Topological degree for supersymmetric chiral models

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For the case of an N-field supersymmetric chiral model we discuss the relationship between the Witten index, the topological degree, the winding number, and the degree of polynomials. Using results of classical analysis we can then place strong constraints on the Witten index of supersymmetric chiral models. For renormalizable chiral models, $2 \leq 2^{\max[n(m(z))]} \leq W \leq 2^{r(\overline{g}g)} \leq 2^N$.

I. INTRODUCTION

In this paper we shall be interested in calculating the Witten index for supersymmetric chiral models. We start by considering the most general supersymmetric chiral model, without imposing renormalizability constraints. Following arguments of Witten, we discuss the equality of Witten's index $\mathrm{tr}\{(-)^F\}$ to the topological degree of the gradient of the superpotential. A brief exposition of computational formulas for the topological degree is presented, along with a discussion of its connection to the winding number. Some general theorems concerning the topological degree are exhibited, and are used to clarify the effect on the Witten index of changes in the superpotential.

We then restrict attention to polynomial interactions. For the case of a single chiral superfield, the topological degree is then exactly the polynomial degree of the derivative of the superpotential. For many chiral superfluids we can, in general, only impose constraints on the topological degree in terms of the degrees of relevant polynomials. Some special cases are described.

Finally, we restrict attention to renormalizable supersymmetric chiral models for which we obtain our most restrictive results.

II. THE GENERAL CHIRAL MODEL

In 3+1 dimensions the most general N-field chiral model is described by the Lagrangian[1]

$$\mathscr{L} = [\overline{\phi}_a \phi_a]_D + 2\,\mathrm{Re}[f(\phi)]_F \ . \tag{1}$$

Here f, the superpotential, is an arbitrary analytic function f: $\mathbb{C}^N \rightarrow \mathbb{C}$. After elimination of the dummy fields, the scalar potential is given by

$$V(\overline{A},A) = |\lambda(A)|^2 \ , \qquad \lambda_a(A) = \left.\frac{\partial f}{\partial A_a}\right|_A \ . \tag{2}$$

The Yukawa couplings are proportional to

$$\mathscr{M}_{ab}(A) = \frac{\partial}{\partial A_a}\lambda_b(A) = \frac{\partial^2}{\partial A_a \partial A_b} f(A) \ . \tag{3}$$

Now consider the Witten index $W[f]$.[2] The Witten index is an integer-valued function of the superpotential.

$$W[f] = \mathrm{tr}\{(-)^F\} = n_B - n_F \ . \tag{4}$$

Cecotti and Giradello[3] have shown that the Witten index is just the winding number of the Nicolai transformation, and that in the ultralocal limit this is just the winding number of $\lambda(A)$. The discussion is perhaps more transparent if we follow the spirit of Witten's paper. According to Witten, $W[f] = W[\epsilon f]$. Further, for ϵ sufficiently small we expect perturbation theory to be valid. In the perturbative regime $n_F = 0$, while n_B is just the number of zeros of $V(\overline{A},A)$. But by definition this is just the topological degree of the analytic map λ: $\mathbb{C}^N \rightarrow \mathbb{C}^N$, so

$$W[f] = d(\lambda, \mathbb{C}^N, 0) \ . \tag{5}$$

This equality having been established in the perturbative regime, Witten's arguments imply that the equality should hold for aribtrary superpotentials. The calculation of the Witten index is then reduced to a relatively simple problem in analysis on $\mathbb{C}^N$.

III. THE TOPOLOGICAL DEGREE

Let λ: $\mathbb{C}^N \rightarrow \mathbb{C}^N$ be an analytic function, Ω an open set in $\mathbb{C}^N$, and ρ a point in $\mathbb{C}^N$. Let us start by assuming that $\lambda^{-1}(\rho)$ is a disjoint union of points with $\lambda^{-1}(\rho) \bigcap \partial\Omega = \varnothing$. We define the topological degree of λ in Ω with respect to ρ as

$$\begin{aligned} d(\lambda, \Omega, \rho) &= \int_\Omega \delta(z - \lambda^{-1}(\rho))\, d^N x\, d^N y \\ &= \int_\Omega \delta(\lambda(z) - \rho) |\det m(z)|^2\, d^N x\, d^N y \ , \end{aligned} \tag{6}$$

