Parametrized-post-Newtonian limit of the scalar-tetradic theory A

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The scalar-tetradic theory A was recently formulated. Here, all the parametrized-post-Newtonian solar system predictions of this theory are proved to be identical to those of general relativity.

I. INTRODUCTION

Recently, we formulated[1] the so-called "scalar-tetradic theory A" which is a generalization of Møller's theory of gravitation.[2] The application of this theory to cosmology has given interesting results;[3] in particular, it has been proved that (i) the missing-matter problem does not hold in the theory A, and (ii) the universe can be closed, flat, or open for any density of energy; however, we have still not presented the parametrized-post-Newtonian (PPN) limit. Its study is the subject of this paper.

We will deal with both the computation of the PPN parameters of the theory A and the derivation of its PPN equations of motion; it requires a particular treatment due to the tetradic formulation of the theory (the expansions and the PPN coordinate conditions must be written in terms of the tetrad). Since we will obtain the same results as in general relativity (GR) the PPN predictions of the theory A agree with the present solar system observational evidences; this fact gives considerable support to the theory and motivates us to look for exact physical solutions in future works.

In Sec. II we give an introduction to theory A. In Sec. III we obtain the PPN limit of the theory and the conclusions are summarized in Sec. IV.

II. THE THEORY A

The gravitational field is described by a tetrad field $h_{\bar{\alpha}}$ and a scalar field ϕ. The partial derivative of a function Ψ with respect to the coordinate x^α is denoted by $\Psi_{,\alpha}$.

From the tetrad, we define the nonsymmetric connection $\Gamma^{\alpha}{}_{\beta\gamma}=h^{\alpha}_{\bar{\mu}}h_{\bar{\mu}\beta,\gamma}$, its torsion $S^{\alpha}{}_{\beta\gamma}$, and the vector $\Gamma_\beta=S^{\alpha}{}_{\beta\alpha}$. D_- and D_+ denote the covariant derivatives defined[1] from the connection Γ, and D expresses the covariant derivative derived from the Christoffel symbols of the metric $g_{\alpha\beta}=h_{\bar{\sigma}\alpha}h_{\bar{\sigma}\beta}$, which has a Lorentz signature because we assume an imaginary vector $h_{\bar{4}}$.

The field equations read

$$\phi(G^{\alpha}_{\beta}+M^{\alpha}_{\beta})-(W/\phi)(\phi^{,\alpha}\phi_{,\beta}-\delta^{\alpha}_{\beta}\phi_{,\gamma}\phi^{,\gamma})+\phi^{,\alpha}\Gamma_\beta$$
$$+\phi_{,\beta}\Gamma^{\alpha}-2(\phi_{,\gamma}\phi^{,\gamma})\delta^{\alpha}_{\beta}-\phi^{,\gamma}(S_{\beta}{}^{\alpha}{}_{\gamma}+S^{\alpha}{}_{\beta\gamma})$$
$$=(8\pi/c^4)T^{\alpha}_{\beta}\ , \quad (2.1)$$

$$\phi F_{\alpha\beta}+\lambda\phi_{,\gamma}(S_{\beta}{}^{\gamma}{}_{\alpha}-S_{\alpha}{}^{\gamma}{}_{\beta}-S^{\gamma}{}_{\beta\alpha})$$
$$+\tfrac{1}{2}(2\Gamma_\beta\phi_{,\alpha}-2\phi_{,\beta}\Gamma_\alpha+D_{+\alpha}\phi_{,\beta}-D_{+\beta}\phi_{,\alpha})=0\ , \quad (2.2)$$

$$2W\phi^{-1}D_{-\gamma}\phi^{,\gamma}-4D_{-\gamma}\Gamma^{\gamma}-4\phi^{-1}\Gamma_\gamma\phi^{,\gamma}=(8\pi\phi^{-1}/c^4)T^{\gamma}_{\gamma}\ , \quad (2.3)$$

where λ and W are dimensionless coupling constants. c is the speed of light. δ is the Kronecker unit matrix. All the greek indices take values from 1 to 4. The scalar ϕ has dimensions of G^{-1} (G being the gravitational constant). G^{α}_{β} is the Einstein tensor and T^{α}_{β} the energy-momentum tensor. M^{α}_{β} and F^{α}_{β} were defined in Ref. 1 and their explicit form is not necessary here. Finally we write $\phi^{,\alpha}=g^{\alpha\beta}\phi_{,\beta}$. The equations of motion

$$D_\alpha T^{\alpha\beta}=0 \quad (2.4)$$

are a consequence of the field equations.

