Chiral Jacobians and two-dimensional QED at finite temperature

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We show that in massless two-dimensional QED there is no restoration of gauge symmetry at finite temperature (because the boson mass is temperature independent), using zeta-function techniques to solve the fermionic determinant exactly.

Since the pioneering work of Dolan and Jackiw[1] on the symmetry behavior of field theories at finite temperature, it has been known that in two-dimensional electrodynamics[2] the gauge symmetry remains broken for all values of the temperature, and that the mass of the gauge boson does not change when the temperature is varied. In Ref. 1, this fact was demonstrated by showing that the vacuum-polarization tensor at finite temperature coincides with the one at zero temperature. In this Brief Report, we rederive these results using a completely different method which is similar to Fujikawa's[3] path-integral approach to chiral anomalies. Because an anomalous divergence of the axial-vector current is responsible for the mass of the "photon" field, these methods are a very convenient way to solve exactly the Schwinger model (massless QED_2) and other two-dimensional theories. This was done in Refs. 4 and 5 for the zero-temperature ($T=0$) case. To extend these calculations to $T>0$, we will use the zeta-function method to regularize functional determinants. As was shown in a recent paper,[6] this regularization scheme is equivalent to Fujikawa's (at least for theories without γ_5 couplings), and has the additional advantage of producing well-defined expressions at every intermediate step of the calculation. This is in contrast with Ref. 3, where cutoff factors have to be introduced explicitly.

We start by writing down the partition function for two-dimensional (Euclidean) electrodynamics at temperature $T=\beta^{-1}$ (for our conventions, see Ref. 6)

$$Z(\beta)=\int_{F_\beta}[dA\,d\psi\,d\bar{\psi}]\exp\left(\int_0^\beta dx^0\int_{-\infty}^{\infty}dx^1[\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/-m)\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}]\right)\ , \tag{1}$$

where gauge-fixing and ghost terms are included in the measure $[dA]$. The path integral has to be extended over the space F_β of functions defined by imposing the (anti-) periodicity conditions in the Euclidean time variable x^0:

$$A_\mu(x^0+\beta,x^1)=A_\mu(x^0,x^1)\ , \tag{2}$$

$$\psi(x^0+\beta,x^1)=-\psi(x^0,x^1)\ , \tag{3a}$$

$$\bar{\psi}(x^0+\beta,x^1)=-\bar{\psi}(x^0,x^1)\ . \tag{3b}$$

Integrating out the fermions in (2) yields

$$Z(\beta)=\int[dA]\det{}_\beta(iD\!\!\!/-m)\exp\left(-\tfrac{1}{4}\int d^2xF_{\mu\nu}F^{\mu\nu}\right)\ , \tag{4}$$

where the subscript β indicates that $(iD\!\!\!/-m)$ is restricted to the space F_β ($D_\mu\equiv\partial_\mu+ieA_\mu$).

As a first step in exactly calculating the fermionic determinant, we now determine the Jacobian for infinitesimal chiral transformations

$$\begin{aligned}\psi(x)&\to\tilde{\psi}(x)=e^{i\alpha(x)\gamma_5}\psi(x)\ ,\\ \bar{\psi}(x)&\to\tilde{\bar{\psi}}(x)=\bar{\psi}(x)e^{i\alpha(x)\gamma_5}\end{aligned} \tag{5}$$

in (1), where the boundary conditions (2), (3) are taken into account. Hereby, α is assumed to be periodic in x^0. Performing the transformations (5) in a Grassmann integral like (1), it is easy to see that the Jacobian $J_\beta[\alpha]$ associated with (5) can be represented as

$$J_\beta[\alpha]=\frac{\det_\beta(iD\!\!\!/-m)}{\det_\beta(iD\!\!\!/-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\alpha\gamma_5)}\ . \tag{6}$$