where we have used the fact that for complex delta functions $\delta(az) = \delta(z)/|a|^2$. Standard arguments[4] may now be used to extend the degree functional to all analytic functions $\mathbb{C}^N \rightarrow \mathbb{C}^N$ with $\lambda^{-1}(\rho) \bigcap \partial\Omega = \varnothing$. A standard computational formula may be obtained by inserting the Gaussian representation for $\delta(z)$,

$$\delta(z) = \lim_{\sigma \rightarrow 0} (2\pi\sigma^2)^{-N} \exp\left(-\frac{\overline{z}z}{2\sigma^2}\right) \ , \tag{7}$$

in which case

$$d(\lambda, \mathbb{C}^N, \rho) = (2\pi\sigma^2)^{-N} \int_{\mathbb{C}^N} e^{-|\lambda(z)-\rho|^2/2\sigma^2} \times |\det(m(z))|^2\, d^N x\, d^N y \ . \tag{8}$$

Note that we do not have to take the limit $\sigma \rightarrow 0$ since the right-hand side is independent of σ.

Suppose now that Ω is a simply connected region so that $\partial\Omega \simeq S^{2N-1}$. Consider the mapping

$$\hat{\lambda}:\ S^{2N-1} \simeq \partial\Omega \rightarrow S^{2N-1}:\ z \rightarrow \frac{\lambda(z)-\rho}{\|\lambda(z)-\rho\|} \ . \tag{9}$$

$\hat{\lambda}$ is essentially a mapping from S^{2N-1} to S^{2N-1}; therefore, it has a well-defined winding number. This winding number is the same as the topological degree.[4]

Having established the identity of the Witten index, topological degree, and winding number, we are now in a position to utilize some of the classical theorems of analysis. Consider, for example, the following theorem.

IV. ROUCHÉ'S THEOREM

If λ and $\delta\lambda$ are analytic functions from $\overline{\Omega}$ to $\mathbb{C}^N$ and

$$\forall z \in \partial\Omega,\quad \|\delta\lambda(z)\| < \|\lambda(z)\| \ ,$$

then

$$d(\lambda,\Omega,0) = d(\lambda+\delta\lambda,\Omega,0) \ . \tag{10}$$

[The proof is trivial; consider the homotopy $H(t,z)=\lambda(z)+t\delta\lambda(z)$.] For our purposes Rouché's theorem may best be viewed as a means for making more precise Witten's statements concerning the stability of his index under changes in the superpotential.[2]

Define

$$K = \{z: \|\delta\lambda(z)\| \geqslant \|\lambda(z)\|\} \ . \tag{11}$$

If $\overline{K}$ is compact, then

$$W[f+\delta f] = d(\lambda+\delta\lambda,\mathbb{C}^N,0) = d(\lambda,\mathbb{C}^N,0) = W[f] \ . \tag{12}$$

As a final comment, before we particularize to polynomial interactions, note that if two sectors of a chiral model mix only weakly, then the Witten index of the combined system is just the product of the indices for the individual sectors. Let $\mathbb{C}^N = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$;

$$\lambda(z) = \lambda(z_1,z_2) = \lambda_1(z_1) + \lambda_2(z_2) + \lambda_{\text{mix}}(z_1,z_2) \ ;$$

define

$$K = \{z=(z_1,z_2):\ \|\lambda_{\text{mix}}(z_1,z_2)\| \geqslant \|\lambda_1(z_1)\| + \|\lambda_2(z_2)\|\} \ .$$

If $\overline{K}$ is compact then

$$W[\lambda] = W[\lambda_1+\lambda_2+\lambda_{\text{mix}}] = W[\lambda_1+\lambda_2] = W[\lambda_1]\,W[\lambda_2] \ . \tag{13}$$

