \right.$$
$$\left. - \int_x \frac{i}{2\xi}(\mathscr{G}+d\mathscr{G})^2 + \int_x \mathscr{J}\mathscr{F} \right] , \tag{4.9}$$

it follows that

$$\frac{dZ}{Z^0[0]} = \int_x \left\langle i\bar{\mathbf{c}}\cdot d\mathscr{N}_\mu \tilde{\mathscr{D}}^{\mu}(A+a)\mathbf{c} - \frac{i}{\xi}\mathscr{G}\cdot d\mathscr{G} \right\rangle . \tag{4.10}$$

But by Eq. (4.8),

$$\frac{dZ}{Z^0[0]} = \int_x \langle i\bar{\mathbf{c}}\cdot d\mathscr{N}_\mu \tilde{\mathscr{D}}^{\mu}(A)\mathbf{c}\rangle$$
$$+ \int_y \left[-id\mathscr{N}_{y\mu}\frac{\delta}{\delta \mathbf{J}_{y\mu}} \right]\cdot\left\langle \bar{c}_y \frac{d}{d\lambda}\int_x \mathscr{J}\delta\mathscr{F} \right\rangle$$
$$= \left\langle i\frac{d}{d\lambda}(\mathscr{J}\delta\mathscr{F})\int_y \bar{c}_y\cdot d\mathscr{G}_y \right\rangle . \tag{4.11}$$

We define $\mathscr{J}_F$ to be $\mathscr{J}$ with all sources except the fermion ones $(\eta,\bar{\eta})$ set equal to zero. Equation (4.11) implies that $Z^{a,\mathscr{G}+d\mathscr{G}}[\mathscr{J}_F] = Z^{a,\mathscr{G}}[\mathscr{J}^*_F]$ where

$$\bar{\eta}^* = \bar{\eta}\left[1 + gc\int_y \bar{c}_y\cdot d\mathscr{G}_y \right] , \tag{4.12a}$$

$$\eta^* = \left[1 - gc\int_y \bar{c}_y\cdot d\mathscr{G}_y \right]\eta . \tag{4.12b}$$

It follows that $Z^{a,G+\Delta G}[\mathscr{J}_F] = Z^{a,G}[\mathscr{J}^*_F]$ where now $\mathscr{J}^*_F$ is given by Eqs. (4.12) with $d\mathscr{G}$ replaced by $\Delta\mathbf{G}$. Replacing $Z^{a,G+\Delta G}[\mathscr{J}_F]$ by $Z^{a,G}[\mathscr{J}^*_F]$ in Eq. (4.5), we see that $X = Y^*$ where Y^* is obtained from Y by replacing $\mathscr{J}$ with $\mathscr{J}^*_F$.

Step 2: Only $\mathbf{a}^-$ matters. Recall in QED the jet is blind to all but the minus polarization of the soft photon cloud, i.e., $(\delta/\delta\bar{\eta})(\delta/\delta\eta)Z$ of Eq. (1.1) is independent of a^μ, $\mu=+,1,2$, and only the minus component of a^μ matters. For QCD we will argue that in $(\delta/\delta\bar{\eta})(\delta/\delta\eta)Z$ of Eq. (4.1) for X, only $\mathbf{J}^-$ matters. Therefore since $X=Y^*$ by step 1, in $(\delta/\delta\bar{\eta})(\delta/\delta\eta)$ of Y^*, only $\mathbf{a}^-$ matters.

The proof that only the minus component of $\mathbf{J}^\mu$ matters is based on results from Ref. 1. In the language of Ref. 1, if we follow the Lorentz index of a soft gluon into the jet it may or may not cap inside the jet. If it caps inside the jet, it may do so with either a wedge or a cross. In both wedge and cross cases, only the minus component of $\mathbf{J}^\mu$ contracted with the cap is leading by power counting. Reference 1 also shows that soft gluons that do not cap inside the jet are disallowed by power counting.

Step 3: Use Eq. (3.18). By virtue of step 2, we may replace $\mathbf{a}_\mu$ by $\mathbf{V}_\mu$ everywhere in Y^*. Indeed, since the jet does not see $\mathbf{a}^\mu$, $\mu=+,1,2$ we can rearrange them to make $\mathbf{a}^\mu$ pure gauge. Our next task is to establish the analog of Eq. (1.4), i.e., that

$$\{ Z^{a,G(A,a)}[\mathscr{J}^*_F] \}_{a\to V} = Z^{0,G(A,0)}[\mathscr{J}'_F] , \tag{4.13}$$

where $\mathscr{J}'_F$ will be defined in the proof below.

