Cosmological perturbations in a universe dominated by a coherent scalar field

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Energy-density perturbations in a universe dominated by coherent oscillations of a scalar field evolve as in a pressureless fluid. The analysis applies in particular to axion models. Axions are thus a candidate for cold dark matter.

It has recently been suggested[1] that the universe might be dominated by axions. The axion field[2] is a new quantum field which is added to particle-physics models in order to solve the strong *CP* problem.[3] At high temperatures the axion is massless. Below the confinement scale it acquires a mass $m_a \sim 10^{-5}$ eV.[1] It would be a big coincidence if the axion field in the region of space which evolves into our observed universe were initially localized at the minimum of the potential. It is thus assumed[1] that the axion field at the confinement scale is localized away from the minimum. Hence its expectation value will oscillate coherently and give an energy density

$$\rho(t) = \tfrac{1}{2} m_a^2 A^2(t) \tag{1}$$

and pressure

$$p(t) = \tfrac{1}{2} m_a^2 \cos(2m_a t) A^2(t) \;; \tag{2}$$

$A(t)$ is the amplitude of oscillation and will decrease in time due to the expansion of the universe. In Eqs. (1) and (2) we have assumed that the expansion is slow compared to the oscillation frequency and thus can be treated adiabatically.

In the past analyses[4] of the growth of structure in a universe which eventually becomes axion-dominated, it has always been assumed that the pressure can be averaged over an oscillation period, and that the axion is in effect a pressureless gas. Thus, structure would form as in the usual cold-dark-matter scenarios.[5]

However, the evolution of energy-density perturbations does not only depend on the instantaneous equation of state of the background, i.e., on $w(t)$,

$$w(t) = \frac{p(t)}{\rho(t)} \;, \tag{3}$$

but also on its change in time, i.e., on

$$c_s^2(t) = \frac{\dot{p}(t)}{\dot{\rho}(t)} \;. \tag{4}$$

Note that c_s^2, the quantity which enters in the linearized relativistic perturbation equations, is not the speed of sound. The notation which suggests this interpretation is unfortunate but by now standard. Since $\dot{p}$ is of the order $m_a p$ and $\dot{\rho}$ of the order $H\rho$, $|c_s^2(t)|$ is typically very large, i.e., of the order $m_a H^{-1}$. Thus it is not obvious that perturbations in a universe dominated by a coherent scalar field grow in the same way as in a cold-dark-matter universe ($w = c_s^2 = 0$).

In this Brief Report we investigate the evolution of fluctuations in a universe dominated by coherent oscillations of a scalar field using linearized relativistic perturbation theory. We demonstrate that deviations between the growth of perturbations in a universe with $w = c_s^2 = 0$ and in one with the equation of state of a coherent field are suppressed by a factor H/m_a. In particular, the Jeans length[6] vanishes, since free streaming[7] of axions in the coherent state, from overdense to lower-density regions, is prevented by the rapidly oscillating forces on the particles. Our results confirm the standard scenario[4] of the evolution of perturbations in an axion-dominated universe, in contradiction to recent claims by Sasaki.[8]

A nonrelativistic analysis clearly illustrates the basic idea of our method. Consider a spatially flat expanding universe with scale factor $a(t)$. The Newtonian analysis yields the following equation of motion for the fractional scalar-field energy-density perturbation δ_a:[9]

$$\ddot{\delta}_a + 2\frac{\dot{a}}{a}\dot{\delta}_a + \left[\frac{v_s^2 k^2}{a^2} - 4\pi G\rho_a\right]\delta_a = 0. \tag{5}$$

ρ_a is the scalar-field energy density, k the wave number of the fluctuation, and $v_s^2 = \Delta p/\Delta\rho$ the equation of state of the inhomogeneities; in our case, $v_s^2 = \cos(2m_a t)$.

