Derivation of the Lee-Yang term via stochastic quantization

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We derive the so called Lee-Yang term for velocity-dependent potentials via stochastic quantization. The derivation shows that there is a subtlety in choosing the appropriate Langevin equation, i.e., the Langevin equation must have a positive-definite Fokker-Planck Hamiltonian. In the derivation, we also make use of the Stratonovic calculus to transform a stochastic equation with an additive white noise to a stochastic equation with multiplicative white noise.

Consider the Lagrangian with a velocity-dependent potential given by

$$L=\tfrac{1}{2}\dot{q}_iA_{ij}(q)\dot{q}_j-V(q)\ , \tag{1}$$

where the dot represents differentiation with respect to t. We will assume for simplicity that A_{ij} is symmetric and that $\det A\neq 0$, i.e., the Lagrangian is nonsingular.

The Euclideanized effective action for (1) was first[1] shown to be

$$S_{\text{eff}}[q]=\int dt[\tfrac{1}{2}\dot{q}_iA_{ij}\dot{q}_j+V(q)]-\tfrac{1}{2}\delta(0)\operatorname{tr}\ln A\ , \tag{2}$$

via canonical quantization. Later on, the same effective action was derived via the path-integral approach to quantization.[2] We would like to derive in this paper this effective action via the stochastic approach to quantization.[3]

In the stochastic approach to quantization, one introduces an extra "time" τ and a random white noise $\eta(t,\tau)$. Then one solves the Langevin equation

$$\frac{\partial q_i}{\partial\tau}=-\frac{\delta S_E}{\delta q_i}+\eta_i(t,\tau)\ , \tag{3}$$

where S_E is the Euclidean action given by the first term in (2). Euclidean quantum theory is given by the prescription

$$\lim_{\tau\to\infty}\langle q_1(t_1,\tau;\eta)\cdots q_n(t_n,\tau;\eta)\rangle_\eta=\langle 0|T(q_1(t_1)\cdots q_n(t_n))|0\rangle\ , \tag{4}$$

where $q(t,\tau;\eta)$ represents a solution of (3) and $\langle\ \rangle_\eta$ means averaging over the white noise η.

In practice, Eq. (4) is verified perturbatively. To prove (4) nonperturbatively, one relies on the Fokker-Planck equation for the distribution function $P[q(t);\tau]$, corresponding to the Langevin equation (3). The distribution $P[q(t);\tau]$ satisfies the boundary condition $P[q(t);\tau=0]=\delta[q(t)]$, and

$$-\frac{\partial P}{\partial\tau}=H_{\text{FP}}[q]P[q(t);\tau]\ , \tag{5}$$

$$H_{\text{FP}}[q]=\int dt\frac{\delta}{\delta q_i(t)}\left[\frac{\delta}{\delta q_i(t)}+\frac{\delta S_E}{\delta q_i}\right]\ . \tag{6}$$

The relation with the Langevin formulation is given by

$$\langle q_1(t_1,\tau;\eta)\cdots q_n(t_n,\tau;\eta)\rangle_\eta=\int[dq(t)]q_i(t_1)\cdots q_n(t_n)P[q(t);\tau]\ . \tag{7}$$

Then, Eq. (4) is proved by showing that

$$\lim_{\tau\to\infty}P[q(t);\tau]\sim e^{-S_E[q(t)]}\ . \tag{8}$$

However, if we naively apply the above procedure to the theory given by the Lagrangian (1), we get into trouble. Writing

$$P[q(t);\tau]=\sum_n C_n\psi_n[q(t)]e^{-E_n\tau}\ , \tag{9}$$

we find that $\psi_n[q(t)]$ satisfies

$$H_{\text{FP}}[q]\psi_n[q]=E_n\psi_n[q]\ . \tag{10}$$

The prescription given by Eqs. (3)–(8) only works if the $H_{\text{FP}}[q]$ given by (6) is positive semidefinite ($E_n\geq 0$), and the ground state $\psi_0[q]$ (with energy $E_0=0$) is unique. If these assumptions are valid, then it follows from (9) and the fact that $e^{-S_E[q]}$ satisfies (10) with $E=0$ that (8) is valid. But for the Lagrangian (1), we know that (8) is the wrong answer, because the canonical and path-integral methods have shown that the correct answer is