The next steps in the evaluation of (6) are the same as for the zero-temperature case in four dimensions, which was extensively discussed in Ref. 6, so that we can skip the details here. Suffice it to say that (6) can be rewritten in terms of a positive, Hermitian operator, to which the ζ-function technique can easily be applied. The result reads[7]

$$J_\beta[\alpha]=\exp\left(-i\int_0^\beta dx^0\int_{-\infty}^{\infty}dx^1\alpha(x)\mathcal{A}_\beta(x)\right)\ , \tag{7}$$

with

$$\mathcal{A}_\beta(x)=2\left.\frac{d}{ds}\right|_0\mathrm{tr}[\gamma_5 sR(s)]\ , \tag{8}$$

where

$$R(s)\equiv\lim_{x'\to x}(D\!\!\!/_x^{\,2}+m^2)^{-s}\delta(x-x')\ , \tag{9}$$

and the analytic continuation of R to the point $s=0$ is understood. The δ function in (9) was obtained via the completeness relation of a set of eigenfunctions of $D\!\!\!/^2$. Because $D\!\!\!/$ is defined on the space of antiperiodic functions (3), these eigenfunctions are also antiperiodic, and thus the Fourier representation of $\delta(x-x')$ reads

$$\delta(x-x')=\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\exp\left(i\frac{2\pi}{\beta}(n+\tfrac{1}{2})(x-x')^0\right)\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\exp[-ik_1(x-x')^1]\ . \tag{10}$$

Now we insert this in (9), introduce a parameter integral for the operator power, and insert a sum of δ functions so that the discrete n summation formally can be replaced by an integral:

$$R(s)=\lim_{x'\to x}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\exp[-(\not{D}_x^2+m^2)t]\int\frac{d^2k}{(2\pi)^2}\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}\delta\left[k_0-\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]e^{ik\cdot(x-x')} \ . \tag{11}$$

Except for the factor $2\pi\beta^{-1}$ times the sum, this is the same expression as for the zero-temperature case. Applying $\not{D}_x^2$ to the last exponential and performing the coincidence limit yields

$$R(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-m^2t}\int\frac{d^2k}{(2\pi)^2}\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}\delta\left[k_0-\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]\exp\left[-\left[-k^2+2ik\cdot D+D^2+\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\right]t\right] . \tag{12}$$

Now we use $\tau=ts^{-2}$ and $k'=ks\sqrt{\tau}$ as new integration variables; with the $(++)$ metric for Euclidean space, this gives

$$R(s)=\frac{1}{\Gamma(s)}(s^2)^s\frac{1}{s^2}\int_0^\infty d\tau\tau^{s-2}e^{-m^2s^2\tau}\int\frac{d^2k'}{(2\pi)^2}\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}\delta\left[\frac{k_0'}{s\sqrt{\tau}}-\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]e^{-k'^2}$$
$$\times\exp\left[2ik\cdot Ds\sqrt{\tau}+\left[D^2-\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\right]s^2\tau\right] \ . \tag{13}$$

Just as in the zero-temperature case, one now expands the last exponential of (13) in powers of $s\sqrt{\tau}$; it turns out that it is sufficient to calculate only the term of order $s^2\tau$:

$$R(s)=\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{\Gamma(s)}(s^2)^s\int_0^\infty d\tau\,\tau^{s-1}e^{-m^2s^2\tau}\int\frac{d^2k}{(2\pi)^2}\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}s\sqrt{\tau}\delta\left[k_0-\frac{2\pi}{\beta}s\sqrt{\tau}(n+\tfrac{1}{2})\right]e^{-k_0^2-k_1^2}[1+O(s\sqrt{\tau})]$$
$$=\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{\Gamma(s)}(s^2)^s\int_0^\infty d\tau\,\tau^{s-1}e^{-m^2s^2\tau}\frac{s\sqrt{\tau}}{\beta}\sum_{n=-\infty}^{\infty}\exp\left[-\left(\frac{2\pi}{\beta}\right)^2 s^2\tau(n+\tfrac{1}{2})^2\right]\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}e^{-k_1^2}[1+O(s\sqrt{\tau})] \ . \tag{14}$$

Here we have omitted terms which vanish upon the trace operation in (8).

