PHYSICAL REVIEW D VOLUME 32, NUMBER 2 15 JULY 1985

Brief Reports

Brief Reports are short papers which report on completed research which, while meeting the usual **Physical Review** *standards of scientific quality, does not warrant a regular article. (Addenda to papers previously published in the* **Physical Review** *by the same authors are included in Brief Reports.) A Brief Report may be no longer than 3½ printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Derivation of a formula in finite-temperature field theory

Pinaki Roy and Rajkumar Roychoudhury
Electronics Unit, Indian Statistical Institute, Calcutta 700035, India
(Received 26 December 1984)

Using an extended version of ζ-function regularization, we derive the one-loop fermion contribution to the effective potential in the presence of finite temperature and a chemical potential.

During the past few years much attention has been focused on finite-temperature quantum field theory (QFT), and a standard algorithm has evolved for calculations in finite-temperature QFT.[1] However, if a chemical potential for fermions (which corresponds to a conserved fermion number) is included, then the calculations are not trivial generalizations of finite-temperature QFT calculations.[2] In the past, several gauge-theory models have been treated in the presence of a chemical potential,[3-5] and in all these papers the contour integration method was used to evaluate the frequency sums. This method is rather involved and cumbersome. In this Brief Report we shall use an extended version of the generalized Riemann zeta-function regularization to evaluate the one-loop fermion contribution to the effective potential. Furthermore, in this method we can directly use the functional-integration formalism, without taking recourse to either the diagrammatic method or the real-time formalism.

For our purpose we need only the fermionic part of the Lagrangian, and this is given by

$$L_f = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \quad . \tag{1}$$

After shifting the scalar fields of the theory by ϕ, the effective fermion mass is given by

$$M = m + g\phi \quad , \tag{2}$$

where g is the Yukawa coupling. The one-loop effective potential is

$$V_1^f = 2i\int \frac{d^4p}{(2\pi)^4}\ln(p^2-M^2) \quad . \tag{3}$$

To perform finite-temperature calculations, one has to make the following replacements:

$$\int \frac{d^4p}{(2\pi)^4} \to 2T\sum_n \int \frac{d^3p}{(2\pi)^3} \quad , \qquad p_0 \to (2n+1)\pi T \quad , \tag{4}$$

and if we incorporate a classical potential μ, then in addition to (4) we should also have

$$p_o \to (2n+1)\pi T - i\mu \quad . \tag{5}$$

The one-loop effective potential is then given by

$$V_1^f = -2T\sum_n \int \frac{d^3p}{(2\pi)^3}\ln\{[(2n+1)\pi T - i\mu]^2 + \mathbf{p}^2 + M^2\} \quad . \tag{6}$$

Instead of evaluating (6) by the contour integration method, we proceed as follows. Let

$$v(M) = \sum_n \ln(X^2+E^2) \quad , \tag{7}$$

where $X = (2n+1)\pi T - i\mu$ and $E = (\mathbf{p}^2+\mu^2)^{1/2}$. Then

$$\frac{dv(M)}{dM^2} = \sum_n \frac{1}{X^2+E^2} = \frac{1}{2iE}\sum_n\left[\frac{1}{X-iE} - \frac{1}{X+iE}\right] \quad . \tag{8}$$

To evaluate this sum, we first consider the first part:

$$\sum_n \frac{1}{X-iE} = \frac{1}{2\pi T}\sum_{n=0}^{\infty}\left[\frac{1}{n+l} - \frac{1}{n+\bar{l}}\right] \quad , \tag{9}$$

where

$$l = \tfrac{1}{2} - i\frac{\mu+E}{2\pi T} \quad , \tag{10}$$

and an overbar denotes complex conjugation. The summation in (9) is clearly divergent. To regularize it, we consider the following:

$$\frac{1}{2\pi T}\sum_{n=0}^{\infty}\left[\frac{1}{(n+l)^s} - \frac{1}{(n+\bar{l})^s}\right] = \frac{1}{2\pi T}[\zeta(s,l) - \zeta(s,\bar{l})] \quad , \tag{11}$$

where $\zeta(s,l)$ stands for the generalized Riemann zeta function.[6] Now using the results[6]

$$\lim_{s\to 1}\left[\zeta(s,\alpha) - \frac{1}{s-1}\right] = -\psi(\alpha) \quad , \tag{12}$$

$$\operatorname{Im}\psi(\tfrac{1}{2}+iy) = \tfrac{1}{2}\pi\tanh(\pi y) \quad , \tag{13}$$

we obtain

$$\sum_n \frac{1}{X-iE} \stackrel{R}{=} -\frac{1}{2iT}\tanh\left(\frac{\mu+E}{2T}\right) \tag{14}$$

$$= -\frac{1}{2iT}\left(2\sum_{n=1}^{\infty}(-)^n e^{-n(\mu+E)/T}+1\right) , \tag{15}$$

where $\stackrel{R}{=}$ stands for the regularized part. (It is thus seen that the infinities appear as poles of the zeta function.) Proceeding similarly we obtain

$$\sum_n \frac{1}{X+iE} \stackrel{R}{=} -\frac{1}{2iT}\tanh\left(\frac{\mu-E}{2T}\right) \tag{16}$$

$$= -\frac{1}{2iT}\left(2\sum_{n=1}^{\infty}(-)^n e^{n(\mu-E)/T}-1\right) . \tag{17}$$