V. POLYNOMIAL INTERACTIONS

For polynomials in a single complex variable, the fundamental theorem of algebra tells us

$$W[f] = d(\lambda,\mathbb{C}^N,0) = d \ , \tag{14}$$

where d is the degree of the polynomial $\lambda(z)$, which is also its winding number. For polynomials in many variables the situation is considerably messier. Let d_a^+ be the highest degree present in the multinomial $\lambda_a(z)$; let $\delta = \max(d_1^+, d_2^+, \ldots, d_N^+)$; let d_a^- be the lowest nontrivial degree present in the multinomial $\lambda_a(z)$. Then

$$\prod_{a=1}^{N} d_a^- = d^- \leqslant d(\lambda,\mathbb{C}^N,0) \leqslant d^+ = \prod_{a=1}^{N} d_a^+ \ . \tag{15}$$

The upper bound is a classical theorem due to Bézout. To the best of my knowledge the lower bound is due to Cronin.[5] The rest of this section will be devoted to various improvements on these bounds.

To settle the notation, write the polynomial $\lambda(z)$ as follows:

$$\lambda_a(z) = \lambda_a^0 + m_{ab}z_b + \frac{1}{2!}g_{abc}z_bz_c + \frac{1}{3!}g^{(3)}_{ab_1b_2b_3}z_{b_1}z_{b_2}z_{b_3} + \cdots + \frac{1}{\delta!}g^{(\delta)}_{ab_1b_2\cdots b_\delta}z_{b_1}z_{b_2}\cdots z_{b_\delta} \ . \tag{16}$$

Define

$$g_-^{(n)} = \min(\|g^{(n)}_{ab_1b_2\cdots b_n}z_{b_1}z_{b_2}\cdots z_{b_n}\|/\|z\|^n). \tag{17}$$

Theorem. If $g_-^{(n)} > 0$ then

$$d(\lambda,\mathbb{C}^N,0) = d\left(\left[\sum_{i=n}^{\delta}\frac{1}{i!}g^{(i)}(z)\right],\mathbb{C}^N,0\right) \tag{18}$$

(that is, we can drop all the terms in λ of degree less than n).

Proof. If $g_-^{(n)} > 0$ then for sufficiently large $\|z\|$, $\|\lambda\| \rightarrow \infty$ at least as rapidly as $(1/n!)g_-^{(n)}\|z\|^n$, thereby swamping all terms of lower degree. Apply Rouché's theorem.

Corollary 1. If $g_-^{(\delta)} > 0$ then $d(\lambda,\mathbb{C}^N,0) = \delta^N$.

Corollary 2. If $\det(m) \neq 0$ then

$$d(\lambda,\mathbb{C}^N,0) = d(\lambda-\lambda^0,\mathbb{C}^N,0) \geqslant 1 \ . \tag{19}$$

Proof. Consider $g_-^{(1)} = \min(\|mz\|/\|z\|)$.

Corollary 2 is by now a well-known result, the first proof being that of Nicolai,[6] while equivalent theorems using different language may be found in Refs. 3 and 7.

Corollary 3. If $\exists z_0: \det(m(z_0)) \neq 0$, then

$$\begin{aligned} d(\lambda(z),\mathbb{C}^N,0) &= d(\lambda(z_0+z),\mathbb{C}^N,0) \\ &= d(\lambda(z_0+z)-\lambda(z_0),\mathbb{C}^N,0) \geqslant 1 \ . \end{aligned} \tag{20}$$

Theorem. $\forall z,\ d^- \geqslant 2^{[n(m(z))]}$.

Proof. Observe that if $\lambda(z)$ is a polynomial, then so is $m(z)$, and likewise $\det(m(z))$. Thus either $\det(m(z)) \equiv 0$ or $\det(m(z)) \neq 0$ almost everywhere. If $\det(m(z)) \equiv 0$, the computational formulas previously exhibited yield

$$W[f] = d(\lambda,\mathbb{C}^N,0) = 0 \ , \tag{21}$$

in agreement with Corollary 3. From now on we shall always set $\det(m(z)) \neq 0$ except on a set of measure zero. Now apply Corollary 3 and diagonalize $m(z_0)$ by a unitary transformation of the type

$$m(z_0) \rightarrow U^T m(z_0) U = m_{\text{diag}} \ . \tag{22}$$

(Note that *any* symmetric complex matrix can be so diagonalized.) Then

$$\begin{aligned} d([\lambda(z)],\mathbb{C}^N,0] &= d([\lambda(z+z_0)-\lambda(z_0)],\mathbb{C}^N,0) \\ &= d([U^T\lambda(Uz+z_0)-U^T\lambda(z_0)],\mathbb{C}^N,0) \\ &= d([U^Tm(z_0)Uz + (\text{higher order})],\mathbb{C}^N,0) \\ &= d([m_{\text{diag}}z + (\text{higher order})],\mathbb{C}^N,0) \ . \end{aligned} \tag{23}$$