III. THE PPN LIMIT OF THE THEORY A

The theory A is, from its variational formulation,[1] a conservative theory without preferred frames. In this section we will obtain the post-Newtonian limit of this new scalar-tetradic theory of graviation. Indeed, we will calculate both the ten PPN parameters and the PPN equations of motion; thus, we will search for the solar system predictions of the theory A.

The existence of the tetrad and the scalar field leads to some peculiarities in the techniques being necessary to find the post-Newtonian limit. Nevertheless, our analysis is similar to that of the metric theories of gravitation as GR.

In this section the greek indices take values from 1 to 4 and the latin ones are restricted to the values 1, 2, and 3. Furthermore ϵ^2 is the maximum value of the Newtonian potential in the solar system[4] and $R_\odot$ is the solar radius.

Since the field equations depend on the tetrad and its derivatives (not only on the metric) the initial expansions and PPN coordinate conditions must be written in terms of the tetrad. Therefore, we cannot assume the PPN coordinate conditions of GR *a priori.*

We write

$$h^{\alpha}_{\bar{\beta}}=\delta^{\alpha}_{\beta}+{}_1h^{\alpha}_{\bar{\beta}}+\cdots\ , \quad (3.1)$$

$$h_{\bar{\alpha}\beta}=\delta_{\alpha\beta}-{}_1h^{\beta}_{\bar{\alpha}}+\cdots\ , \quad (3.2)$$

$$\phi={}_0\phi+{}_1\phi+\cdots\ , \quad (3.3)$$

where

$$ {}_1h^{i}_{\bar j}=O(\epsilon^2),\quad {}_1h^{4}_{\bar 4}=O(\epsilon^2),\quad i\,{}_1h^{4}_{\bar k}=O(\epsilon^3)\ , $$
$$ i\,{}_1h^{k}_{\bar 4}=O(\epsilon^3),\quad {}_1\phi=O(\epsilon^2)\ . \tag{3.4} $$

Here we have assumed that the vector $h_{\bar 4}$ and the x^4 coordinate are imaginary. Furthermore, the PPN coordinate conditions are assumed to be

$$ {}_1h^{i}_{\bar j,j}=\tfrac{1}{2}\,{}_1h^{\sigma}_{\bar\sigma,i}+O(\epsilon^4/R_\odot)\ , \tag{3.5} $$

$$ {}_1h^{4}_{\bar j,j}=\tfrac{1}{2}\,{}_1h^{k}_{\bar k,4}+O(\epsilon^5/R_\odot)\ . \tag{3.6} $$

After these general considerations, we will study the field equations of the theory A. Indeed, we will begin with Eqs. (2.2), which are not similar to any equation of GR.

By using the definitions of $\Gamma^{\alpha}{}_{\beta\gamma}$, $S^{\alpha}{}_{\beta\gamma}$, and Γ_α (without considering the coordinate conditions), Eqs. (2.2) can be written as

$$ ({}_1h^{j}_{\bar i,k}-{}_1h^{j}_{\bar k,i}+{}_1h^{k}_{\bar j,i}-{}_1h^{k}_{\bar i,j}-{}_1h^{i}_{\bar j,k}+{}_1h^{i}_{\bar k,j})_{,j} =0+O(\epsilon^4/R_\odot{}^2)\ , \tag{3.7} $$

$$ ({}_1h^{j}_{\bar 4,k}-{}_1h^{j}_{\bar k,4}+{}_1h^{k}_{\bar j,4}-{}_1h^{k}_{\bar 4,j}-{}_1h^{4}_{\bar j,k}+{}_1h^{4}_{\bar k,j})_{,j} =0+O(\epsilon^5/R_\odot{}^2)\ . \tag{3.8} $$

If we define the skew-symmetric quantities $\zeta^{\alpha}_{\bar\beta}$ by

$$ \zeta^{\alpha}_{\bar\beta}={}_1h^{\alpha}_{\bar\beta}-{}_1h^{\beta}_{\bar\alpha} \tag{3.9} $$

