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Gauging of Dirac-Kahler fermions on the cubic lattice

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We propose a new method of gauging of Dirac-Kahler fermions on the cubic lattice. Instead of gauging the finite-difference operators we gauge the incidence numbers of the cell complex in terms of which the cubic lattice is defined.

I. INTRODUCTION

Dirac-Kahler fermions on the lattice are a geometrically natural reinterpretation of Susskind fermions.[1] They provide an alternative to Dirac fermions even in the continuum.[2] For example, unlike the usual Dirac spinors in the continuum they can be defined on all pseudo-Riemannian manifolds. On the lattice the Dirac-Kahler fermions permit one to use a natural geometrical language of cell complexes and thus have an additional appeal. Their gauging, however, has not achieved the same degree of elegance and naturalness on the lattice as has the free-fermion formalism.

In this paper we present some results in this direction. For background information and more on the subject of the geometrical meaning of Dirac-Kahler fermions, the reader should consult an excellent exposition by Becher and Joos.[1]

II. GAUGING

The usual gauging with Wilson factors[1] consists of gauging finite-difference operators $\partial^{\pm}$,

$$\partial_\mu^+ f(x)=f(x+e_\mu)-f(x)\ ,$$

$$\partial_\mu^- f(x)=f(x)-f(x-e_\mu)\ ,$$

by substitutions $\partial^{\pm}\rightarrow D^{\pm}$,

$$D_\mu^+ f(x)\equiv U^{-1}(x,e_\mu)f(x+e_\mu)-f(x)\ .$$

Unfortunately $\partial^{\pm}$ do not have the nice direct geometrical meaning. In contrast, our formalism can be extended on any cubic and simplicial cell complexes where the prescription just described will certainly not work. This will be considered elsewhere.

Let $\bar{d}$ be exterior differentiation, $\bar{\partial}$ its adjoint with respect to the scalar product on forms $(w,w')=\int w\wedge * w'$ ($\wedge$ is the wedge product and $*$ is the Hodge star operator). Then the Dirac-Kahler equation[1,3] is

$$i(\bar{d}-\bar{\partial})w=0\ . \tag{1}$$

On the Euclidean lattice Eq. (1) takes the form

$$(d+\partial)f=0\ , \tag{2}$$

where d is the boundary operator, and ∂ is its adjoint with respect to the scalar product on cochains (C_p is a p-cube)

$$(f,g)=\left[\sum\alpha_p C_p,\sum\beta_q C_q\right]=\sum\alpha_p\beta_p\ . \tag{3}$$

The boundary operator d is defined by

$$df=d\left[\sum_p\alpha_p C_p\right]=\sum_{p,C_{p+1}} I(C_p,C_{p+1})\alpha_p C_{p+1}\ , \tag{4}$$

where $I(C_p,C_{p+1})=\pm1,0$ are the incidence numbers, with $I(C_p,C_{p+1})\neq 0$ only when C_p is a face of C_{p+1} (denoted by $C_p<C_{p+1}$). For the cubic lattice one can take

$$I(d^{x,H_p},d^{y,H_{p+1}})=\sum_{i=1}^{p+1}(-1)^i(\delta^{x,y}-\delta^{x,y+e_{\mu_i}})\times\delta^{\nu_1\mu_1}\cdots\delta^{\nu_{i-1}\mu_{i-1}}\delta^{\nu_i\mu_{i+1}}\cdots\delta^{\nu_p\mu_{p+1}}\ . \tag{5}$$

In the notation used d^{x,H_p} is a unit p-cube built at the site x with p unit vectors $e_{\mu_1},\ldots,e_{\mu_p}$. $(\mu_1,\ldots,\mu_p)\equiv H_p$, $\mu_1<\cdots<\mu_p$. The most important property of the boundary operator is $d^2=0$.

The cup product $\cup$ on cochains is defined through the cup product on p-cubes,

$$d^{x,H}\cup d^{y,L}=\rho_{H,L}\delta^{x+e_H,y}d^{x,H\cup L}\ , \tag{6}$$

where $\rho_{H,L}$ is the sign of the permutation that puts the sequence of numbers $H\cup L$ (when dealing with number sequences, $H\cup L$ means the set theoretical union of H and L, while $H\setminus L$ means the complement of L in H) in increasing order. The cup product is associative. On cohomology classes the cup product is the lattice analog of the wedge product on forms.

The left and the right cap products $\cap_L$ and $\cap_R$ are defined by

$$d^{y,K}\cap_L d^{z,L}=\rho_{L\setminus K,K}\delta^{y,z+e_{L\setminus K}}d^{z,L\setminus K}\ , \tag{7}$$

$$d^{z,L}\cap_R d^{x,H}=\rho_{H,L\setminus H}\delta^{z,x}d^{x+e_H,L\setminus H}\ . \tag{8}$$

They are dual with respect to the cup product, i.e.,

$$(C_q\cap_R C_p,C_r)=(C_q,C_p\cup C_r),\quad q\geq p$$

$$(C_r,C_q\cap_L C_p)=(C_r\cup C_q,C_p),\quad q\leq p\ .$$

When $q=p$, in general, $\cap_R\neq\cap_L$. This is the source of the problems encountered with the definition of a lattice analog of the Clifford product on differential forms. The left cap product is similar to the operation of contraction used by Becher and Joos,[1] when the left factor in the product in Eq. (7) is a constant cochain.