One can change variables in the path integral on the left-hand side of Eq. (4.13) so that $\mathscr{F}\to\mathscr{F}'$. Equation (3.18) can be applied immediately to $\mathscr{L}_{\text{eff}}(\mathscr{F}',V)$. After the change of integration variables on the left-hand side of Eq. (4.13), it gets new source terms given by

$$\int_x \mathscr{J}^*_F\mathscr{F}' = \int_x \bar{\eta}\left[1 + gc'\int_y \bar{c}'_y\cdot(-ig\mathbf{V}_{y\mu}\times\mathbf{A}'^{\mu}_y) \right]\psi'$$
$$+ \int_x \bar{\psi}'\left[1 - gc'\int_y \bar{c}'_y\cdot(-ig\mathbf{V}_{y\mu}\times\mathbf{A}'^{\mu}_y) \right]\eta . \tag{4.14}$$

The right-hand side of Eq. (4.14) can be written in more convenient form as follows. By definition $\bar{\eta}\psi' = \bar{\eta}U^\dagger_F\psi$. Using the identity $U_F T^c U^\dagger_F = T^b U^{bc}_G$, one can show $\bar{\eta}c'\psi' = \bar{\eta}U^\dagger_F c\psi$.

Theorem 4.1. $\bar{\mathbf{c}}'\cdot\mathbf{V}_\mu\times\mathbf{A}'^{\mu} = if^{abc}\bar{c}^{\,a}(U_G\partial_\mu\beta)^b A^{c\mu}$.

Proof. Show $\bar{\mathbf{c}}'\cdot\mathbf{V}_\mu\times\mathbf{A}'^{\mu} = \bar{\mathbf{c}}'\cdot(U_G\tilde{V}_\mu U^\dagger_G)\mathbf{A}^\mu$. $\mathbf{A}^\mu$. Now use the result of Appendix D to replace V^a_μ by $\partial_\mu\beta^a$ where, as always,

$$\beta = \frac{1}{v\cdot\partial - \epsilon}v\cdot V .$$

Finally use $U_G\tilde{T}^a U^\dagger_G = \tilde{T}^b U^{ba}_G$ to establish the theorem. Define

$$R(x) = g^2 f^{abc}c_x\int_y \bar{c}^{\,a}_y(U_G\partial_\mu\beta)^b_y A^{c\mu}_y ,$$
$$\bar{\eta}' = \bar{\eta}U^\dagger_F(1+R) , \tag{4.15}$$
$$\eta' = (1-R)U_F\eta .$$

Then we have shown that

$$\int_x \mathscr{J}_F^* \mathscr{F}' = \int_x (\bar{\eta}'\psi + \bar{\psi}\eta') \,. \tag{4.16}$$

Equation (4.13) follows from Eqs. (3.18) and (4.16).

If we define

$$\frac{1}{Z^0[0]} \frac{\delta}{\delta\bar{\eta}(x_2)} \frac{-\delta}{\delta\eta(x_1)} Z^{a,G+\Delta G}[\mathscr{J}] \Bigg|_{\mathscr{J}=0} = \langle \psi(x_2)\bar{\psi}(x_1) \rangle^a \,, \tag{4.17}$$

then

$$\langle \psi(x_2)\bar{\psi}(x_1)\rangle^a = U_F^\dagger(x_2) \langle [1+R(x_2)]\psi(x_2)\bar{\psi}(x_1)[1-R(x_1)] \rangle^{a=0} \times U_F(x_1) \,, \tag{4.18}$$

where by $a=0$ we mean that a is set to zero everywhere *except* in R. In Fig. 3 we illustrate Eq. (4.18) in the case of a single soft gluon (analytic expressions for pencil and ruler symbols given in Appendix C). Figure 4 gives the g^3 part of Fig. 3. Defining

$$R_2(x_2,x_2') = \langle \psi(x_2) R(x_2)(-ig) A\!\!\!/(x_2')\bar{\psi}(x_2') \rangle^{a=0} \,,$$
$$R_1(x_1',x_1) = \langle \psi(x_1')(-ig) A\!\!\!/(x_1') R(x_1)\bar{\psi}(x_1) \rangle^{a=0} \,, \tag{4.19}$$

it is clear from Figs. 3 and 4 that one can rewrite Eq. (4.18) in the alternative form

$$\langle \psi(x_2)\bar{\psi}(x_1)\rangle^a = U_F^\dagger(x_2) \int_{x_1'x_2'} [\delta(x_2-x_2') + R_2(x_2,x_2')] \times \langle \psi(x_2')\bar{\psi}(x_1') \rangle^{a=0} \times [\delta(x_1'-x_1) + R_1(x_1',x_1)] U_F(x_1) \,. \tag{4.20}$$

(We require R_1 and R_2 to be one particle irreducible in the x_1' and x_2' legs, respectively.)

Equation (4.20) is the main result of this paper. It generalizes Eq. (1.6) to QCD. The exponentials of Eq. (1.6) have become time-ordered exponentials. In addition, the R terms have appeared. (It can be shown that the sum of the last two diagrams on the right side of Fig. 4 is leading by power counting. Hence, at least at order g^3, the R terms produce leading contributions and must be included.)

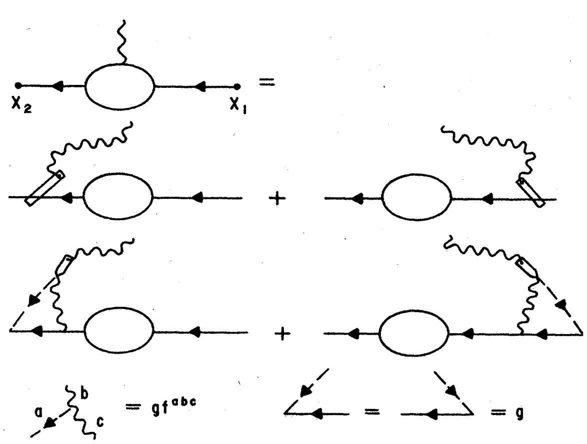

FIG. 3. Equation (4.18) for the case of one soft gluon.