For cold dark matter $v_s^2 = 0$. The basic idea is to estimate the deviation between the evolution of $\delta_a(t)$ for a coherent scalar field and for cold dark matter perturbatively by considering the $v_s^2 k^2$ term in Eq. (5) as a source term. We consider the initial period in which the universe is radiation dominated ($\rho >> \rho_a$). In this case the last term in Eq. (5) is negligible. The solution for vanishing source is

$$\delta_a^{(0)}(t) = \delta_a(t_0) + \dot{\delta}_a(t_0)\ln\frac{t}{t_0} \;, \tag{6}$$

with initial conditions specified at t_0. If we evaluate the source term using $\delta_a^{(0)}(t)$, we can solve for the perturbation $\delta_a^{(1)}(t)$ using the Green's function method:

$$\delta_a(t) = \delta_a^{(0)}(t) + \delta_a^{(1)}(t) \;, \tag{7}$$

$$\delta_a^{(1)}(t) = -f_1(t)\int_{t_0}^{t} I(t')\epsilon(t')f_2(t')\,dt' + f_2(t)\int_{t_0}^{t} I(t')\epsilon(t')f_1(t')\,dt' \;. \tag{8}$$

$I(t)$ is the source term

$$I(t) = -\frac{k^2}{a^2}\cos(2m_a t)\delta_a^{(0)}(t) \;, \tag{9}$$

$f_1(t) = \ln(t/t_0)$ and $f_2(t) = 1$ are two independent solutions of the homogeneous (sourceless) equation, and

$$\epsilon(t) = [f_2(t)\dot{f}_1(t) - f_1(t)\dot{f}_2(t)]^{-1} = t \;. \tag{10}$$

It is crucial that the integrands in Eq. (8) are rapidly oscillating (period m_a^{-1}) with a slowly varying amplitude (period H^{-1}). Hence the integrals will be suppressed by at least a factor H/m_a. The deviation $\delta_a^{(1)}(t)$ is negligible compared to $\delta_a^{(0)}(t)$.

The above Newtonian analysis is incomplete. While the average velocities are zero, their average magnitude is of the order 1 in natural units. Hence a nonrelativistic analysis is not justified. In addition, we would like to focus on the evolution of fluctuations once the oscillating scalar field dominates the energy-momentum tensor. We will therefore proceed to a relativistic analysis. The basic method, however, will be the same.

The general relativistic description of the growth of energy-density perturbations in an expanding universe is given by linearizing the Einstein equations about a background Friedmann-Robertson-Walker solution. Since we are mainly interested in the evolution of perturbations inside the Hubble radius, we adopt the synchronous-gauge formalism of Press and Vishniac.[10]

We first determine the equation of state of a coherently oscillating scalar field ϕ to lowest nonvanishing order in H/m_a, starting with the adiabatic solution

$$\phi(t)=a^{-1}(t)\phi_0\cos mt \ . \tag{11}$$

ϕ_0 is the initial amplitude at time t_0, $a(t)$ is normalized such that $a(t_0)=1$ and $m=m_a$. The result is

$$\begin{aligned} w(t)&=-\cos 2mt \ , \\ c_s^2(t)&=-\frac{m}{H}\sin 2mt \ . \end{aligned} \tag{12}$$

The technical difficulties in the following analysis stem from the fact that $c_s^2(t)$, albeit its time average vanishes, has a large average magnitude of the order m/H.

Energy-density perturbations couple only to scalar-metric perturbations. In the synchronous gauge there are four degrees of freedom for scalar-metric perturbations: h, $\dot h$, δ, and ϕ. h is the trace of the metric tensor h_{ij}, defined by

$$g_{\mu\nu}=g_{\mu\nu}^{(0)}+a^2h_{\mu\nu}, \quad h\equiv h_{ii} \ . \tag{13}$$

δ is the fractional energy-density perturbation ρ_1/ρ_0, and ϕ is proportional to the divergence of the fluid velocity u [and should not be confused with the scalar field $\phi(x,t)$],

$$u^\alpha=(-1,0,0,0)+u_1^\alpha, \quad \phi\equiv\left(\frac{\dot a}{a}\right)^{-1}au^j{}_{,j} \ . \tag{14}$$

In the following, a prime denotes $d/d\eta=(\dot a/a)^{-1}d/dt$. Press and Vishniac[10] derive the following equations of motions for h, δ, and ϕ:

$$\begin{aligned} &h''+\tfrac{1}{2}(1-3w)h'=3(1+3c_s^2)\delta \ , \\ &\delta'+(1+w)(\phi-\tfrac{1}{2}h')=3(w-c_s^2)\delta \ , \\ &\phi'+(\tfrac{1}{2}-\tfrac{3}{2}w-3c_s^2)\phi=\frac{c_s^2k^2\delta}{H^2(1+w)} \ . \end{aligned} \tag{15}$$