Equations (3.7) and (3.8) become

$$ (\zeta^{j}_{\bar i,k}+\zeta^{k}_{\bar j,i})_{,j}+(\zeta^{k}_{\bar i,j})_{,j}=0+O(\epsilon^4/R_\odot{}^2)\ , \tag{3.10} $$

$$ (\zeta^{j}_{\bar 4,k}+\zeta^{k}_{\bar j,4})_{,j}+(\zeta^{k}_{\bar 4,j})_{,j}=0+O(\epsilon^5/R_\odot{}^2)\ , \tag{3.11} $$

and by carrying out a derivative of these last equations with respect to the x^k coordinate we obtain

$$ (\zeta^{k}_{\bar i,k})_{,j,j}=0+O(\epsilon^4/R_\odot{}^3)\ , \tag{3.12} $$

$$ (\zeta^{k}_{\bar 4,k})_{,j,j}=0+O(\epsilon^5/R_\odot{}^3)\ . \tag{3.13} $$

If we assume the natural boundary conditions

$$ {}_1h^{\alpha}_{\bar\beta}=0,\quad {}_1h^{\alpha}_{\bar\beta,\gamma}=0 \tag{3.14} $$

the unique solution of Eqs. (3.12) and (3.13) being continuous at any point of the space is

$$ \zeta^{k}_{\bar i,k}=-\zeta^{i}_{\bar k,k}=0+O(\epsilon^4/R_\odot)\ , \tag{3.15} $$

$$ \zeta^{k}_{\bar 4,k}=-\zeta^{4}_{\bar k,k}=0+O(\epsilon^5/R_\odot)\ . \tag{3.16} $$

From Eqs. (3.10), (3.11), (3.15), and (3.16) we find

$$ (\zeta^{k}_{\bar i})_{,j,j}=0+O(\epsilon^4/R_\odot{}^2)\ , \tag{3.17} $$

$$ (\zeta^{k}_{\bar 4})_{,j,j}=0+O(\epsilon^5/R_\odot{}^2)\ , \tag{3.18} $$

and taking into account the above boundary conditions, the unique continuous solution of Eqs. (3.17) and (3.18) is

$$ \zeta^{k}_{\bar i}=0+O(\epsilon^4)\ , \tag{3.19} $$

$$ \zeta^{k}_{\bar 4}=0+O(\epsilon^5)\ , \tag{3.20} $$

hence, by virtue of Eqs. (3.4) we can write

$$ {}_1h^{\alpha}_{\bar\beta}={}_1h^{\beta}_{\bar\alpha} \tag{3.21} $$

and

$$ g_{\alpha\beta}=\delta_{\alpha\beta}-2\,{}_1h^{\alpha}_{\bar\beta}+\cdots\ . \tag{3.22} $$

It is worth pointing out that the solution (3.21) has been derived taking into account Eqs. (3.4) and the boundary conditions (3.14); nevertheless, no coordinate conditions have been used. Therefore, we can assume four arbitrary coordinate conditions to solve the remaining field equations if they are compatible with the boundary conditions. In particular, the conditions (3.5) and (3.6) are admissible.

Furthermore, taking into account Eqs. (3.21) and (3.22) [which are a direct consequence of the field equations (2.2)], we see that the coordinate conditions (3.5) and (3.6) and the standard PPN coordinate conditions of GR (Ref. 4) are identical. Therefore, we can conclude that the PPN coordinate conditions of GR are satisfied in the theory A. This fact will be important later on.

On the other hand, in order to study the field equations (2.1) and (2.3), we are going to write them in a suitable manner.

By contracting Eqs. (2.1) with the metric tensor we get

$$ \phi(G^{\alpha}_{\alpha}+M^{\alpha}_{\alpha})+W\phi^{-1}(\phi_{,\alpha}\phi^{,\alpha})-4\Gamma_\alpha\phi^{,\alpha}=(8\pi/c^4)T^{\alpha}_{\alpha} \tag{3.23} $$

and by using Eqs. (3.23), the field equations (2.1) and (2.3) can be written as follows:

$$ R_{\alpha\beta}-(8\pi\phi^{-1}/c^4)(T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T^{\gamma}_{\gamma})=-(M_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}M^{\gamma}_{\gamma})+W\phi^{-2}\phi_{,\alpha}\phi_{,\beta}-\phi^{-1}\phi_{,\alpha}\Gamma_\beta-\phi^{-1}\phi_{,\beta}\Gamma_\alpha +\phi^{-1}\phi^{,\gamma}(S_{\beta\alpha\gamma}+S_{\alpha\beta\gamma})\ , \tag{3.24} $$

$$ -(G^{\alpha}_{\alpha}+M^{\alpha}_{\alpha})+2W\phi^{-1}D_{-\alpha}\phi^{,\alpha}-4D_{-\alpha}\Gamma^{\alpha}-W\phi^{-2}\phi_{,\alpha}\phi^{,\alpha}=0\ . \tag{3.25} $$