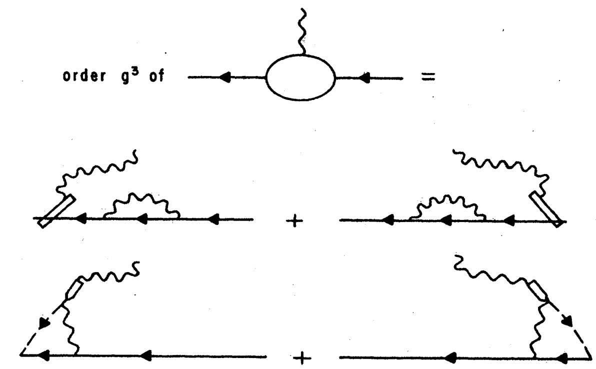

FIG. 4. Order-g^3 part of Fig. 3.

V. TREE WARD IDENTITIES

Equations (2.15) and (3.18) together imply useful exact identities connecting U_F, U_G, and $\mathbf{V}_\mu$. To obtain these identities one sets independently to zero the part of $\mathscr{L}_{\text{eff}}(\mathscr{F}',V_\mu) - \mathscr{L}_{\text{eff}}(\mathscr{F},0)$ with a given number of ψ, $\bar{\psi}$, A, c, $\bar{c}$ fields. The identities are of two types: those which contain inverse propagators and those which do not. The ones that do not are

$$\bar{\psi}' A\!\!\!/' \psi' = \psi A\!\!\!/ \psi \,, \tag{5.1}$$

$$Y_{123} A'^1 A'^2 A'^3 + H_{1234} A'^1 A'^2 A'^3 V^4 = Y_{123} A^1 A^2 A^3 \,, \tag{5.2}$$

$$H_{1234} A'^1 A'^2 A'^3 A'^4 = H_{1234} A^1 A^2 A^3 A^4 \,, \tag{5.3}$$

$$E_{123} \bar{c}'^1 c'^2 A'^3 + K_{(14)(23)} \bar{c}'^1 c'^2 A'^3 V^4 = E_{123} \bar{c}^1 c^2 A^3 \,. \tag{5.4}$$

One can cancel the field factors out of Eqs. (5.1) to (5.4). What remains are identities satisfied by U_F, U_G, $\mathbf{V}_\mu$, and the vertices. For example, Eq. (5.2) becomes

$$Y^{abc}_{\alpha\beta\gamma}(i\partial_{x_1}, i\partial_{x_2}, i\partial_{x_3})[U_G^{\dagger a\bar{a}}(x_1) U_G^{\dagger b\bar{b}}(x_2) U_G^{\dagger c\bar{c}}(x_3) \times e^{-ik_1x_1 - ik_2x_2 - ik_3x_3}] + H^{abcd}_{\alpha\beta\gamma\delta} U_G^{\dagger a\bar{a}}(x_1) U_G^{\dagger b\bar{b}}(x_2) U_G^{\dagger c\bar{c}}(x_3) V^{d\delta}(x_4) = Y^{\bar{a}\bar{b}\bar{c}}_{\alpha\beta\gamma}(i\partial_{x_r}) e^{-i\sum k_j x_j} \tag{5.5}$$

which is an identity first proposed in Ref. 5.

In addition to Eqs. (5.1) to (5.4), one finds three other identities, each one with a different inverse propagator:

$$\bar{\psi}' \left[\frac{i\partial\!\!\!/ - m}{i} + igV\!\!\!\!/ \right] \psi' = \bar{\psi} \left[\frac{i\partial\!\!\!/ - m}{i} \right] \psi \,, \tag{5.6}$$

$$-\frac{i}{4}(\partial_\mu \mathbf{A}'_\nu-\partial_\nu \mathbf{A}'_\mu)^2-\frac{i}{2\xi}(\partial_\mu \mathbf{A}'^\mu)^2$$
$$+(Y+Y')_{123}\frac{A'^1A'^2V^3}{2}+(H+H')_{1234}\frac{A'^1A'^2V^3V^4}{4}$$
$$=-\frac{i}{4}(\partial_\mu \mathbf{A}_\nu-\partial_\nu \mathbf{A}_\mu)-\frac{i}{2\xi}(\partial_\mu \mathbf{A}^\mu)^2 , \quad (5.7)$$

$$-\bar{\mathbf{c}}'\frac{\Box}{i}\mathbf{c}'+(E+E')_{123}\bar{c}'^1c'^2V^3$$
$$+(K_{(14)(23)}+K_{(13)(24)})\frac{\bar{c}'^1c'^2V^3V^4}{2}=-\bar{\mathbf{c}}\frac{\Box}{i}\mathbf{c} . \quad (5.8)$$