As in the nonrelativistic analysis, the idea is to determine the deviations from the solution for cold dark matter ($w=c_s^2=0$) by treating the w- and c_s^2-dependent terms as source terms and by then applying the Green's-function method. In vector form with

$$\mathbf{v}=\begin{pmatrix}\chi\\ \delta\\ \phi\end{pmatrix}, \quad \chi\equiv h' \ , \tag{16}$$

we can rewrite Eq. (15) as

$$\mathbf{v}'=A\mathbf{v}+S(t)\mathbf{v} \ , \tag{17}$$

A is a matrix of constants and $S(t)$ the matrix which represents the terms proportional to w and c_s^2 in Eq. (15). We expand $\mathbf{v}(t)$ about the cold-dark-matter solution $\mathbf{v}^{(0)}(t)$,

$$\mathbf{v}(t)=\mathbf{v}^{(0)}(t)+\mathbf{v}^{(1)}(t) \ , \tag{18}$$

and determine $\mathbf{v}^{(1)}(t)$ by considering $\mathbf{I}(t)=S(t)\mathbf{v}^{(0)}(t)$ as source terms in Eq. (17).

The problem now becomes a straightforward excercise in linear algebra. We diagonalize the matrix A in order to be able to apply the one-dimensional Green's-function formula to each of the eigenmodes. We then consider the effect of nonvanishing $w(t)$ and $c_s^2(t)$ on the growing-mode solution $g(t)$. After some algebra,[11] we find the correction term

$$g^{(1)}(t)=g^{(0)}(t)\int_{t_0}^{t}dt'\left(\frac{\dot a}{a}\right)(t')[\tfrac{5}{2}w(t')+\tfrac{15}{4}c_s^2(t')] \ . \tag{19}$$

In order to prove that $g^{(1)}(t)$ is negligible compared with $g^{(0)}(t)$ it is sufficient to show that the integral is much smaller than zero. But this follows almost immediately since both integrands are rapidly oscillating functions with a slowly varying amplitude. We will focus on the second term, since it contains an additional prefactor of the order m/H. But integration yields a suppression factor $(H/m)^2$. One factor arises from integrating a rapidly oscillating function, the other from cancellations arising since the amplitude is only slowly varying. To leading order in H/m we obtain by integrating over a full period $\Delta t=N\pi m^{-1}$ (N chosen such that $Nm^{-1}\simeq H^{-1}$),

$$\begin{aligned} \int_{t_0}^{t_0+\Delta t}dt'\,t'^{-1}c_s^2(t')&=\frac{m}{Ht_0}\int_0^{\Delta t}dt'\frac{\sin(2mt')}{1+t'/t_0} \\ &\simeq\frac{m}{Ht_0}\frac{1}{2m}\int_0^{2m\Delta t}d\tau\,\sin\tau\,\frac{\tau}{2mt_0} \\ &\simeq\frac{m}{H}\left(\frac{1}{2mt_0}\right)^2 \ . \end{aligned} \tag{20}$$

Since t_0 is of the order H^{-1}, it follows that $g^{(1)}(t)$ is suppressed compared with $g^{(0)}(t)$ by a factor H/m.

To conclude, we have demonstrated that the deviations in the growth of cosmological fluctuations between the cold-dark-matter scenario $[w(t)=c_s^2(t)=0]$ and a universe dominated by coherent scalar-field oscillations is negligible. The difference is suppressed by a factor H/m, m being the oscillation frequency and H the Hubble expansion parameter. Our analysis applies in particular to the case of an axion-dominated universe. Axions behave as cold dark matter. Our analysis verifies the standard lore[4] obtained by averaging the stress-energy tensor in time to obtain an effective equation of state.

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[11]Further details are given in an earlier version of this paper. See R. Brandenberger, Institute of Theoretical Physics, University of California, Santa Barbara Report No. NSF-ITP-84-144, 1984 (unpublished).