Now, we are going to study the field equation (3.25), which has no counterpart in GR. Since. Eq. (3.25) involves the tensor $M^{\alpha\beta}$ we must study its expansion. This tensor was defined in Ref. 1, and as a direct consequence of its definition the order of all the components $M^{\alpha\beta}$ is at least $\epsilon^4/R_\odot{}^2$. Furthermore, one easily finds

$$ M_{ir}=\tfrac{1}{2}({}_1h^{k}_{\bar i,j}-{}_1h^{k}_{\bar j,i})({}_1h^{k}_{\bar r,j}-{}_1h^{k}_{\bar j,r})-\tfrac{1}{2}({}_1h^{k}_{\bar i,j}-{}_1h^{k}_{\bar j,i})({}_1h^{j}_{\bar r,k}-{}_1h^{j}_{\bar k,r})-\tfrac{1}{4}({}_1h^{r}_{\bar k,j}-{}_1h^{r}_{\bar j,k})({}_1h^{i}_{\bar k,j}-{}_1h^{i}_{\bar j,k}) +\delta_{ir}[({}_1h^{j}_{\bar p,s}-{}_1h^{j}_{\bar s,p})({}_1h^{s}_{\bar p,j}-{}_1h^{s}_{\bar j,p})-\tfrac{1}{2}({}_1h^{j}_{\bar p,s}-{}_1h^{j}_{\bar s,p})({}_1h^{j}_{\bar p,s}-{}_1h^{j}_{\bar s,p})]+O(\epsilon^6/R_\odot{}^2)\ , \tag{3.26} $$

$$ M_{i4}=\tfrac{1}{2}({}_1h^{k}_{\bar i,j}-{}_1h^{k}_{\bar j,i})({}_1h^{k}_{\bar r,j}-{}_1h^{k}_{\bar j,r})-\tfrac{1}{2}({}_1h^{k}_{\bar i,j}-{}_1h^{k}_{\bar j,i})({}_1h^{j}_{\bar 4,k}-{}_1h^{j}_{\bar k,4})-\tfrac{1}{4}({}_1h^{4}_{\bar k,j}-{}_1h^{4}_{\bar j,k})({}_1h^{i}_{\bar k,j}-{}_1h^{i}_{\bar j,k})+O(\epsilon^7/R_\odot{}^2) \tag{3.27} $$

and by using Eqs. (3.21) we see that all the terms contained in formulas (3.26) and (3.27) cancel among them. Therefore we can write

$$M_{ir}=M_{ri}=0+O(\epsilon^6/R_\odot^2)\ , \tag{3.28}$$

$$M_{i4}=M_{4i}=0+O(\epsilon^7/R_\odot^2)\ . \tag{3.29}$$

Finally M_{44} has the form exhibited in Eq. (3.26) if we set $i=r=4$. Hence

$$M_{44}=0+O(\epsilon^6/R_\odot^2)\ . \tag{3.30}$$

Furthermore, taking into account Eqs. (3.28), (3.29), (3.30), and the PPN coordinate conditions we get

$$-(G^\alpha_\alpha+M^\alpha_\alpha)=({}_1h^\alpha_{\overline{\alpha}})_{,i,i}+O(\epsilon^4/R_\odot^2)\ , \tag{3.31}$$

$$4D_{-\alpha}\Gamma^\alpha=({}_1h^\alpha_{\overline{\alpha}})_{,i,i}+O(\epsilon^4/R_\odot^2)\ , \tag{3.32}$$

$$D_{-\alpha}\phi^{,\alpha}=({}_1\phi)_{,i,i}+O(\epsilon^4/R_\odot^2)\ . \tag{3.33}$$