Because one may ignore terms of order $s\sqrt{\tau}$, the sum over n can be calculated by an Euler-Maclaurin expansion, where only the integral contributes to the desired order:

$$\sum_{n=-\infty}^{\infty}\exp\left[-\left(\frac{2\pi}{\beta}\right)^2 s^2\tau(n+\tfrac{1}{2})^2\right]=\frac{\sqrt{\pi}\beta}{2\pi s\sqrt{\tau}}[1+O(s\sqrt{\tau})] \ . \tag{15}$$

Inserting this in (14), it is trivial to perform the τ integration and to convince oneself that the $O(s\sqrt{\tau})$ terms indeed do not contribute. Then, doing the s differentiation and the trace in (8), we end up with

$$\mathscr{A}_\beta(x)=-\frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x) \ , \tag{16}$$

which, when inserted in (7), gives the Jacobian for chiral transformations for path integrals which are extended over antiperiodic fields. Obviously, (16) is completely independent of $\beta=T^{-1}$ and coincides with the result of Roskies and Schaposnik[4] for the zero-temperature case.

The anomalous Jacobian derived so far is valid only to first order in α; what we need to solve the fermionic determinant, however, is the Jacobian for finite α, i.e., we have to iterate a sequence of infinitesimal transformations. This iteration could be done as described in Refs. 4 and 5 for the $T=0$ case. Nevertheless, we present an alternative way here which might be a little more transparent. To this end, we choose a path α_τ, $\tau\in[0,1]$, in the space of parameter functions which connects $\alpha(x)\equiv 0$ with the $\alpha(x)$ under consideration: $\alpha_0(x)\equiv 0$, $\alpha_1(x)=\alpha(x)$. Limiting ourselves to the massless theory now, Eq. (6) for noninfinitesimal α can be written as

$$J_\beta[\alpha]=\lim_{N\to\infty}\prod_{n=0}^{N-1}\frac{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_{n/N}\gamma_\mu\gamma_5)}{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_{(n+1)/N}\gamma_\mu\gamma_5)} \ , \tag{17}$$

with all the intermediate factors canceling. Taking the logarithm of (17), one gets

$$\ln J_\beta[\alpha]=\lim_{N\to\infty}\sum_{n=0}^{N-1}\ln\frac{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_{n/N}\gamma_\mu\gamma_5)}{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_{(n+1)/N}\gamma_\mu\gamma_5)}$$
$$=\lim_{N\to\infty}N\int_0^1 d\tau\ln\frac{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_\tau\gamma_\mu\gamma_5)}{\det_\beta(i\not{\partial}-e\not{A}-\partial^\mu\alpha_{\tau+1/N}\gamma_\mu\gamma_5)} \ , \tag{18}$$

because for $N\to\infty$, we may replace the sum by an integral. Now one exploits the important property of the two-dimensional γ matrices

$$\gamma_\mu\gamma_5=-i\epsilon_{\mu\nu}\gamma^\nu \tag{19}$$

to write

$$-e\not{A}-\partial^\mu\alpha_\tau\gamma_\mu\gamma_5=-e\not{A}_\tau \tag{20}$$

and

$$-e\not{A}-\partial^\mu\alpha_{\tau+1/N}\gamma_\mu\gamma_5=-e\not{A}_\tau-\gamma_\mu\gamma_5\partial^\mu\dot{\alpha}_\tau\frac{1}{N}+O\left[\frac{1}{N^2}\right] , \tag{21}$$

where the dot denotes differentiation with respect to τ, and where we have set

$$A_{\nu\tau}\equiv A_\nu+\frac{i}{e}\epsilon_{\nu\mu}\partial^\mu\alpha_\tau \ . \tag{22}$$