Equations (5.6) to (5.8) are most useful after taking their inverse (which makes them nonlocal). For example, consider Eq. (5.6). Let $\{\,|x\rangle\,|\,x\in\mathbb{R}^4\}$ be a vector space with linear functionals $\langle y|$ acting on it such that $\langle y\,|\,x\rangle=\delta(x-y)$. We define U_F to be a matrix (in color and Dirac space) of operators acting on $|x\rangle$ with matrix elements

$$\langle x\,|\,\hat{U}_F^{A\alpha,B\beta}\,|\,y\rangle=U_F^{AB}(x)\delta(x-y)\delta_{\alpha\beta} , \quad (5.9)$$

where $A=1,2,3$ and $\alpha=1,\ldots,4$. Replacing $\bar{\psi}(x)$ by $\delta(z-x)$ and $\psi(x)$ by $\delta(x-y)$ in Eq. (5.6) yields

$$\langle z\,|\,\hat{U}_F\,|\,x\rangle\left[\frac{i\partial\!\!\!/_x-m}{1}+ig V\!\!\!\!/(x)\right]\langle x\,|\,\hat{U}_F^\dagger\,|\,y\rangle$$
$$=\delta(z-x)\left[\frac{i\partial\!\!\!/_x-m}{i}\right]\delta(x-y) . \quad (5.10)$$

At this point it is convenient to define operators $\hat{x}_\mu$ and $\hat{\partial}_\mu$ on the space $\{\,|x\rangle\,|\,x\in\mathbb{R}^4\}$

$$\hat{x}_\mu=\int_x |x\rangle x_\mu\langle x| , \quad (5.11)$$
$$\hat{\partial}_\mu=\int_x |x\rangle(\partial_\mu^x\langle x|) . \quad (5.12)$$

It follows that the electron propagator $S(z-y)=[i/(i\partial\!\!\!/_z-m)]\delta(z-y)$ equals $\langle z\,|\,\hat{S}\,|\,y\rangle$ where $\hat{S}=i/(i\hat{\partial\!\!\!/}-m)$. In the convenient hat notation, Eq. (5.10) becomes

$$\langle z\,|\,\hat{U}_F\,|\,x\rangle\left\langle x\left|\left[\frac{1}{\hat{S}}+igV\!\!\!\!/(\hat{x})\right]\hat{U}_F^\dagger\right|y\right\rangle$$
$$=\langle z\,|\,x\rangle\left\langle x\left|\frac{1}{\hat{S}}\right|y\right\rangle . \quad (5.13)$$

Integrating over x and taking the inverse of both sides yields

$$\frac{1}{1/\hat{S}+igV\!\!\!\!/(\hat{x})}=\hat{U}_F^\dagger\hat{S}\,\hat{U}_F . \quad (5.14)$$

Equation (5.14) can be used in conjunction with $1/(A-B)=1/A+(1/A)B(1/A)+\cdots$, where $A=1/\hat{S}$. Figure 5 illustrates the order-g^2 part of Eq. (5.14). Equation (5.14) is very similar to an identity proposed by Collins and Soper (CS) in Ref. 4. However, note that $V\!\!\!\!/$ is not just the wedge (GY substitution) as in the CS identity but the sum of all pencils (which equals the wedge at lowest order in g). Whereas the CS identity is approximate, Eq. (5.14) is exact. Combining Eq. (5.14) with Appendix D, one easily arrives at the CS identity.

Defining

$$\langle x\,|\,\hat{U}_G^{a\alpha,b\beta}\,|\,y\rangle=U_G^{a\alpha,b\beta}(x)\delta(x-y)g^{\alpha\beta} , \quad (5.15)$$

$$L=\int_{x_1x_2x_3x_4}\delta(x-x_1)\delta(x-x_2)\delta(x-x_3)\delta(x-x_4) , \quad (5.16)$$

Eq. (5.7) can be written as

$$L\left[-\frac{i}{2}\partial_\mu^{x_1}\partial^{x_2\mu}g_{\alpha\beta}\delta_{ab}+\frac{i}{2}\partial_\beta^{x_1}\partial_\alpha^{x_2}\delta_{ab}-\frac{i}{2\xi}\partial_\alpha^{x_1}\partial_\beta^{x_2}\delta_{ab}+(Y+Y')_{123}\frac{V^3}{2}\right.$$
$$\left.+(H+H')_{1234}\frac{V^3V^4}{4}\right]\langle z\,|\,\hat{U}_G^{\bar{a}\bar{\alpha},a\alpha}\,|\,x_1\rangle\langle x_2\,|\,\hat{U}_G^{\dagger b\beta,\bar{b}\bar{\beta}}\,|\,y\rangle$$
$$=L\left[-\frac{i}{2}\partial_\mu^{x_1}\partial^{x_2\mu}g^{\bar{\alpha}\bar{\beta}}+\frac{i}{2}\partial^{\bar{\beta}x_1}\partial^{\bar{\alpha}x_2}-\frac{i}{2\xi}\partial^{\bar{\alpha}x_1}\partial^{\bar{\beta}x_2}\right]\delta_{\bar{a}\bar{b}}\delta(z-x_1)\delta(y-x_2) . \quad (5.17)$$