We have already obtained all the information necessary to study Eq. (3.25). In fact, on account of Eqs. (3.28)–(3.33), this equation reduces to

$$({}_1\phi)_{,i,i}=0+O(\epsilon^4/R_\odot^2)\ . \tag{3.34}$$

The last equation holds outside and inside of the perfect fluid studied by means of the PPN formalism, and consequently, any admissible solution of this equation must be continuous at any point of the space. This fact, together with the natural boundary condition ${}_1\phi=0$, finds the following unique solution:

$${}_1\phi=0+O(\epsilon^4) \tag{3.35}$$

which will be assumed in order to compute the ten PPN parameters of theory A. However, from a mathematical point of view, Eq. (3.34) also has the solution

$$\begin{gathered}{}_1\phi=A+(B/r)+O(\epsilon^4)\ ,\\ r^2=x^1x^1+x^2x^2+x^3x^3\ ,\end{gathered} \tag{3.36}$$

A and B being arbitrary constants. This last solution must be rejected in the PPN analysis because it is not continuous at $r=0$. However, it is helpful to point out its existence to prevent future apparent contradictions (see Sec. IV). From Eq. (3.35) we conclude that the order of the first nonvanishing correction to ${}_0\phi$ [see Eq. (3.3)] is at least ϵ^4.

So far our analysis is characteristic of the scalar-tetradic theory A, while the comparisons with GR will be very profitable henceforth. Now we will study Eqs. (3.24), which generalize Einstein's equations in a evident manner; thus we will obtain the ten PPN parameters determining the PPN metric of the theory A.

From Eqs. (3.28)–(3.30) and (3.35) the following "proposition A" is easily proved: "All the terms of the right-hand side of Eqs. (3.24) are at least of order $\epsilon^6/R_\odot^2$." This fact will be basic in order to determine the PPN parameters of the theory A.

The proposition A also states that Eqs. (3.24) and Einstein's field equations become identical up to order $\epsilon^4/R_\odot^2$. This proposition is based on the choice of the solution (3.35) of Eq. (3.34), and it is not satisfied if we choose the solution (3.36) in order to do a formal calculation.

Since we know that (i) the standard PPN coordinate conditions of GR hold in theory A (it has been already proved) and (ii) the energy-momentum tensor is defined as in GR, we easily find the following.

(a) In the linearized order (Newtonian energy-momentum tensor), the terms of orders $\epsilon^4/R_\odot^2$ and higher are neglected, and as an obvious consequence of the proposition A, Eqs. (3.24) and Einstein's field equations appear to be identical and lead to $\gamma=\Delta_1=\Delta_2=1$.

(b) In order to obtain the remaining PPN parameters, we must particularize Eqs. (3.24) for $\alpha=\beta=4$, retaining the terms of order $\epsilon^4/R_\odot^2$ and neglecting the terms of order $\epsilon^6/R_\odot^2$, i.e., all the right-hand side of the equation in accordance with the proposition A. Furthermore, in this calculation we must use the post-Newtonian correction to the energy-momentum tensor [see Eqs. (39) and (42) of Ref. 4] with $\gamma=1$, which has the same form as the corresponding correction used in GR. After these considerations, it appears to be evident that Eqs. (3.24) become identical to the Einstein equations for $\alpha=\beta=4$ to the order of approximation considered; hence, both give the same PPN parameters $\beta=\beta_1=\beta_2=\beta_3=\beta_4=1$ and $\xi=\eta=0$ (see Ref. 4, pp. 1089–1090). We can therefore conclude that the PPN parameters of the theory A are identical to those of GR.

In order to obtain the PPN equations of motion we must use the standard equations of motion (2.4), the post-Newtonian energy-momentum tensor with $\gamma=1$, and the Christoffel symbols of the PPN metric depending on the ten PPN parameters (see Ref. 4). Since all these elements are the same as in GR, we can state that the PPN equations of motion of theory A and its PPN solar system predictions are also identical to those of GR. No explicit calculations are necessary because they would reduce to those of GR.

IV. CONCLUSIONS AND COMMENTS

The ten PPN parameters, the PPN equations of motion, and consequently all the solar system predictions of the theory A have been proved to be identical to those of GR. It is helpful to point out the possible existence of exact static spherically symmetric solutions of the field equations of the theory A having $\beta\neq1$; it is due to the existence of the solution (3.36) of Eq. (3.34), from which we could calculate ten fictitious PPN parameters (they do not describe the asymptotic behavior of the solar field) which would be different from the true parameters. In particular, we would find $\beta\neq1$. All the possible solutions having $\beta\neq1$ would be consistent with Eqs. (3.36) but not with Eq. (3.35).

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