$$\lim_{\tau\to\infty}P[q(t);\tau]\sim e^{-S_{\text{eff}}[q]}\ , \tag{11}$$

with $S_{\text{eff}}[q]$ given by (2). So, where is the problem? The problem lies in the fact that the H_{FP} for (1) is not positive semidefinite,

$$H_{\text{FP}}[q]=\int dt\frac{\delta}{\delta q_i(t)}\left[\frac{\delta}{\delta q_i(t)}+A_{ij}\ddot{q}_j+\frac{1}{2}\left(2\frac{\partial A_{ij}}{\partial q_k}-\frac{\partial A_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k+\frac{\partial V}{\partial q_i}\right]\ . \tag{12}$$

The third term in the brackets is the source of the nonpositive definiteness of H_{FP}.

Because of the above problem, we want to argue in this paper that the Langevin equation (3) is not the correct starting point for the stochastic quantization. We will derive another Langevin equation and show that it is the correct starting point for stochastically quantizing (1) by deriving (11) from it.

Let us define a new coordinate x by[4] $q_i=q_i(x)$ such that (1) becomes

$$L=\tfrac{1}{2}\dot{x}_i\dot{x}_i-V'(x)\ , \tag{13}$$

and $V'(x)=V(q(x))$. This means that

$$h_{lk}(q)h_{jl}(q)A_{ij}(q)=\delta_{kl} \ , \tag{14}$$

where

$$h_{ij}=\frac{\partial q_i}{\partial x_j} \ . \tag{15}$$

Note that we can always do the above transformation because the theory is nonsingular.

The Lagrangian given by (13) is now in a regular form. The Langevin equation for x is

$$\frac{\partial x_i}{\partial \tau}=-\frac{\delta S_E'}{\partial x_i}+\eta_i(t,\tau) \ , \tag{16}$$

where

$$S_E'[x]=\int dt[\tfrac{1}{2}\dot{x}_i\dot{x}_i+V'(x)] \ . \tag{17}$$

The corresponding Fokker-Planck equation for the distribution $P'[x(t),\tau]$ is

$$-\frac{\partial P'}{\partial \tau}=\int dt\frac{\delta}{\delta x_i(t)}\left[\frac{\delta}{\delta x_i(t)}+\frac{\delta S_E'}{\delta x_i(t)}\right]P'[x(t);\tau] \ . \tag{18}$$

Since $\delta S_E'/\delta x_i=\ddot{x}_i+\partial V'/\partial x_i$, then the Fokker-Planck Hamiltonian $H_{\rm FP}'[x]$ given by (18) is positive semidefinite. Because of this, we know that

$$\lim_{\tau\to\infty}P'[x(t);\tau]\sim\psi_0'[x]=e^{-S_E'[x]} \ . \tag{19}$$

The correct starting point for stochastically quantizing (1) is starting from (16) and using $x_i=x_i(q)$. From (16), and using the Stratonovic calculus,[5,6] we derive the Langevin equation for q_i given by

$$\frac{\partial q_i}{\partial \tau}+h_{ik}h_{jk}\frac{\delta S_E}{\delta q_j}=h_{ij}\eta_j \ . \tag{20}$$

We claim that the Langevin equation given by (20) is the correct starting point. This Langevin equation has a multiplicative white noise, or, in the language of Namiki, Ohba, and Okano,[7] a distorted random noise.

We would like to emphasize that (20) reduces to the usual Langevin equation for velocity-independent potentials, because in this case the h_{ij}'s are constants.

The Fokker-Planck Hamiltonian corresponding to the Langevin equation (20) is

$$\tilde{H}_{\rm FP}[q]=\int dt\frac{\delta}{\delta q_i(t)}\left[A_{ik}^{-1}(q)\left[\frac{\delta S_E}{\delta q_k}+\frac{\delta}{\delta q_k}-\tfrac{1}{2}\delta(0)A_{lm}^{-1}\frac{\partial A_{ml}}{\partial q_k}\right]\right] \ . \tag{21}$$

To prove that (21) is the Fokker-Planck Hamiltonian corresponding to the Langevin equation (20), one can start from (7), take its derivative with respect to τ, use (21), and one will get the Langevin equation (20). Or, one can start from (7), evaluate $g[q_\eta(t,\tau)]=q_1(t_1,\tau;\eta)\cdots q_n(t_n,\tau;\eta)$ at time $\tau+\Delta\tau$ in two ways, i.e.,

$$\begin{aligned}\langle g[q(t)]_{\tau+\Delta\tau}\rangle&=\int dq(t)P[q(t);\tau+\Delta\tau]g[q(t)]\\&=\left\langle\int dq(t)g[q_\eta(t)]_{\tau+\Delta\tau}P[q(t),\tau]\right\rangle_\eta \ ,\end{aligned} \tag{22}$$

and Eq. (22) will give the Fokker-Planck Hamiltonian $\tilde{H}_{\rm FP}$.

