Several theorems about the interference of particle beams and the theoretical interpretation of Collela-Overhauser-Werner and Marton experiments

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A new theorem about the interference between particle beams is established. The Collela-Overhauser-Werner and Marton experiments are theoretically explained with a theorem established by Chiu and Stodolsky. The Feynman theory is applied to obtaining the propagator for a particle beam in a gravitational field, and the Collela-Overhauser-Werner experiment is further analyzed. Another theorem, established by Chiu and Stodolsky, and our new theorem are proved with the method of Feynman's path integrals.

I. INTRODUCTION

It is a well-known fact that interferometers have been widely applied to optics. The x-ray interferometer[1] and neutron interferometer,[2] based on Bragg reflection and Laue diffraction, were made long ago. The former was reviewed in detail by Bonse,[3,4] Hart,[5,6] etc. Since the first neutron interferometer was made, scientific researchers working in the areas of solid-state physics, nuclear physics, particle physics, gravitation theory, etc., have been very interested in it.[7] However, it should be mentioned that Rauch[8] first worked on the theory of the neutron interferometer.

Until now a lot of research on the interference of particles, e.g., electrons and neutrons, was based on amplitude coherence, mainly using scattering theory.[9–11] Some physicists discussed the interference of particle beams from the viewpoint of the relative phase difference between coherent particle beams. Overhauser and Collela[12] and Collela, Overhauser, and Werner[13] (COW) experimentally studied the interference of the relative phase difference between two coherent beams caused by the gravitational effect. Werner[14] and Rauch[15] independently pointed out that a 360-degree rotation of a neutron interferometer in the magnetic field changed the sign of the neutron wave function. In 1976 Werner[16] designed a new experiment and made theoretical analyses. Greenberger[17] also did important work. However, all their analyses are restricted by the requirement that particles are considered as plane waves.

Using the principle of least action and the concept of phase coherence, Chiu and Stodolsky[18] established two general theorems about the interference of particle beams. In this paper, we establish a new theorem which further discusses the influence of the disturbance at the initial points of coherent particle beams on "coherent fringe shifts." Then the famous COW and Marton experiments[19–31] are theoretically interpreted using the Chiu-Stodolsky Theorem II. The results agree with these experiments. The Feynman theory is applied to obtaining the propagator for particles in the gravitational field, and the COW experiment is analyzed further. Our results agree with the experimental results. Finally, the Chiu-Stodolsky Theorem I and our new theorem are proved with the method of Feynman's path integrals.

II. SEVERAL THEOREMS ABOUT THE INTERFERENCE OF PARTICLE BEAMS

Dirac[32] pointed out the following: "For any dynamical system with a classical analog, a state for which the classical description is valid as an approximation is represented in quantum mechanics by a wave packet, all the coordinates and momenta having approximate values, whose accuracy is limited by Heisenberg's principle of uncertainty. Now Schrödinger's wave equation fixes how such a wave packet varies with time, so in order that the classical description may remain valid, the wave packet should remain a wave packet and should move according to the laws of classical dynamics."

The above-mentioned ideas of Dirac are applied to particle beams. The states of a particle can be described by wave functions. Each wave function is a superposition of many plane waves with the form

$$\psi(\mathbf{x},t)=a_0 e^{i(\mathbf{p}\cdot\mathbf{x}-Ht)/\hbar}$$

(the plane waves describe the states of free particles in quantum mechanics). Therefore such a wave function is a wave packet representing a state of a particle.

We can assume that the wave function of a particle is $\psi=Ae^{iS/\hbar}$, in which A and S are functions of q and t.

The action is defined as[33]

$$S(x_I,x_F)=\int_{x_I}^{x_F} L\,dt=\int_{x_I}^{x_F} p_\mu dx_\mu=\int_{x_I}^{x_F}(\mathbf{p}\cdot d\mathbf{x}-E\,dt)\ . \tag{2.1}$$

The path of the packet motion is just the path of the par-

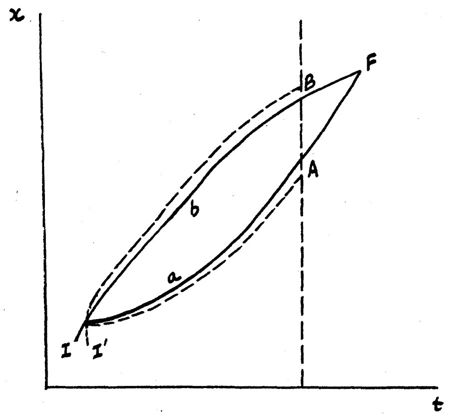

FIG. 1. Changes of classical paths after disturbing the incident particles.

ticle motion in accordance with the classical laws. Such a path is called a "classical path." In this "semiclassical limit," the size of the wave packet is large compared to the wavelength.

Consider the condition in which energy remains constant. The phase of the wave packet representing the state of a particle is

$$\phi=\frac{1}{\hbar}\int(\mathbf{p}\cdot d\mathbf{x}-E\,dt)\ . \tag{2.2}$$

If two coherent particle beams starting from point I move along their classical paths, respectively, and meet at point F, then the phase difference between the two packets is

$$\Delta\phi=\phi_{IF}^{(a)}-\phi_{IF}^{(b)}\ . \tag{2.3}$$

Generally, the particles do not move along their classical paths from I to F. If the final point F is not at the classical path and the phase at some point, e.g., A, in the neighborhood of the point F is known, then

$$\phi_{IF}=\phi_{IA}+\frac{1}{\hbar}\mathbf{p}\cdot\delta\mathbf{x}_{AF}\ . \tag{2.4}$$

Assume in Fig. 1 that the solid curves represent the original paths and the dashed curves represent the classical paths after disturbing the incident particles. Chiu and Stodolsky[18] established two theorems as follows:

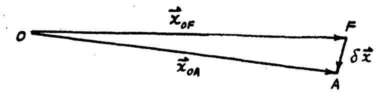

FIG. 2. Because of disturbance, the end point moves from F to A.

Theorem I. In the neighborhood of the ideal configuration the effect on $\Delta\phi$ of changing the incident beam vanishes to first order in the variation.

Theorem II. For two paths in the neighborhood of classically crossing paths the variation of the phase difference due to a change in the applied fields $\delta\beta_j$ is given by the explicit change only:

$$\Delta\phi_{IF}\rightarrow\Delta\phi_{IF}+\left[\int^{(a)}\frac{\partial L}{\partial\beta_j}\delta\beta_j dt-\int^{(b)}\frac{\partial L}{\partial\beta_j}\delta\beta_j dt\right]\ , \tag{2.5}$$

where the integral is along the classical crossing paths.