Then

$$d_a^- = 1 \quad \text{for } a = 1, 2, \ldots, r(m(z_0)) ,$$
$$d_a^- \geq 2 \quad \text{for } a = r(m(z_0)) + 1, \ldots, N , \tag{24}$$

where

$$r(X) = \text{ rank of } X ,$$
$$n(X) = \text{ nullity of } X = N - r(X) . \tag{25}$$

Thus

$$\forall z, \quad d^- \geq 2^{n(m(z))} . \tag{26}$$

Finally, note that $\det(m(z))$ must have at least one zero, so

$$\max[n(m(z))] \geq 1 \quad \text{and } d^- \geq 2 . \tag{27}$$

Theorem.

$$d^+ \leq \delta^{r(\bar{g}g)} (\delta - 1)^{n(\bar{g}g)} . \tag{28}$$

Proof. Define the Hermitian matrix $\bar{g}g$ by

$$(\bar{g}g)_{ab} = \bar{g}\,^{(\delta)}_{ac_1c_2 \cdots c_\delta}\, g^{(\delta)}_{bc_1c_2 \cdots c_\delta} . \tag{29}$$

Since $\bar{g}g$ is a Hermitian matrix, it is diagonalizable, after diagonalization;

$$d_a^+ \leq \delta \quad \text{for } a = 1, 2, \ldots, r(\bar{g}g) ,$$
$$d_a^+ \leq \delta - 1 \quad \text{for } a = r(\bar{g}g) + 1, \ldots, N . \tag{30}$$

Thus

$$d^+ \leq \delta^{r(\bar{g}g)} (\delta - 1)^{n(\bar{g}g)} \leq \delta^N . \tag{31}$$

Combining these results, we summarize.

(1)

$$\text{If } g_-^{(\delta)} > 0, \quad \text{then } d(\lambda, \mathbb{C}^N, 0) = \delta^N .$$

(2)

$$2 \leq 2^{\max[n(m(z))]} \leq d(\lambda, \mathbb{C}^N, 0) \leq \delta^{r(\bar{g}g)} (\delta - 1)^{n(\bar{g}g)} \leq \delta^N . \tag{32}$$

This is as far as I have been able to take arguments for general polynomials, though doubtless further progress is possible. The results simplify somewhat for renormalizable models ($\delta = 2$ in $3+1$ dimensions).

VI. RENORMALIZABLE CHIRAL MODELS

For nontrivial renormalizable models in $3+1$ dimensions, $\delta = 2$,

$$\lambda_a(z) = \lambda_a^0 + m_{ab} z_b + \frac{1}{2!} g_{abc} z_b z_c . \tag{33}$$

In the case of a single chiral field, then

$$W[f] = d(\lambda, \mathbb{C}, 0) = 2 , \tag{34}$$

in agreement with Witten.[2]

For N interacting chiral fields we observe that, provided $\det(m(z)) \not\equiv 0$,

$$2 \leq 2^{\max[n(m(z))]} \leq d(\lambda, \mathbb{C}^N, 0) \leq 2^{r(\bar{g}g)} \leq 2^N . \tag{35}$$

[Note that $\max[n(m(z))]$ cannot be greater than $r(\bar{g}g)$ without violating the constraint on the determinant.] Otherwise, if $\det(m(z)) \equiv 0$, $W[f] = d(\lambda, \mathbb{C}^N, 0) = 0$. These results represent the best limits I have been able to achieve to date.

VII. CONCLUSION

We can place reasonably good constraints on the Witten index for N-field supersymmetric chiral models. Unfortunately, an explicit evaluation of $W[f]$ is, as yet, elusive.

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