At this point it is convenient to introduce the following operators,

$$(\hat{D}^{-1})_{a\alpha,b\beta}=-\frac{i}{2}\delta_{ab}\left[\hat{\partial}^2g_{\alpha\beta}+\left[-1+\frac{1}{\xi}\right]\hat{\partial}_\beta\hat{\partial}_\alpha\right]$$
$$(\langle x\,|\,\hat{D}\,|\,y\rangle \text{ is the gluon propagator}) , \quad (5.18)$$

$$\hat{Y}_{a\alpha,b\beta}=\int_{xx_1x_2x_3}\delta(x-x_1)\delta(x-x_2)\delta(x-x_3)$$
$$\times(Y+Y')_{a\alpha,b\beta,c\gamma}(i\partial_{x_1},i\partial_{x_2},i\partial_{x_3})\frac{V^{c\gamma}(x_3)}{2}\,|\,x_1\rangle\langle x_2\,| , \quad (5.19)$$

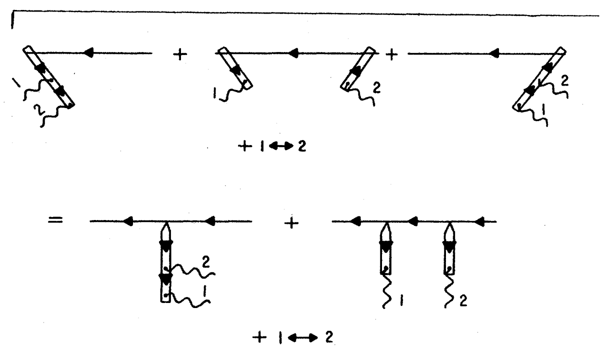

FIG. 5. Order-g^2 part of Eq. (5.14).

$$\hat{H}_{a\alpha,b\beta}=(H+H')_{a\alpha,b\beta,c\gamma,d\delta}V^{c\gamma}(\hat{x})V^{d\delta}(\hat{x}) . \tag{5.20}$$

In terms of the above operators, (5.17) becomes

$$\frac{1}{1/\hat{D}-\hat{Y}-\hat{H}}=\hat{U}_G^\dagger \hat{D}\,\hat{U}_G . \tag{5.21}$$

The final identity [Eq. (5.8)] is inverted by the same techniques employed above. The result is

$$\frac{1}{(\hat{\bar{\Delta}})^{-1}-\hat{E}-\hat{K}}=\hat{U}_G^\dagger \hat{\bar{\Delta}}\,\hat{U}_G , \tag{5.22}$$

$$(\hat{\bar{\Delta}}^{-1})_{ab}=\frac{\hat{\partial}^2}{i}\delta_{ab} \tag{5.23}$$

[$\langle x\,|\,\bar{\Delta}\,|\,y\rangle$ is the ghost propagator $(+i/\Box)\,\delta(x-y)$. We put a dash over the Δ because we reserve $\Delta(x-y)$ for $(-i/\Box)\delta(x-y)$, the scalar propagator],

$$\begin{aligned}(\hat{E})_{ab}=\int_{xx_1x_2x_3}&\delta(x-x_1)\delta(x-x_2)\delta(x-x_3)\\&\times(E+E')_{a,b,c\gamma}(i\partial_{x_1},i\partial_{x_2})\\&\times V^{c\gamma}(x_3)\,|\,x_1\rangle\langle x_2\,| ,\end{aligned} \tag{5.24}$$

$$\hat{K}_{ab}=[K_{(14)(23)}+K_{(13)(24)}]V^{c\gamma}(\hat{x})V^{d\delta}(\hat{x}) . \tag{5.25}$$

The identities Eqs. (5.1) to (5.4), (5.14), (5.21), and (5.22) which we have just derived are in 1-1 correspondence with the diagrammatic identities used by 't Hooft[11] to show the renormalizability of non-Abelian gauge theories. In fact, the order-g part of our identities are exactly 't Hooft's identities. Thus 't Hooft's identities can be understood as describing a jet under an infinitesimal (first order in g) soft background, whereas ours describe the more general situation of a jet under a finite (arbitrarily high order in g) soft background.

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APPENDIX A: DEFINITION OF $(v\cdot\partial)^{-1}$

For finite $\epsilon>0$ one can show by contour integration that

$$\frac{i}{k\cdot v+i\epsilon}=\int_0^\infty d\eta\, e^{i\eta(k\cdot v+i\epsilon)} . \tag{A1}$$

Letting $k\to -k$ in (A1) yields

$$\frac{-i}{k\cdot v-i\epsilon}=\int_0^\infty d\eta\, e^{-i\eta(k\cdot v-i\epsilon)} . \tag{A2}$$

We define

$$\begin{aligned}\frac{1}{v\cdot\partial\pm\epsilon}f(x)&=\frac{1}{v\cdot\partial\pm\epsilon}\int\frac{d^4k}{(2\pi)^4}e^{-ikx}f(k)\\&=\int\frac{d^4k}{(2\pi)^4}\frac{1}{-iv\cdot k\pm\epsilon}e^{-ik\cdot v}f(k) .\end{aligned} \tag{A3}$$