We can deduce from Theorem I that the diffusion of wave packets cannot cause changes of the interference fringes but only affects the clearness of the interference fringes. This deduction can be strictly proved by the path-integral method in our later work. Of course, it is true to first order in the variation. Now we establish a new theorem as follows:

Theorem III. In the neighborhood of the ideal configuration the change of the incident particle beam affects $\Delta\phi$ for the second-order variation.

Under the condition that the particles are free and energy remains constant, the wave function of a free particle is $\psi=Ae^{i(\mathbf{p}\cdot\mathbf{x}-E_0t)/\hbar}$. Let

$$\mathbf{p}\,|_{t=0,\mathbf{x}=\mathbf{x}_0}=\mathbf{p}_0\ . \tag{2.6}$$

Then for states $\psi=e^{i(\mathbf{p}\cdot\mathbf{x}-E_0t)/\hbar}$, $\mathbf{p}$ has the definite value $\mathbf{p}_0$, shown in Fig. 2. Thus

$$\begin{aligned}\hbar\phi_{0F}&=-E_0t+\int_0^F\mathbf{p}\cdot d\mathbf{x}=-E_0t+\mathbf{p}_{0F}\cdot\mathbf{x}_{0F}\\&=-E_0t+p_0x_0\ .\end{aligned}\tag{2.7}$$

Because of disturbance, the end point moves from F to A. Then

$$\begin{aligned}\hbar\phi_{0A}&=-E_0t+\int_0^A\mathbf{p}\cdot d\mathbf{x}=-E_0t+\mathbf{p}_{0A}\cdot\mathbf{x}_{0A}=-E_0t+p_0\,|\,\mathbf{x}_{0F}+\delta\mathbf{x}\,|\\&=-E_0t+p_0[(\mathbf{x}_{0F}+\delta\mathbf{x})\cdot(\mathbf{x}_{0F}+\delta\mathbf{x})]^{1/2}=-E_0t+p_0x_0\left[1+\frac{2\mathbf{x}_{0F}\cdot\delta\mathbf{x}+\delta x^2}{x_0^{\,2}}\right]^{1/2}\\&\simeq-E_0t+p_0x_0+\mathbf{p}_{0F}\cdot\delta\mathbf{x}+\mathbf{p}_0\cdot\delta x^2/2x_0-\frac{1}{2r_0^{\,3}}p_0\eta x_0^{\,2}\delta x^2\\&=-E_0t+p_0x_0+\mathbf{p}_{0F}\cdot\delta\mathbf{x}+(p_0/2x_0-\eta p_0/2x_0)\delta x^2\\&=-E_0t+p_0x_0+\mathbf{p}_{0F}\cdot\delta\mathbf{x}+(p_0/2r_0)\xi\,dx^2\ ,\end{aligned}\tag{2.8}$$

where η is a constant whose value is determined by the angle between $\mathbf{x}_{0F}$ and $\delta\mathbf{x}$, and $\xi=1-\eta$. From (2.7) and (2.8), we have

$$\phi_{0F}=\phi_{0A}-\frac{1}{\hbar}\left[\mathbf{p}_{0F}\cdot\delta\mathbf{x}_{AF}+\frac{p_0}{2r_0}\xi\delta x_{AF}{}^2\right] . \tag{2.9}$$

If 0 is replaced by I' in the above expression, then we obtain with Fig. 1

$$\hbar\Delta\phi_{IF}=S_{IF}^{(a)}-S_{IF}^{(b)} , \tag{2.10}$$

$$\hbar\Delta\phi_{I'F}=\hbar\phi_{I'A}-\mathbf{p}_F^{(a)}\cdot\delta\mathbf{x}_{AF}-\frac{p_0^{(a)}}{2x_b^{(a)}}\xi^{(a)}\delta x_{AF}{}^2-\hbar\phi_{I'B}+\mathbf{p}_F^{(b)}\cdot\delta\mathbf{x}_{BF}+\frac{p_0^{(b)}}{2x_0^{(b)}}\xi^{(b)}\delta x_{BF}{}^2$$

$$=S_{I'A}^{(a)}-S_{I'B}^{(b)}-\left[\mathbf{p}_F^{(a)}\cdot\delta\mathbf{x}_{AF}+\frac{p_0^{(a)}}{2x_0^{(a)}}\xi^{(a)}\delta x_{AF}{}^2-\mathbf{p}_F^{(b)}\cdot\delta\mathbf{x}_{BF}-\frac{p_0^{(b)}}{2x_0^{(b)}}\xi^{(b)}\delta x_{BF}{}^2\right] . \tag{2.11}$$

According to the variational principle and considering second-order effects,

$$S_{I'A}^{(a)}=S_{I'I}+S_{IF}^{(a)}+\delta S_{IF}^{(a)}+\delta S_{IF}^{(a)2} \tag{2.12}$$

and

$$\delta S_{IF}^{(a)2}=\int_{t_0}^{t_1}(Q\delta x^2+R\delta p^2)dt+\frac{\partial^2 L}{\partial\mathbf{x}\partial\mathbf{p}}\delta r^2\bigg|_{\substack{\mathbf{x}=\mathbf{x}_0\\ \mathbf{p}=\mathbf{p}_0}}-\frac{\partial^2 L}{\partial\mathbf{x}\partial\mathbf{p}}\delta r^2\bigg|_{\substack{\mathbf{x}=\mathbf{x}_1\\ \mathbf{p}=\mathbf{p}_1}} , \tag{2.13}$$

where

$$Q=\frac{1}{2}\left[\frac{\delta^2 L}{\partial\mathbf{x}\,\partial\mathbf{x}}-\frac{d}{dt}\frac{\partial^2 L}{\partial\mathbf{x}\,\partial\mathbf{p}}\right],\quad R=\frac{1}{2}\frac{\partial^2 L}{\partial\mathbf{p}\,\partial\mathbf{p}} .$$