Given a one-cochain U with values in some group representation, we define the right gauged boundary operator as

$$d_U f = \sum_{C_p} \alpha_p C_p \cup I_U(C_p, C_{p+1}) \ , \tag{9}$$

where I_U is the one-cochain

$$I_U(C_p, C_{p+1}) = I(C_p, C_{p+1}) U(C_{p+1} \cap_R C_p) \ . \tag{10}$$

The left gauged boundary operator can be defined similarly by changing the order in the cup product in Eq. (9) and defining U in Eq. (9) on $C_p \cap_L C_{p+1}$. The operator d_U has the following properties [Eqs. (11)—(14)] with 1 referring to the unit one-cochain:

$$d_1 = d \ . \tag{11}$$

For any f-cochain and for some two-cochain F_U,

$$d_U{}^2 f = f \cup F_U \ . \tag{12}$$

Under a gauge transformation

$$\begin{aligned} &U([ab]) \rightarrow \Phi(a) U([ab]) \Phi^{-1}(b) \ , \\ &F_U \rightarrow \Phi(a') F_U \Phi^{-1}(b') \ , \end{aligned} \tag{13}$$

where $[ab]$ is an oriented link (one-cube) and a,b,a',b' are some sites (zero cubes). (For the relationship between a,b and a',b', see below.) When the lattice spacing α goes to zero

$$F_U \rightarrow F_{\mu\nu} dx^\mu \wedge dx^\nu \ . \tag{14}$$

The left boundary operator has similar properties.

In order to prove Eqs. (11)—(14) we give an explicit expression for F_U. First, we note that for any C_p and C_{p+2}, with $C_p < C_{p+2}$, there exist exactly two $(p+1)$-cubes such that

$$C_p < C_{p+1}^i < C_{p+2} \ , \tag{15}$$

which has the meaning

$$\sum_{i=1,2} I(C_{p+1}^i, C_{p+2}) I(C_p, C_{p+1}^i) = 0 \ . \tag{16}$$

Second, the two-cube (square), which is the cup product of $U(C_{p+1}^1, C_{p+2})$ and $U(C_p, C_{p+1}^1)$, is the same as $U(C_{p+1}^2, C_{p+2}) \cup U(C_p, C_{p+1}^2)$. Then, using associativity of the cup product, we arrive at

$$\begin{aligned} &d_U{}^2 = \cup F_U \ , \\ &F_U = I_U \cup I_U \ , \end{aligned} \tag{17}$$

where I_U is the one-cochain of Eq. (10). Calculation of Eq. (17) is simple. Let

$$\begin{aligned} &d^{x,\mu_p}, \quad \mu_1 < \cdots < \mu_p \\ &d^{x,\mu_{p+1}^1}, \quad \mu_1 < \cdots < \mu_r < \cdots < \mu_p \\ &d^{x,\mu_{p+1}^2}, \quad \mu_1 < \cdots < \mu_q < \cdots < \mu_p \\ &d^{x,\mu_{p+2}}, \quad \mu_1 < \cdots < \mu_r < \cdots < \mu_q < \cdots < \mu_p \end{aligned}$$

be the four cubes in Eq. (15). Then

$$\begin{aligned} &I_U \cup I_U = - \sum_{y,\mu_r\mu_q} \{ U([y, y+e_{\mu_r}]) U([y+e_{\mu_r}, y+e_{\mu_r}+e_{\mu_q}]) - U([y, y+e_{\mu_q}]) U([y+e_{\mu_q}, y+e_{\mu_q}+e_{\mu_r}]) \} d^{y,\mu_r\mu_q} \ , \\ &y = x + \sum_{i=1}^{p} e_{\mu_i} \ . \end{aligned} \tag{18}$$

It follows from Eq. (18) that when $U=1$, $d_U{}^2=0$ and therefore $d_1 = d$. Since the points $y, y+e_{\mu_r}, y+e_{\mu_q}$, $y+e_{\mu_q}+e_{\mu_r}$ lie on the same two-cube $d^{y,\mu_r\mu_q}$, it is obvious that each term of the sum in Eq. (18) is the difference between Wilson phase factors along paths beginning and ending at the same point. From this follows Eqs. (13) and (14). The exact meaning of Eq. (14) is

$$\lim_{\alpha \to 0} \frac{1}{\alpha^2} (I_U \cup I_U, d^{x,\nu\mu}) = F_{\mu\nu} \ , \tag{19}$$

where $F_{\mu\nu}$ is the field strength of the gauge field $A_\mu : U = \exp(i\alpha A_\mu e_\mu)$.

It is interesting to compare the usual definition of a gauged boundary operator given in Ref. 1 by Eq. (3.61) and ours. The easiest way to see the difference between the two is to compare the gauge field strengths $F_{\mu\nu}$ corresponding to the two operators. One finds from (3.61) of Ref. 1 and Eqs. (18) and (19) that if the usual field strength of Ref. 1 is $F_{\mu\nu}(x)$ then ours is shifted in the argument and is $F_{\mu\nu}(x + \sum_{i \in H_p} e_i)$. Clearly, in general, the gauged boundary operators cannot be the same.

III. CONCLUSIONS

We presented a new way to gauge Dirac-Kahler fermions on the lattice. This was accomplished by introduction of right (left) gauged boundary operators in a natural geometrical manner, using only the notions of the incidence number, the cup, and the cap products. Unlike the conventional approach,[1] our method has extensions on arbitrary cubic and simplicial cell complexes. This has significance for inclusion of matter in the Regge calculus and description of dimensional phase transitions.[4] The gauged boundary operators have correct gauge covariance properties and the correct continuum limit.

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