By substituting (A1) [or (A2)] and $f(k)=\int d^4x\, e^{ikx}f(x)$ into (A3) one can show that

$$\frac{1}{v\cdot\partial+\epsilon}=\int_0^\infty d\eta\, e^{-\eta(v\cdot\partial+\epsilon)} , \tag{A4}$$

$$\begin{aligned}&\frac{1}{v\cdot\partial-\epsilon}=-\int_0^\infty d\eta\, e^{\eta(v\cdot\partial-\epsilon)}\\&[e^{\eta v\cdot\partial}f(x)=f(x+\eta v)] .\end{aligned} \tag{A5}$$

Identity (A4) may be obtained nonrigorously by replacing k by $i\partial$ in (A1). (A5) follows from (A4) if we let $x\to -x$.

So far we have kept ϵ finite. As $\epsilon\to 0$, $1/(v\cdot\partial+\epsilon)\neq 1/(v\cdot\partial-\epsilon)$ so it is important to write the ϵ and not just $1/(v\cdot\partial)$.

APPENDIX B: PROOF OF EQ. (3.18)

In this appendix we prove Eq. (3.18) directly using Eqs. (3.16), (3.17), and (3.11) for U_F, U_G, and $\mathbf{V}_\mu$, respectively.

Theorem B1. $\partial_\mu U_F^\dagger=-igV_\mu U_F^\dagger$.

Proof. Express $U_F^\dagger$ and V_μ in terms of $v\cdot a(x+\eta v)$. Then use $e^XYe^{-X}=e^{[X,\,]}Y$ (Baker-Campbell-Hausdorff formula).

Theorem B2. $U_F\mathscr{D}_\mu(V)U_F^\dagger=\partial_\mu$ and $U_G\bar{\mathscr{D}}(V)U_G^\dagger=\partial_\mu$.

Proof. This follows trivially from Theorem B1.

Theorem B3. $U_F\mathscr{D}_\mu(A'+V)U_F^\dagger=\mathscr{D}_\mu(A)$ and $U_G\bar{\mathscr{D}}_\mu(A'+V)U_G^\dagger=\bar{\mathscr{D}}_\mu(A)$.

Proof. Use Theorem B2 and $\bar{\psi}'A'\psi'=\bar{\psi}A\psi$.

The proof of Eq. (3.18) is an almost immediate consequence of the above theorems:

$$\begin{aligned}F^{\mu\nu}(A'+V)&=-\frac{i}{g}[\mathscr{D}^\mu(A'+V),\mathscr{D}^\nu(A'+V)]\\&=U_F^\dagger F^{\mu\nu}(A)U_F\end{aligned}$$

by Theorem B3. Thus

$$\mathrm{Tr}[F^{\mu\nu}(A'+V)F_{\mu\nu}(A'+V)]=\mathrm{Tr}[F^{\mu\nu}(A)F_{\mu\nu}(A)] .$$

Similarly

$$F^{\mu\nu}(V)=-(i/g)[\mathscr{D}^\mu(V),\mathscr{D}^\nu(V)]=U_F^\dagger[\partial^\mu,\partial^\nu]U_F=0$$

by Theorem B2. The fact that $\mathbf{F}^{\mu\nu}(V)=0$ is not a surprise since $\mathbf{F}^{\mu\nu}(V)$ is gauge covariant and $\mathbf{V}_\mu$ can be gauge transformed to zero. By Theorem B3

$$\bar{\psi}'[i\mathscr{D}(A'+V)-m]\psi'=\psi[i\mathscr{D}(A)-m]\psi .$$

Also by Theorem B3,

$$\mathbf{G}(A',V)=\bar{\mathscr{D}}_\mu(V)\mathbf{A}'^\mu=U_G^\dagger\partial_\mu\mathbf{A}^\mu$$

so

$$\mathbf{G}(A',V)^2=(\partial_\mu A^\mu)^2 .$$

Finally, $\bar{\mathbf{c}}'\bar{\mathscr{D}}_\mu(V)\bar{\mathscr{D}}^\mu(A'+V)\mathbf{c}'=\bar{\mathbf{c}}\partial_\mu\bar{\mathscr{D}}^\mu(A)\mathbf{c}$.

APPENDIX C: FEYNMAN RULES FOR PENCILS AND RULERS

From Eq. (3.16) one gets

$$U_F(x)=\sum_{N=0}^{\infty}\int_0^{\infty}d\lambda_1\cdots\int_0^{\infty}d\lambda_N(-ig)v\cdot V(x+v(\lambda_1+\cdots+\lambda_N))$$
$$\times(-ig)v\cdot V(x+v(\lambda_2+\cdots+\lambda_N))\cdots(-ig)v\cdot V(x+v\lambda_N)\,. \tag{C1}$$