Since $\delta p^2=0$, $L=p^2/2\mu$, then $\partial^2 L/\partial\mathbf{x}^2=0$, $\partial^2 L/\partial\mathbf{x}\,\partial\mathbf{p}=0$. Thus $\delta S_{IF}^{(a)2}=0$ and

$$S_{I'A}^{(a)}=S_{I'I}+S_{IF}^{(a)}+\delta S_{IF}^{(a)},\quad \delta S_{IF}^{(a)}=\mathbf{p}_F^{(a)}\cdot\delta\mathbf{x}_{AF} , \tag{2.14}$$

$$S_{I'B}^{(b)}=S_{I'I}+S_{IF}^{(b)}+\delta S_{IF}^{(b)},\quad \delta S_{IF}^{(b)}=\mathbf{p}_F^{(b)}\cdot\delta\mathbf{x}_{BF} . \tag{2.15}$$

We conclude that

$$\Delta\phi_{I'F}=\Delta\phi_{IF}-\frac{p_0^{(a)}}{2x_0^{(a)}}\xi^{(a)}\delta r_{AF}{}^2+\frac{p_0^{(b)}}{2x_0^{(b)}}\xi^{(b)}\delta r_{BF}{}^2 , \tag{2.16}$$

i.e., considering the second-order variation, we have obtained the effect of the change of the incident particle beam.

III. INTERPRETATION OF COLLELA-OVERHAUSER-WERNER AND MARTON EXPERIMENTS

Now we interpret theoretically the Collela-Overhauser-Werner and Marton experiments by means of Theorem II in Sec. II.

A. Collela-Overhauser-Werner (COW) experiment

Overhauser and others made similar experiments independently in 1974 (Ref. 12) and 1975.[13] The principle of these experiments is shown in Fig. 3. A neutron beam A is divided into two neutron beams (I) and (II), at the point B of silicon crystal sheet BE, which are reflected at silicon crystal sheets C and D and meet at E. The apparatus is placed in the gravitational field. Let the apparatus rotate through 180 degrees about BD and fringe shifts will be found at F. For example, if $\lambda=1.42$ Å and the area of parallelogram BCED is 6 cm^2, then a shift of 10.5 fringes will occur.

From Theorem II,[18]

$$\delta S^{(\mathrm{I})}=\int^{(\mathrm{I})}\frac{\partial L}{\partial\beta_j}\delta\beta_j dt+(p_i^F\delta x_i^F-p_i^B\delta x_i^B) , \tag{3.1}$$

$$\delta S^{(\mathrm{II})}=\int^{(\mathrm{II})}\frac{\partial L}{\partial\beta_j}\delta\beta_j dt+(p_i^F\delta x_i^F-p_i^B\delta x_i^B) . \tag{3.2}$$

In the gravitational field,[34]

$$\delta S=-mc\int\delta(ds)=-mc\int\left[\left[g_{il}\frac{d^2x^i}{ds^2}+\Gamma_{l,ik}\frac{dx^i}{ds}\frac{dx^k}{ds}\right]ds+mcg_{ik}\frac{dx^i}{ds}\delta x^k\right] . \tag{3.3}$$

In our problem, the velocity is small compared with the velocity of light and the apparatus belongs to a small scale. The Lagrangian of a particle in the gravitational field can be written as

$$L=-mc^2+\frac{mv^2}{2}-m\phi \; . \tag{3.4}$$

Thus

$$\delta S^{(\mathrm{I})}-\delta S^{(\mathrm{II})}=-\left[\int^{(\mathrm{I})} m\,\delta\phi\,dt-\int^{(\mathrm{II})} m\,\delta\phi\,dt\right] , \tag{3.5}$$

$$\delta S^{(\mathrm{I})}=-\int^{(\mathrm{I})} m\,\delta\phi\,dt=mg\left[\int_0^{t_1}[(v\sin\theta)t-\tfrac{1}{2}gt^2]dt+\int_{t_1}^{t_2}[y_1-\tfrac{1}{2}g(t-t_1)^2]dt\right]$$

$$=mg[\tfrac{1}{2}v\sin\theta t_1{}^2-\tfrac{1}{6}gt_1{}^3+y_1(t_2-t_1)-\tfrac{1}{6}g(t_2-t_1)^3] , \tag{3.6}$$

$$\delta S^{(\mathrm{II})}=-\int^{(\mathrm{II})} m\,\delta\phi\,dt=mg\left[\int_0^{t_1'}(-\tfrac{1}{2}gt^2)dt+\int_{t_1'}^{t_2}(v_1\sin\theta_1' t-\tfrac{1}{2}gt^2)dt\right]$$

$$=mg[\tfrac{1}{2}v_1\sin\theta_1'(t_2-t_1')^2-\tfrac{1}{6}g(t_2-t_1')^3-\tfrac{1}{6}gt_1'{}^3] , \tag{3.7}$$

$$\delta S^{(\mathrm{I})}-\delta S^{(\mathrm{II})}=mg[\tfrac{1}{2}v\sin\theta t_1{}^2-\tfrac{1}{2}v_1\sin\theta'(t_2-t_1')^2+y_1(t_2-t_1)-\tfrac{1}{6}gt_1{}^3+\tfrac{1}{6}gt_1'{}^3-\tfrac{1}{6}g(t_2-t_1)^3+\tfrac{1}{6}g(t_2-t_1')^3] , \tag{3.8}$$

where

$$y_1=r=v\sin\theta t_1-\tfrac{1}{2}gt_1{}^2 , \tag{3.9}$$

$$t_1=\frac{l}{v_1} , \tag{3.10}$$

$$v_1{}^2=v^2+(gt_1')^2=v^2+\left[\frac{gl}{v}\right]^2 . \tag{3.11}$$

The solution of Eq. (3.9) is

$$\begin{aligned} t_1&=\tfrac{1}{2}[2v\sin\theta/g-2(v^2\sin^2\theta-2rg)^{1/2}/g]\\ &=v\sin\theta/g-(v^2\sin^2\theta-2rg)^{1/2}/g\\ &=\frac{1}{g}[v\sin\theta-(v^2\sin^2\theta-2rg)^{1/2}] .\end{aligned} \tag{3.12}$$

Making an approximate calculation, only terms less than second order in $1/v$ remain:

$$\begin{aligned} &t_1'\cong l/v ,\\ &v\cong v_1,\quad \theta\cong\theta' ,\\ &t_1\cong r/v\sin\theta ,\\ &t_2\cong l/v+r/v\sin\theta .\end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned}\delta S^{(\mathrm{I})}-\delta S^{(\mathrm{II})}&=mg[\tfrac{1}{2}v\sin\theta t_1{}^2-\tfrac{1}{2}v_1\sin\theta'(t_2-t_1')^2+r(t_2-t_1)-\tfrac{1}{6}gt_1{}^3+\tfrac{1}{6}gt_1'{}^3-\tfrac{1}{6}g(t_2-t_1)^3+\tfrac{1}{6}g(t_2-t_1')^3]\\ &\cong mgr(t_2-t_1)=mgr\frac{l}{v}=m^2gr\frac{l}{p}=m^2grl\frac{\lambda}{h} ,\end{aligned} \tag{3.14}$$

i.e.,

$$\Delta\phi=2\pi m^2grl\frac{\lambda}{h^2} , \tag{3.15}$$

where h is Planck's constant.