Now consider

$$\tilde{\psi}_0[q]\sim\exp\{-S_E[q]+\tfrac{1}{2}\delta(0)\operatorname{tr}\ln A(q)\} \ . \tag{23}$$

This satisfies

$$\tilde{H}_{\rm FP}[q]\tilde{\psi}_0[q]=0 \ . \tag{24}$$

To complete the proof that

$$\begin{aligned}\lim_{\tau\to\infty}\tilde{P}[q(t),\tau]&\sim\tilde{\psi}_0[q]\\&\sim\exp\{-S_E[q]+\tfrac{1}{2}\delta(0)\operatorname{tr}\ln A(q)\} \ ,\end{aligned} \tag{25}$$

we only have to show that $\tilde{H}_{\rm FP}[q]$ is a positive-semidefinite operator.

We know that $H_{\rm FP}'[x]$ given by (18) is a positive-semidefinite operator. If we can show that $\tilde{H}_{\rm FP}[q]$ is a similarity transform of $H_{\rm FP}'[x]$, then $\tilde{H}_{\rm FP}[q]$ and $H_{\rm FP}'[x]$ have the same set of eigenvalues, and their eigenfunctions are also related by the similarity transformation. This means

$$\tilde{H}_{\rm FP}[q]=UH_{\rm FP}'[x(q)]U^{-1} \ , \tag{26a}$$

$$\tilde{\psi}_n[q]=U\psi_n'[x(q)] \ . \tag{26b}$$

One can easily verify that

$$U=\exp[+\tfrac{1}{2}\delta(0)\operatorname{tr}\ln A(q)] \tag{27}$$

will satisfy (26a). And, therefore, from (26b) and (19) we find that

$$\begin{aligned}\lim_{\tau\to\infty}\tilde{P}[q(t),\tau]&\sim\tilde{\psi}_0[q]\\&\sim\exp\{-S_E[q]+\tfrac{1}{2}\delta(0)\operatorname{tr}\ln A(q)\} \ .\end{aligned} \tag{28}$$

Or a more general statement is that

$$\tilde{P}[q(t),\tau]=UP'[x(t)\tau] \ . \tag{29}$$

This completes the derivation of the Lee-Yang term via stochastic quantization.

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[1]T. D. Lee and C. N. Yang, Phys. Rev. **128**, 2082 (1962).
[2]See, for example, E. S. Abers and B. W. Lee, Phys. Rep. **9C**, 1 (1973).
[3]G. Parisi and Wu Yong-Shi, Sci. Sin. **24**, 483 (1981).

[4]We assume here that the transformation from q to x is a transformation from curvilinear to Cartesian coordinates.

[5]L. Stratonovic, J. Control Optim. **4**, 362 (1966); R. E. Mortensen, J. Stat. Phys. **1**, 271 (1969); K. L. C. Hunt and J. Ross, J. Chem.

Phys. **75**, 976 (1981).

[6]Had we used the Ito calculus [K. Ito, Mem. Am. Math Soc. **4**, 1 (1951); see also Hunt and Ross in Ref. 5], we would get instead the Langevin equation

$$\frac{\partial q_i}{\partial \tau} + h_{ik} h_{jk} \frac{\delta S_E}{\delta q_j} - \frac{\partial h_{kj}}{\partial q_j} h_{ik} = h_{ik} \eta_k \ .$$

However, as argued by Hunt *et al.* (see Ref. 4) the choice of the calculus is equivalent to choosing the parameter α in deriving the path-integral representation for the distribution function. This parameter α eventually drops out in the final expression for the distribution function; i.e., the distribution function is eventually independent of the calculus used.

[7]M. Namiki, I. Ohba, and K. Okano, Prog. Theor. Phys. **72**, 350 (1984).