Next we use

$$\frac{i}{(q_1+\cdots+q_i)\cdot v+i\epsilon}=\int_0^{\infty}d\lambda_i\exp\{i\lambda_i[(q_1+\cdots+q_i)\cdot v+i\epsilon]\}\,, \tag{C2}$$

$$V(x)=\sum_{q_i}V(-q_i)e^{iq_ix} \tag{C3}$$

to get

$$U_F(0)=\sum_{N=0}^{\infty}\sum_{q_1\cdots q_N}(-ig)v\cdot V(-q_1)\frac{i}{q_1\cdot v+i\epsilon}(-ig)v\cdot V(-q_2)$$
$$\times\frac{i}{(q_1+q_2)\cdot v+i\epsilon}\cdots(-ig)v\cdot V(-q_N)\frac{i}{(q_1+\cdots+q_N)\cdot v+i\epsilon}\,. \tag{C4}$$

[Whenever we write $\sum_q$ we mean a discrete sum over four-momenta and thus $(1/V_4)\sum_q$ is identical to $\int d^4q/(2\pi)^4$ and $V_4\delta_K(q)=(2\pi)^4\delta^{(4)}(q)$, where δ_K is the Kronecker delta. For convenience, we set the volume of four-dimensional space-time V_4 to 1. This is all right since no physical result may depend on it.]

Replacing V by $\tilde{V}$ above, one gets

$$U_G^{b'b}(0)=\sum_{N=0}^{\infty}\sum_{q_1\cdots q_N}(-g)v_{\alpha_1}V_{a_1}^{\alpha_1}(-q_1)f^{a_1b'b_2}\frac{i}{q_1\cdot v+i\epsilon}(-g)v_{\alpha_2}V_{a_2}^{\alpha_2}(-q_2)f^{a_2b_2b_3}$$
$$\times\frac{i}{(q_1+q_2)\cdot v+i\epsilon}\cdots(-g)v_{\alpha_N}V_{a_N}^{\alpha_N}(-q_N)f^{a_Nb_Nb}\frac{i}{(q_1+q_2+\cdots+q_N)\cdot v+i\epsilon}\,. \tag{C5}$$

By similar techniques, one shows

$$V_8^d(0)=\sum_{N=1}^{\infty}\sum_{q_1\cdots q_N}[-iq_{18}v^{\alpha_1}\delta^{a_1r_1}V_{\alpha_1}^{a_1}(-q_1)]\frac{i}{q_1\cdot v+i\epsilon}[-gf^{a_2r_1r_2}v^{\alpha_2}V_{\alpha_2}^{a_2}(-q_2)]$$
$$\times\frac{i}{(q_1+q_2)\cdot v+i\epsilon}[-gf^{a_3r_2r_3}v^{\alpha_3}V_{\alpha_3}^{a_3}(-q_3)]\cdots[-gf^{a_Nr_{N-1}r_N}v^{\alpha_N}V_{\alpha_N}^{a_N}(-q_N)]$$
$$\times\frac{i}{(q_1+q_2+\cdots+q_N)\cdot v\;\;i\epsilon}\delta^{r_Nd}\,. \tag{C6}$$

Equations (C4), (C5), and (C6) can be represented diagrammatically by introducing pencil and ruler symbols to be used in conjunction with the usual Feynman rules. Reading off the g^2 term of Eq. (C4) yields

$$\text{Fig. 6(a)}=(-igv^{\alpha_1})(T^{a_1})_{ir}\frac{i}{q_1\cdot v+i\epsilon}(-igv^{\alpha_2})(T^{a_2})_{rj}\frac{i}{(q_1+q_2)\cdot v+i\epsilon}\,. \tag{C7}$$

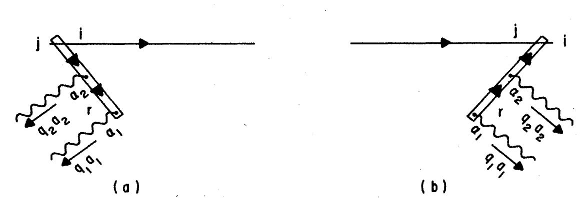

FIG. 6. Diagrammatic representation of order-g^2 parts of (a) U_F, (b) $U_F^\dagger$.

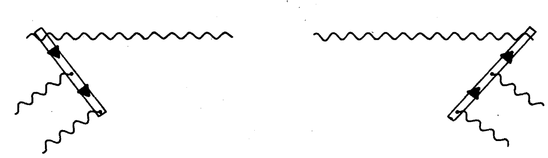

FIG. 7. Same as Fig. 6 but for U_G.

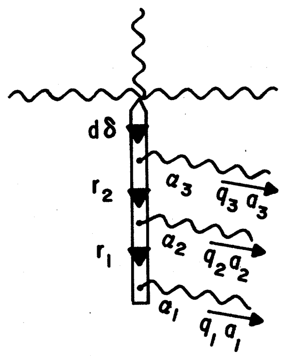

FIG. 8. Diagrammatic representation of order-g^2 part of V^d_δ.

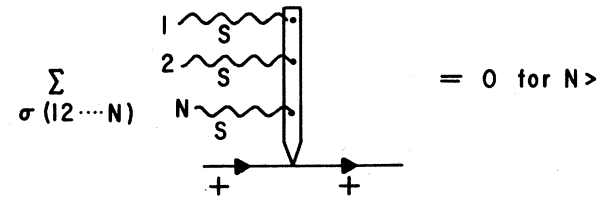

FIG. 9. A pencil couples $N>1$ soft gluons to a jet + fermion. If we sum over all permutations of the soft gluons, we get zero.