After rotating the apparatus through 180 degrees about AD, we have

$$\Delta N=2\frac{\Delta\phi}{2\pi}=2m^2grl\lambda/h^2 . \tag{3.16}$$

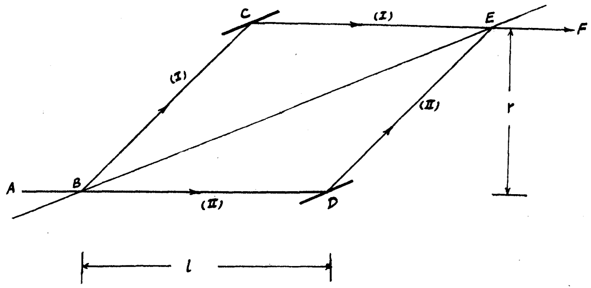

FIG. 3. Sketch of COW experiment.

Inserting $\lambda=1.42$ Å and $rl=6\ \text{cm}^2$ in (3.16), we obtain

$$\Delta N=10.7\ . \tag{3.17}$$

This result corresponds quite closely to the COW experiment. However, according to the interpretation of the Marton experiment by Werner in 1960, the fringe shifts cannot occur because the phase difference due to the sliding effect, i.e., the slide of end points caused by the applied field, balances the phase difference due to the bending effect, i.e., the bend of paths caused by the applied field. It is apparent that Werner's interpretation conflicts with the result of the COW experiment. However, the COW experiment can be interpreted by Theorem II.

B. Marton experiment

The principle of Marton's experiment is sketched in Fig. 4. The apparatus is placed in a static electric field or a static magnetic field and the fringes can be observed at C. If the apparatus is placed in a slowly changing electric field or a slowly changing magnetic field, the fringe shift can be observed at C. For example, if the apparatus is placed in a slowly changing electric field, then we have

$$\delta S^{(\mathrm{I})}-\delta S^{(\mathrm{II})}=-e\left[\int^{(\mathrm{I})}\delta\phi\,dt-\int^{(\mathrm{II})}\delta\phi\,dt\right], \tag{3.18}$$

where ϕ is the potential of the field.

In the Marton experiment, Marton and others canceled coherent fringes caused by nonparallelism of these thin copper sheets. However, as Marton informed us, the stray 60-cycle magnetic fields were present in his electron interferometer. The interference fringes would have been shifted back and forth sixty times each second by such large amounts that they could not possibly have been seen.

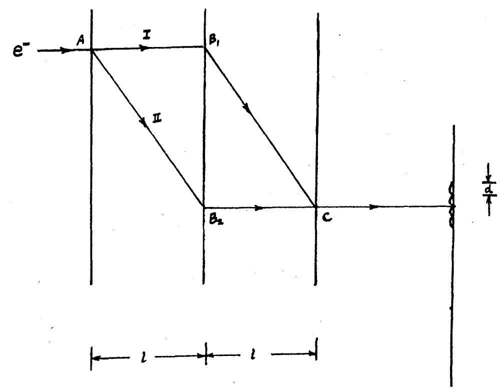

FIG. 4. Sketch of Marton experiment.

This is what caused Werner to make a wrong interpretation. An explanation for this experiment is given in Ref. 17. The mistake in Werner's "interpretation" was pointed out in Refs. 17, 18, and 38.

IV. PROPAGATOR OF PARTICLE BEAM IN GRAVITATIONAL FIELD AND COW EXPERIMENT

In Sec. III the COW experiment was explained by Theorem II. In this section the COW experiment will be discussed strictly by means of the path integration of quantum mechanics. The propagator of the particle beam in the gravitational field will be derived and the wave function will be obtained.

From quantum mechanics,[35–37] the propagator of a particle beam, which moves from $\mathbf{r}'$ to $\mathbf{r}''$ in time interval $\tau=t''-t'$, is

$$K(\mathbf{r}'',\mathbf{r}',\tau)=\lim_{N\to\infty}A_N\int\exp\left[-\frac{2}{l}\sum_{j=1}^{N}S(\mathbf{r}_j,\mathbf{r}_{j-1})\right]\prod_{j=1}^{N-1}d\mathbf{r}_j\ , \tag{4.1}$$

where $l=2i\hbar/\mu$, μ is the mass of a particle, A_N is a normalizing factor satisfying the condition

$$\int K(\mathbf{r}'',\mathbf{r}',\tau)d\mathbf{r}''=1\ . \tag{4.2}$$

The action is

$$S(\mathbf{r}_j,\mathbf{r}_{j-1})=\frac{\epsilon}{2}\left[\frac{x_j-x_{j-1}}{\epsilon}\right]^2+\frac{\epsilon}{2}\left[\frac{y_j-y_{j-1}}{\epsilon}\right]^2-\epsilon g y_j$$

$$=\frac{1}{2\epsilon}(x_j{}^2+x_{j-1}{}^2-2x_jx_{j-1})+\frac{1}{2\epsilon}(y_j{}^2+y_{j-1}{}^2-2y_jy_{j-1})-\epsilon g y_j\ . \tag{4.3}$$

ϵ is the time interval in which the particle moves from $\mathbf{r}_{j-1}$ to $\mathbf{r}_j$ and the equal-time interval $\epsilon=\tau/N$ is adopted. Thus we have

$$\exp\left[-\frac{2}{l}S(\mathbf{r}_j,\mathbf{r}_{j-1})\right]=\exp\left[(x_j^2+x_{j-1}^2-2x_jx_{j-1})\left[-\frac{1}{\epsilon l}\right]\right]\exp\left[-\frac{1}{\epsilon l}(y_j^2+y_{j-1}^2-2y_jy_{j-1})+\frac{2\epsilon}{l}gy_j\right]. \tag{4.4}$$