Upon inserting this in (18), one obtains an integral over Jacobians J_β^{inf} for infinitesimal transformations which we have already determined in the above:

$$\ln J[\alpha] = \lim_{N\to\infty} N \int_0^1 dt \ln \frac{\det_\beta(i\partial\!\!\!/ - eA\!\!\!/_\tau)}{\det_\beta[i\partial\!\!\!/ - eA\!\!\!/_\tau - \partial^\mu(\dot\alpha_\tau/N)\gamma_\mu\gamma_5]}$$
$$= \lim_{N\to\infty} N \int_0^1 dt \ln J_\beta^{\inf}\left[\frac{\dot\alpha_\tau}{N}; A^\mu \to A_\tau^\mu\right]$$
$$= \lim_{N\to\infty} N \int_0^1 dt \frac{ie}{2\pi} \int d^2x \frac{\dot\alpha_\tau(x)}{N} \epsilon_{\mu\nu} F^{\mu\nu}(A_\tau(x)) \ . \tag{23}$$

Inserting A_τ^μ and integrating over τ finally leads to the result

$$J_\beta[\alpha] = \exp\left[\frac{ie}{2\pi}\int_0^\beta dx^0 \int dx^1 \alpha(x)\left[\epsilon_{\mu\nu}F^{\mu\nu}(x) - \frac{i}{e}\partial^2\alpha(x)\right]\right] \tag{24}$$

which, except for the limited x^0-integration interval, is identical to its zero-temperature analog.[4]

If we now work in the Lorentz gauge where every $A_\nu(x)$ can be represented as

$$A_\nu(x) = \frac{i}{e}\epsilon_{\nu\mu}\partial^\mu\alpha(x) \ , \tag{25}$$

Eq. (24) together with (6) is sufficient to establish the A_μ dependence of the fermionic determinant:[8]

$$\det_\beta(i\partial\!\!\!/ - eA\!\!\!/) = \det_\beta(i\partial\!\!\!/) J[\alpha; F_{\mu\nu}=0]^{-1}$$
$$= \det_\beta(i\partial\!\!\!/) \exp\left(\frac{1}{2\pi}\int d^2x \partial_\mu\alpha\partial_\mu\alpha\right)$$
$$= \det_\beta(i\partial\!\!\!/) \exp\left(-\frac{e^2}{2\pi}\int d^2x A_\mu A_\mu\right) \ . \tag{26}$$

From (26) we can see that, due to its interaction with the fermions, the photon gets a mass $e/\sqrt{\pi}$, which is independent of β, i.e., at arbitrarily high temperatures gauge symmetry is not restored. This is in contrast with theories where the symmetry breaking is due to Higgs scalars; it is known that in such models, above a certain critical temperature, symmetry can be restored.[1] The temperature independence in the Schwinger model reflects the fact that the mass-generating mechanism is a chiral anomaly which has its origin in operator-product singularities at short distances, i.e., large momenta, whereas temperature modifies a theory only near the mass shell.

Now we have confirmed the results of Ref. 1 concerning massless QED_2, using ζ-function regularization and a simple iteration technique. These methods can be used equally well for other two-dimensional models, such as QCD_2 or the chiral Gross-Neveu model.[5,9] In all cases, the temperature independence of the anomaly factors can be explicitly demonstrated. [The only difference is that in non-Abelian theories, the iteration can be done in a simple way only for the path $\alpha_\tau(x) = \tau\alpha(x)$.] Despite the fact that the fermionic determinant cannot be solved exactly, the same is true in four dimensions. Here, for QCD at $T > 0$, for example, the Jacobian of infinitesimal transformations similar to (5) is given by

$$J_\beta[\alpha] = \exp\left(-i\int_0^\beta dx^0 \int d^3x\, \alpha(x) \frac{g^2}{8\pi^2} \mathrm{tr} F_{\mu\nu}{}^*F^{\mu\nu}\right) \ . \tag{27}$$

Finally, we should mention that our Jacobians for (anti-) periodic boundary conditions are useful to study not only possible finite-temperature effects but also field theories at $T=0$ on which nontrivial, Casimir-type, spatial boundary conditions are imposed.

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[7] See Eqs. (2.24)–(2.26) of Ref. 6.

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