Now inserting (15) and (17) in (6) and using the formula[7]

$$\int_m^{\infty}(x^2-m^2)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2m}{\mu}\right)^{\nu-1/2}\sqrt{\nu}\,K_{\nu-1/2}(m\mu) , \tag{18}$$

we finally arrive at

$$\frac{dV_1^f}{dM^2} = -\frac{MT}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\cosh\left(\frac{n\mu}{T}\right)K_1\left(\frac{nM}{T}\right) . \tag{19}$$

This expression agrees with the results obtained in Refs. 4 and 5. Further, on using the formula[6]

$$\int z^{\nu-1}K_\nu(z)\,dz = -z^{\nu+1}K_{\nu+1}(z) , \tag{20}$$

we get

$$V_1^f = \frac{2M^3T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^n}{n^2}\cosh\left(\frac{n\mu}{T}\right)K_2\left(\frac{nM}{T}\right) . \tag{21}$$

It should be noted that both (19) and (21) are exact expressions, and no approximation has been made in deriving them. However, in many situations one needs approximate formulas, and in such cases the following expression is useful:

$$\frac{dV_1^f}{dM^2} = \frac{1}{2\pi^2}\int_M^{\infty}dx\,(x^2-M^2)^{1/2} \times\left(\frac{1}{e^{(x+\mu)/T}+1}+\frac{1}{e^{(x-\mu)/T}+1}\right) . \tag{22}$$

[Equation (22) can be obtained through the use of (14) and (16).] From this expression, one can obtain results for various approximations like $|\mu| > M$ or $|\mu| < M$, etc. Finally, we briefly describe the usual contour integration method. In this case one chooses the contour shown in Fig. 1. Then $\int_{C_\mu} = \int_{C_0} + \int_{\delta C}$. The C_0 integral is μ independent

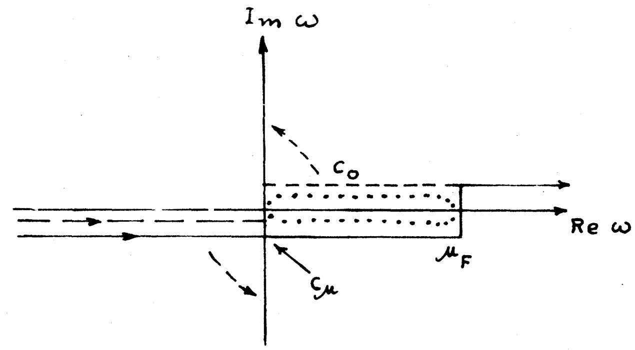

FIG. 1. Integration contour in the complex frequency plane for zero chemical potential (dashed line) and nonzero chemical potential (solid line).

and infinite, while the second integral is nonzero. The general formula for such sums is given by[3]

$$T\sum_n \phi(w_n) = \int_{-i\infty}^{i\infty}\frac{dw}{2\pi i}\phi(w) \sum_{\mathrm{Re}\,w_a>0}\frac{\mathrm{res}\,\phi(w_a)}{\exp(w_a/T)+1} + \sum_{\mathrm{Re}\,w_a<0}\frac{\mathrm{res}\,\phi(w_a)}{\exp(-w_a/T)+1} .$$

The integration contour with and without chemical potential at finite temperature is shown in Fig. 2.

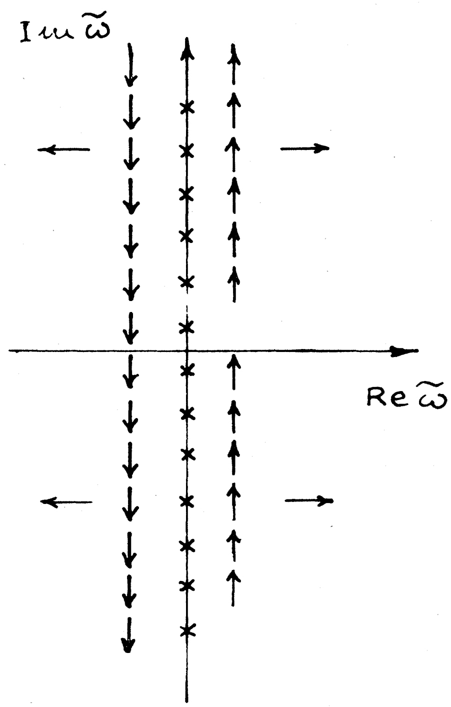

FIG. 2. Integration contour in the complex frequency plane with and without chemical potential at finite temperature.

DISCUSSION

In this Brief Report we have illustrated how one can use the generalized Riemann zeta-function technique in finite-temperatue QFT (coupled with a chemical potential) calculations. It appears that this method is simpler than the conventional contour integration technique and would likely prove useful in higher-loop calculations and other situations as well.

[1]L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974); S. Weinberg, *ibid.* **9**, 3357 (1974).
[2]M. B. Kislinger and P. D. Morley, Phys. Rev. D **13**, 2765 (1976).
[3]E. V. Shuryak, Phys. Rep. **61**, 72 (1980).
[4]B. J. Harrington and A. Yildiz, Phys. Rev. Lett. **33**, 324 (1974).
[5]A. Yildiz, Phys. Rev. D **16**, 3450 (1977).
[6]W. Magnus *et al.*, *Formulas and Theorems of Special Functions of Mathematical Physics,* 3rd ed. (Springer, Berlin, 1966).
[7]I. M. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).