Let

$$I_1=\int\exp\left[\sum_{j=1}^{N}\left[-\frac{1}{\epsilon l}(y_j^2+y_{j-1}^2)+\frac{2}{l\epsilon}y_jy_{j-1}+\frac{2\epsilon}{l}gy_j\right]\right]\prod_{j=1}^{N-1}dy_j\ , \tag{4.5}$$

i.e.,

$$I_1=\int\exp\left[-\frac{1}{\epsilon l}(y_1^2+y_0^2)+\frac{2}{l\epsilon}(y_0+y_2)y_1+\frac{2\epsilon}{l}gy_1-\frac{1}{\epsilon l}(y_2^2+y_1^2)+\frac{2\epsilon}{l}gy_2\right.$$
$$\left.+\cdots-\frac{1}{\epsilon l}(y_N^2+y_{N-1}^2)+\frac{2}{l\epsilon}y_Ny_{N-1}+\frac{2\epsilon}{l}gy_N\right]\prod_{j=1}^{N-1}dy_j\ . \tag{4.6}$$

The integral (4.6) is a multiple integral. First, it can be integrated over y_1,

$$\int_{-\infty}^{\infty}\exp\left[-\frac{2}{\epsilon l}y_1^2+\frac{2}{\epsilon l}(y_2+y_0)y_1+\frac{2\epsilon}{l}gy_1\right]dy_1=\int_{-\infty}^{\infty}\exp\left[-\frac{2}{\epsilon l}[y_1-\tfrac{1}{2}(y_2+y_0+\epsilon^2g)]^2+\frac{1}{2\epsilon l}(y_2+y_0+\epsilon^2g)^2\right]dy_1$$
$$=(\epsilon\pi l/2)^{1/2}\exp\left[\frac{1}{2\epsilon l}(y_2+y_0+\epsilon^2g)^2\right]. \tag{4.7}$$

Then it is integrated over y_2 and the terms of ϵ^2 are neglected,

$$\int_{-\infty}^{\infty}\left[\frac{\epsilon l\pi}{2}\right]^{1/2}\exp\left[\frac{1}{2\epsilon l}(y_2+y_0+\epsilon^2g)^2-\frac{2}{\epsilon l}y_2^2+\frac{2}{\epsilon l}y_3y_2+\frac{2\epsilon}{l}gy_2\right]dy_2$$
$$=\left[\frac{\epsilon l\pi}{2}\right]^{1/2}\exp\left[\frac{y_0^2}{2\epsilon l}+\frac{g\epsilon}{l}y_0\right]\int_{-\infty}^{\infty}\exp\left[\frac{y_2^2}{2\epsilon l}+\frac{y_0y_2}{\epsilon l}+\frac{3g\epsilon}{l}y_2-\frac{2}{\epsilon l}y_2^2+\frac{2}{\epsilon l}y_3y_2\right]dy_2$$
$$=\left[\frac{\epsilon l\pi}{2}\right]^{1/2}\exp\left[\frac{y_0^2}{2\epsilon l}+\frac{g\epsilon}{l}y_0\right]\int_{-\infty}^{\infty}\exp\left[-\frac{3}{2\epsilon l}y_2^2+\left[\frac{y_0}{\epsilon l}+\frac{3g\epsilon}{l}+\frac{2}{\epsilon l}y_3\right]y_2\right]dy_2$$
$$=\left[\frac{\epsilon l\pi}{2}\right]^{1/2}\left[\frac{2\epsilon l\pi}{2}\right]^{1/2}\exp\left[\frac{y_0^2}{2\epsilon l}+\frac{2\epsilon g}{l}y_0\right]\exp\left[\frac{\epsilon l}{2\times 3}\left[\frac{y_0}{\epsilon l}+\frac{2g\epsilon}{l}+\frac{2}{\epsilon l}y_3\right]^2\right]. \tag{4.8}$$

By means of mathematical induction we can obtain

$$I_1=\frac{(\epsilon l\pi)^{N/2}}{N^{1/2}}\exp\left[\frac{y_0^2}{2\epsilon l}+\frac{y_0^2}{2\times 3\epsilon l}+\cdots+\frac{y_0^2}{(N-1)N\epsilon l}\right]\exp\left[(N-1)\frac{g\epsilon}{l}y_0\right]\exp\left[-\frac{y_N^2}{N\epsilon l}\right]$$
$$=\left[\frac{(\epsilon l\pi)^N}{N}\right]^{1/2}\exp\left[\frac{-y_0^2}{N\epsilon l}-\frac{y_N^2}{N\epsilon l}\right]\exp\left[(N-1)\frac{g\epsilon}{l}y_0\right]. \tag{4.9}$$

Let

$$I_2=\int\exp\left[-\frac{2}{l\epsilon}\sum_{j=1}^{N}(x_j-x_{j-1})^2\right]\prod_{j=1}^{N-1}dx_j\ ; \tag{4.10}$$

(4.10) can then be easily calculated:

$$I_2=(\pi l\epsilon)^{N/2}(l\pi N\epsilon)^{-1/2}\exp\left[-\frac{(x_N-x_0)^2}{N\epsilon l}\right]. \tag{4.11}$$

Substituting r_0, r_N, and $N\epsilon$ in I_1 and I_2 by r', r'', and τ, respectively, we can obtain from (4.1)

$$K(\mathbf{r}'',\mathbf{r}',\tau)=\lim_{N\to\infty} A_N I_1 I_2=\lim_{N\to\infty} A_N(\pi l\epsilon)^N(Nl\pi\tau)^{-1/2}\exp\left[-\frac{1}{l\tau}[y'^2+y''^2+(x''-x')^2]+\frac{g\tau}{l}y'\right].$$

Let $A_N=N^{1/2}/(\pi l\epsilon)^N$, then the above expression becomes

$$K(\mathbf{r}'',\mathbf{r}',\tau)=(l\pi\tau)^{-1/2}\exp\left[-\frac{1}{l\tau}[y'^2+y''^2+(x''-x')^2]+\frac{g\tau}{l}y'\right]. \tag{4.12}$$

Thus the wave function at $t=t''$ and $\mathbf{r}=\mathbf{r}''$ can be found from the wave function at $t=t'$ and $\mathbf{r}=\mathbf{r}'$:

$$\begin{aligned}\psi(\mathbf{r}'',t'')&=\int K(\mathbf{r}'',\mathbf{r}',\tau)\psi(\mathbf{r}',t')d\mathbf{r}'\\&=\frac{1}{l\pi\tau}\int\exp\left[-\frac{1}{l\tau}[y''^2+y'^2+(x''-x')^2]+\frac{g\tau}{l}y'\right]\psi(\mathbf{r}',t')d\mathbf{r}'.\end{aligned} \tag{4.13}$$

Since the path of a particle in the gravitational field is a complicated curve, it is difficult to calculate (4.13). However, we are able to estimate that the phase of wave function (4.13) depends on the path of integration. This result is in accordance with the conclusion in Sec. III.