From the Hermitian transpose of Eq. (C5), one gets

$$\text{Fig. 6(b)}=\frac{i}{-(q_1+q_2)\cdot v+i\epsilon}igv^{\alpha_2}(T^{a_2})_{ir}\frac{i}{-q_1\cdot v+i\epsilon}igv^{\alpha_1}(T^{a_1})_{rj}\ . \tag{C8}$$

The analytic expressions corresponding to Figs. 7(a) and 7(b) can be obtained from Eqs. (C7) and (C8) by replacing T^a by $\tilde{T}^a$. Finally, reading off from Eq. (C6) we get

$$\text{Fig. 8}=(-iq_{1\delta}v^{\alpha_1}\delta^{a_1r_1})\frac{i}{q_1\cdot v+i\epsilon}(-gf^{a_2r_1r_2}v^{\alpha_2})\frac{i}{(q_1+q_2)\cdot v+i\epsilon}(-gf^{a_3r_2d}v^{\alpha_3})\frac{i}{(q_1+q_2+q_3)\cdot v+i\epsilon}\ . \tag{C9}$$

APPENDIX D: $\mathbf{V}^\mu$ EFFECTIVELY LONGITUDINAL

If $\mathbf{V}^\mu(x)$ is coupled to a jet, then according to step 2 of Sec. IV only the minus component of $\mathbf{V}^\mu$ is seen by the jet. Define the longitudinal BF $\mathbf{a}_L^\mu$ by

$$\mathbf{a}_L^\mu(x)=-\int_0^\infty d\eta'\partial^\mu v\cdot\mathbf{V}(x+\eta' v)\ .$$

Note that

$$\mathbf{a}_L^-(x)=-\int_0^\infty d\eta'\frac{d}{d\eta'}v\cdot\mathbf{V}(x+\eta' v)=\mathbf{V}^-(x)\ .$$

Since $\mathbf{a}_L^\mu$ and $\mathbf{V}^\mu$ have the same minus components, we can replace $\mathbf{V}^\mu$ by $\mathbf{a}_L^\mu$. Thus the jet sees only the longitudinal part of $\mathbf{V}^\mu$ [i.e., the $N=1$ term or zeroth order in g term of Eq. (C6)]. A direct consequence of this observation is the diagrammatic identity[12] given in Fig. 9. In this figure, a pencil couples N soft gluons to a jet + fermion. If $N>1$ and we sum over all permutations of the soft gluons, the result is zero to leading order in m/Q.

[1]R. Tucci, Argonne Report No. ANL-HEP-PR-84-37 (unpublished).

[2]G. Grammer and D. Yennie, Phys. Rev. D **8**, 4332 (1973).

[3]Actually Ref. 2 replaces all photon propagators (not just the soft ones) by $g^{\mu\nu}/k^2=G^{\mu\nu}/k^2+K^{\mu\nu}/k^2$. G and K are the transverse and longitudinal parts *if* the photon is soft and connects at the ends to jet + and jet − fermions. Roughly, in Ref. 2

$$K^{\mu\nu}=\frac{v_i^\mu}{v_i\cdot k}\frac{v_f^\nu}{v_f\cdot k}$$

so these workers use $v^\mu/(v\cdot k)$ at both ends of the propagator. In justifying the GY approximation, the possibility of Glauber photons with large k_T must be dealt with (see Refs. 6 and 7). The GY approximation has also been used to describe the coupling of jet + and jet − particles instead of soft and collinear. For examples and justification of this, see Refs. 6 and 7.

[4]J. Collins and D. Soper, Nucl. Phys. **B193**, 381 (1981).

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[6]J. Collins, D. Soper, and G. Sterman, Nucl. Phys. **B223**, 381 (1983); Phys. Lett. **109B**, 388 (1982); **126B**, 275 (1983); **134B**, 263 (1984); University of Oregon Report No. OITS287 (unpublished).

[7]G. Bodwin, Phys. Rev. D **31**, 2616 (1985).

[8]For a clear set of lectures see L. Abbott, Acta Phys. Pol. **B13**, 33 (1982). Original references cited therein.

[9]For the analog in one-dimensional quantum mechanics, see R. Glauber, in *Lectures in Theoretical Physics,* Vol. I, Boulder, 1958, edited by W. Brittin and L. Dunham (Interscience, New York, 1959).

[10]Our argument relating $Z^{a,G+\Delta G}$ to $Z^{a,G}$ is a trivial modification of the proof due to B. Lee and J. Zinn-Justin which shows that S-matrix elements of non-Abelian gauge theories are independent of gauge fixing. See B. Lee and J. Zinn-Justin, Phys. Rev. D **7**, 1049 (1973); B. Lee, in *Methods in Field Theory,* 1975 Les Houches Lectures, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[11]G. 't Hooft, Nucl. Phys. **B33**, 173 (1971). For an excellent set of lectures on these identities, see P. Cvitanovic, *Field Theory* (NORDITA, Copenhagen, Denmark, 1983).

[12]I thank G. Bodwin for suggesting this identity to me.