V. QUANTUM PROOFS OF THEOREMS I AND III

Chiu and Stodolsky established Theorems I and II with the semiclassical method. In Sec. II we have established Theorem III with the same method. Now we prove Theorems I and III with the method of Feynman's path integrals.

Figure 5 shows that a particle beam I incident to the crystal sheet M_0 at $t=T$ is divided into the particle beams II′ and II″ by reflection and refraction, respectively. Then II′ and II″ are reflected at the crystal sheets M_1 and M_2 into the particle beams III′ and III″. Finally III′ and III″ are superposed at D.

Now the phase and amplitude at D along the paths II′—III′ and II″—III″ will be calculated by the method of path integration. We consider that the incident beam is reflected and refracted at the slit of which the breadth is $2b$ (or at the lattice of which the spacing is $2b$).

From quantum mechanics, the wave functions in the beams II′ and II″ are, respectively,

$$\begin{aligned}\psi(\mathbf{r}_1'')=\int K(\mathbf{r}_1''+\mathbf{r}_0,\tau+T,\mathbf{r}_0+\mathbf{r},T)\\\times K(\mathbf{r}_0+\mathbf{r},T,0,0)d\mathbf{r}\,,\end{aligned} \tag{5.1}$$

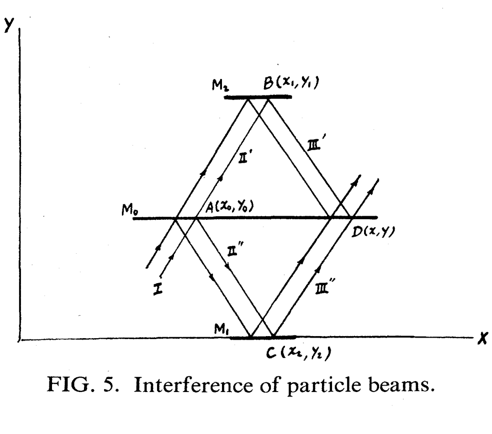

FIG. 5. Interference of particle beams.

$$\begin{aligned}\psi(\mathbf{r}_2'')=\int K(\mathbf{r}_2''+\mathbf{r}_0,\tau+T,\mathbf{r}_0+\mathbf{r},T)\\\times K(\mathbf{r}_0+\mathbf{r},T,0,0)d\mathbf{r}\,.\end{aligned} \tag{5.2}$$

It should be stated that the particles in the beam are emitted at $t=0$, $\mathbf{r}=0$.

From Fig. 5, we can find particles only in the position $x\in[x_0-b,x_0+b]$, $y=y_0$ at the instant T. The propagator of free particles is

$$K(\mathbf{r}'',\mathbf{r}')=\left[\frac{2\pi i\hbar(t''-t')}{m}\right]^{-3/2}\exp\left[\frac{im(\mathbf{r}''-\mathbf{r}')^2}{2\hbar(t''-t')}\right], \tag{5.3}$$

where m is the mass of a particle, $\mathbf{r}'$ and $\mathbf{r}''$ the coordinates of the initial and final points, t' and t'' the initial and final times, respectively. In the two-dimensional case

$$\begin{aligned}\psi(x',y')&=\left[\frac{m}{2\pi i\hbar}\right]^2\frac{1}{T\tau'}\\&\quad\times\int_{-b}^{b}\exp\left[\frac{im[(x'-x)^2+(y'-y_0)^2]}{2\hbar\tau'}\right]\\&\quad\times\exp\left[\frac{im[(x_0+x)^2+(y_0+y_0)^2]}{2\hbar T}\right]dx\,,\end{aligned} \tag{5.4}$$

where $\tau'=t''-t'$. Let

$$G(x)=\begin{cases}1, & x\in[-b,b]\\0 & \text{elsewhere}\,.\end{cases} \tag{5.5}$$

The function $\exp(-x^2/2b^2)$ satisfies approximately (5.5), i.e.,

$$G(x)\cong\exp(-x^2/2b^2)\,, \tag{5.6}$$

(5.4) can be written in the form

$$\psi(x_1'',y_1'')=\left[\frac{m}{2\pi i\hbar}\right]^2(T\tau')^{-1}\int_{-\infty}^{\infty}G(x)\exp\left[\frac{im[(x_1''-x)^2+(y_1''-y_0)^2]}{2\hbar\tau'}\right]\left[\frac{im[(x_0+x)^2+(2y_0)^2]}{2\hbar T}\right]dx$$

$$=\left[\frac{m}{2\pi i\hbar}\right]^{3/2}\left[T\tau'\left[T+\tau'+T\tau'\frac{i\hbar}{mb^2}\right]\right]^{-1/2}\exp\left[\frac{im}{2\hbar}\left[\frac{(y_1''-y_0)^2}{\tau'}+\frac{4y_0{}^2}{T}\right]\right]$$

$$\times\exp\left[\frac{im}{2\hbar}\left[\frac{x_0{}^2}{T}+\frac{x_1''^2}{\tau'}\right]+\frac{(m^2/2\hbar^2\tau'^2)(x_1''-x_0\tau'/T)^2}{(m/\hbar)(i/T+i/\tau'')-1/b^2}\right]. \tag{5.7}$$

Similarly, the amplitude function of particles in beam II″ is

$$\psi(x_2'',y_2'')=\left[\frac{m}{2\pi i\hbar}\right]^{3/2}\left[T\tau''\left[T+\tau''+T\tau''\frac{i\hbar}{mb^2}\right]\right]^{-1/2}\exp\left[\frac{im}{2\hbar}\left[\frac{(y_2''-y_0)^2}{\tau''}+\frac{4y_0{}^2}{T}\right]\right]$$

$$\times\exp\left[\frac{im}{2\hbar}\left[\frac{x_0{}^2}{T}+\frac{x_2''^2}{\tau''}\right]+\frac{(m^2/2\hbar^2\tau''^2)(x_2''-x_0\tau''/T)^2}{(m/\hbar)(i/T+i/\tau'')-1/b^2}\right], \tag{5.8}$$

where $\psi(\mathbf{r}_1'')$ and $\psi(\mathbf{r}_2'')$ represent the amplitude functions of particles passing through the paths II′ and II″, respectively, t_1', t_1'' and t_2', t_2'' the initial and final instants of beams II′ and II″, $\tau'=t_1''-t_1'$, $\tau''=t_2''-t_2'$.

The probability density of finding particles on the crystal sheet M_2 at the instant $t=T+\tau'$ is

$$P(\mathbf{r}_1'')=\psi^*(\mathbf{r}_1'')\psi(\mathbf{r}_1'')$$

$$=\left[\frac{m}{2\pi\hbar}\right]^3(T\tau')^{-1}\left[\left[T+\tau'+T\tau'\frac{i\hbar}{mb^2}\right]\left[T+\tau'-T\tau'\frac{i\hbar}{mb^2}\right]\right]^{-1/2}\exp\left[-\frac{(x_1-x_0\tau'/T)^2}{\lambda^2}\right], \tag{5.9}$$

where

$$\lambda^2=b^2\left[1+\frac{\tau'}{T}\right]+\frac{\tau'^2\hbar^2}{m^2b^2}=b_1{}^2+\frac{\tau'^2\hbar^2}{mb^2}. \tag{5.10}$$

The semibreadth of the wave function at the initial point is b and the semibreadth at $\mathbf{r}_1''$ is b_1, so wave packets diffuse when moving.

The wave functions $\psi(\mathbf{r}_3'')$ and $\psi(\mathbf{r}_4'')$ of the particles in the beams III′ and III″ are calculated as

$$\psi(\mathbf{r}_3'')=\int K(\mathbf{r}_3'',\mathbf{r}_1')\psi(\mathbf{r}_1'')d\mathbf{r}_1'', \tag{5.11}$$

$$\psi(\mathbf{r}_4'')=\int K(\mathbf{r}_4'',\mathbf{r}_2'')\psi(\mathbf{r}_2'')d\mathbf{r}_2'', \tag{5.12}$$

$$\psi(\mathbf{r}_3'')=\left[\frac{m}{2\pi i\hbar}\right]^{3/2}\left[T\tau_1'\left[T+\tau_1'+T\tau_1'\frac{i\hbar}{mb^2}\right]\right]^{-1/2}$$

$$\times\exp\left[\frac{im}{2\hbar}\left[\frac{(y_1''-y_0)^2}{\tau_1'}+\frac{4y_0{}^2}{T}\right]+\frac{im}{2\hbar\tau'}(y_3''-y_1'')^2\right]\frac{m\sqrt{\pi}}{2\pi i\hbar\tau'}\left[\frac{m^2/2\hbar^2\tau_1'^2}{\frac{1}{b^2}-\frac{m}{\hbar}\left[\frac{i}{T}+\frac{i}{\tau'}\right]}-\frac{im}{\hbar}\left[\frac{1}{\tau'}+\frac{1}{\tau_1'}\right]\right]^{-1/2}$$

$$\times\exp\left\{-\frac{1}{4}\left[\frac{im}{\hbar\tau'}x_3''+\left[\frac{m^2}{T\hbar^2\tau_1'^2}\right]\frac{x_0\tau_1'}{\frac{m}{\hbar}\left[\frac{i}{T}+\frac{i}{\tau_1'}\right]-\frac{1}{b^2}}\right]^2\left[\frac{im}{2\hbar\tau'}+\frac{im}{2\hbar\tau_1'}+\frac{m^2/2\hbar^2\tau_1'^2}{\frac{m}{\hbar}\left[\frac{i}{T}+\frac{i}{\tau_1'}\right]-\frac{1}{b^2}}\right]^{-1}\right\}, \tag{5.13}$$

$$\psi(\mathbf{r}_4'')=\left[\frac{m}{2\pi i\hbar}\right]^{3/2}\left[T\tau_1''\left[T+\tau_1''+T\tau_1''\frac{i\hbar}{mb^2}\right]\right]^{-1/2}$$
$$\times\exp\left[\frac{im}{2\hbar}\left[\frac{(y_2''-y_0)^2}{\tau_1''}+\frac{4y_0^{\,2}}{T}\right]+\frac{im}{2\hbar\tau''}(y_4''-y_2'')^2\right]\frac{m\sqrt{\pi}}{2\pi i\hbar\tau''}\left[\frac{m^2/2\hbar^2\tau_1''^2}{\dfrac{1}{b^2}-\dfrac{m}{\hbar}\left[\dfrac{i}{T}+\dfrac{i}{\tau''}\right]}-\frac{im}{\hbar}\left[\frac{1}{\tau_1''}+\frac{1}{\tau''}\right]\right]^{-1/2}$$
$$\times\exp\left\{-\frac{1}{4}\left[\frac{im}{\hbar\tau''}x_4''+\left[\frac{m^2}{T\hbar^2\tau_1''^2}\right]\frac{x_0\tau_1''}{\dfrac{im}{\hbar}\left[\dfrac{1}{T}+\dfrac{1}{\tau''}\right]-\dfrac{1}{b^2}}\right]^2\left[\frac{im}{2\hbar}\left[\frac{1}{\tau_1''}+\frac{1}{\tau''}\right]+\frac{m^2/2\hbar^2\tau_1''^2}{\dfrac{im}{\hbar}\left[\dfrac{1}{T}+\dfrac{1}{\tau_1''}\right]-\dfrac{1}{b^2}}\right]^{-1}\right\},\tag{5.14}$$

where τ_1' and τ_1'' are time differences between the initial and final instants of the paths III′ and III″, i.e., $\tau_1'=t_1'''-t_1''$, $\tau_1''=t_2'''-t_2''$.

Supposing the particle beams III′ and III″ meet at the point D at the instant $t=T+\tau_0$. From the superposition principle of wave functions, the wave function at the point D and the instant $t=T+\tau_0$ is $\psi(\mathbf{r}_D)=\psi(\mathbf{r}_3'')+\psi(\mathbf{r}_4'')$, where $\mathbf{r}_D=\mathbf{r}_3''=\mathbf{r}_4''$. The phase difference between $\psi(\mathbf{r}_3'')$ and $\psi(\mathbf{r}_4'')$ at D is

$$i\Delta\phi_D=\frac{im}{2\hbar}\left[\frac{(y_1''-y_0)^2}{\tau_1'}-\frac{(y_2''-y_0)^2}{\tau_1''}\right]+\frac{im}{2\hbar}\left[\frac{(y_D-y_1'')^2}{\tau'}-\frac{(y_D-y_2'')^2}{\tau''}\right]$$
$$-\frac{1}{4}\left[\frac{im}{\hbar\tau'}x_D+\left[\frac{m^2}{T\hbar^2\tau_1'^2}\right]\frac{x_0\tau_1'}{\dfrac{im}{\hbar}\left[\dfrac{1}{T}+\dfrac{1}{\tau_1'}\right]-\dfrac{1}{b^2}}\right]^2\left[\frac{im}{2\hbar}\left[\frac{1}{\tau'}+\frac{1}{\tau_1'}\right]+\frac{m^2/2\hbar^2\tau_1'^2}{\dfrac{m}{\hbar}\left[\dfrac{i}{T}+\dfrac{i}{\tau_1'}\right]-\dfrac{1}{b^2}}\right]^{-1}$$
$$+\frac{1}{4}\left[\frac{im}{\hbar\tau''}x_D+\left[\frac{m^2}{T\hbar^2\tau_1''^2}\right]\frac{x_0\tau_1''}{\dfrac{im}{\hbar}\left[\dfrac{1}{T}+\dfrac{1}{\tau_1''}\right]-\dfrac{1}{b^2}}\right]^2\left[\frac{im}{2\hbar}\left[\frac{1}{\tau''}+\frac{1}{\tau_1''}\right]+\frac{m^2/2\hbar^2\tau_1''^2}{\dfrac{im}{\hbar}\left[\dfrac{1}{T}+\dfrac{1}{\tau_1''}\right]-\dfrac{1}{b^2}}\right]^{-1}.\tag{5.15}$$

Making the substitution $x_0\to x_0+\delta x_0$ in (5.15) and letting $\tau_1'+\tau'=\tau_1''+\tau''=\tau_0$, the coefficient of δx_0 is

$$\eta=imx_D\Big/T\hbar\left[\frac{\tau''}{\tau_1''}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\right]-imx_D\Big/T\hbar\left[\frac{\tau'}{\tau_1'}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\right]$$
$$+mx_0\tau''\Big/\hbar^2T^2\tau_1''\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\left[\frac{\tau''}{\tau_1''}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\right]$$
$$-mx_0\tau'\Big/\hbar^2T^2\tau_1'\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\left[\frac{\tau'}{\tau_1'}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\right]$$
$$=im\left[\frac{\tau'}{\tau_1'}-\frac{\tau''}{\tau_1''}+i\tau_0\frac{i}{\tau_1'}-i\tau_0\frac{i}{\tau_1''}\right]x_D\Big/T\hbar\left[\frac{\tau''}{\tau_1''}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\right]\left[\frac{\tau'}{\tau_1'}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\right]$$
$$+\frac{mx_0}{\hbar T^2}\left\{1\Big/\frac{\tau_1''}{\tau''}\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\left[\frac{\tau''}{\tau_1''}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1''}-\frac{\hbar^2}{mb^2}\right]\right]\right.$$
$$\left.-1\Big/\frac{\tau_1'}{\tau'}\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\left[\frac{\tau'}{\tau_1'}+i\tau_0\left[\frac{i}{T}+\frac{i}{\tau_1'}-\frac{\hbar^2}{mb^2}\right]\right]\right\}.\tag{5.16}$$

For brevity, let $x_0 \to 0$ and $T \to 0$, then the first term on the right-hand side of (5.16) is

$$\eta_1 = \lim_{\substack{T\to 0\\ x_0\to 0}} im \left[\frac{\tau'}{\tau_1'} - \frac{\tau''}{\tau_1''} - \frac{\tau'}{\tau_1'} + \frac{\tau''}{\tau_1''}\right] x_D \Big/ T\hbar \left[\frac{\tau''}{\tau_1''} + i\tau_0\left[\frac{i}{T} - \frac{i}{\tau_1''} - \frac{\hbar^2}{mb^2}\right]\right]\left[\frac{\tau'}{\tau_1'} + i\tau_0\left[\frac{i}{T} + \frac{i}{\tau_1'} - \frac{\hbar^2}{mb^2}\right]\right]$$

$$=0 \tag{5.17}$$

and the second term is

$$\eta_2 = \lim_{\substack{T\to 0\\ x_0\to 0}} \frac{mx_0}{\hbar}\left\{\frac{1}{\dfrac{\tau_1''}{\tau''}\left[iT + \dfrac{iT^2}{\tau_1''} - \dfrac{\hbar^2 T^2}{mb^2}\right]\left[\dfrac{\tau''}{\tau_1''} + i\tau_0\left[\dfrac{i}{T} + \dfrac{i}{\tau_1''} - \dfrac{\hbar^2}{mb^2}\right]\right]} - \frac{1}{\dfrac{\tau_1'}{\tau'}\left[iT + \dfrac{iT^2}{\tau_1'} - \dfrac{\hbar^2 T^2}{mb^2}\right]\left[\dfrac{\tau'}{\tau_1'} + i\tau_0\left[\dfrac{i}{T} + \dfrac{i}{\tau_1'} - \dfrac{\hbar^2}{mb^2}\right]\right]}\right\}$$

$$= \lim_{\substack{T\to 0\\ x_0\to 0}} \frac{mx_0}{\hbar}\left[\frac{1}{\dfrac{\tau_1''}{\tau''} iTi^2 \dfrac{\tau_0}{T}} - \frac{1}{\dfrac{\tau_1'}{\tau'}\dfrac{i^3\tau_0 T}{T}}\right] = 0 \tag{5.18}$$

so

$$\eta = \eta_1 + \eta_2 = 0 \; . \tag{5.19}$$

The phase difference between the wave functions of two particle beams is not affected by the variation of the initial point in the first-order approximation. Further calculation shows that the coefficient of $\delta x_0{}^2$ does not vanish. Therefore we have proved Theorems I and III by the method of path integrals of quantum mechanics.

In the semiclassical limit, Chiu and Stodolsky had proved Theorem I in Ref. 18, and we have proved Theorem III in Sec. II. We have extended the scope of application of Theorem I and Theorem III from the semiclassical limit into the quantum area in this section.

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