Fermion-monopole system reexamined. III

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A detailed derivation is presented for the fermion-monopole Hamiltonian, paying particular attention to the physical significance of various surface terms. The version of the s-wave Hamiltonian given by Callan is found to require amendment for the case of massive fermions. We also note that the dyon (rotor) degree of freedom cannot be interpreted as a localized charge, but should rather be regarded as a charge on a capacitor.

I. INTRODUCTION

Interest in the monopole-fermion system has been greatly revived since the suggestion of Rubakov[1] and Callan[2] that monopoles may catalyze proton decay at strong-interaction rates.[3] In the proposal, the fermions in the $j=0$ partial wave were found to play a crucial role; in particular, Callan has derived a reduced Hamiltonian which describes the dynamics of this sector.

We have found, however, that the Hamiltonian given in Ref. 2 actually requires amendment if the fermions are massive. It turns out that surface terms must be carefully treated to obtain the correct Hamiltonian in this case, because the vector potential for a dyon has components which do not vanish at spatial infinity. Since the point is relevant to a discrepancy noted for the case of a Higgs mass between Refs. 2 and 4 on the one hand and Refs. 5 and 6 on the other, we have decided to present the derivation in some detail, although the general procedure for treating the dyon (rotor) degree of freedom has already been given some time ago.[7,8]

The paper is organized as follows. In Sec. II we give a derivation based on the field energy. We also note that the dyon degree of freedom should be regarded as a charge on a capacitor, rather than a localized charge. In Sec. III we rederive the results by a canonical procedure. Quantum corrections are briefly discussed, although not treated in full. In Sec. IV we confirm our results in the bosonic language.[1–4] The appendix is devoted to a comparison of our results with the existing literature. The notation is essentially that of our previous paper (II).[5] We received Refs. 6 and 9–11 which deal with related matters after first submittal of this paper.

II. FIELD ENERGY DERIVATION

Let us choose a gauge in which the spatial components are static and have the form[12,13]

$$e\,\mathbf{A}(\mathbf{x},t)=(\hat{\mathbf{x}}\times\tau)\frac{K(r)-1}{2r}\ , \tag{2.1}$$

where $K(r)$ is equal to[14] $1+rA(r)$ in Ref. 2 and behaves as

$$K(r)=1+O(r^2)\ \ (r\to 0)\ ,\qquad K(r)=O(e^{-m_W r})\ \ (r\to\infty)\ . \tag{2.2}$$

The time component then has the form

$$eA_0(\mathbf{x},t)=(\hat{\mathbf{x}}\cdot\tau)\frac{J(r)}{2r}\ , \tag{2.3}$$

$$\frac{J(r)}{r}=O(r)\ \ (r\to 0)\ ,\qquad \frac{J(r)}{r}\sim const\ \ (r\to\infty) \tag{2.4}$$

as in the Julia-Zee solution.[12] As noted by various authors,[9,10,15] the existence of an electrostatic gradient between $r=0$ and $r=\infty$ is of considerable physical importance.

For a path along the radial direction

$$P\exp\left[ie\int d\mathbf{x}\cdot\mathbf{A}(\mathbf{x})\right]=1 \tag{2.5}$$

so the fermion field ψ in this gauge is the gauge-invariant fermion field. (It may sound strange that a noninvariant object ψ in a particular gauge is gauge invariant. However, it is the same as saying that the energy in the c.m. frame is equal to the invariant $\sqrt{s}$. See also the appendix.)

From (2.1) and (2.3), we may find the field strengths

$$E_i^a(\mathbf{x})=-(\delta_{ia}-\hat{\mathbf{x}}_i\hat{\mathbf{x}}_a)\frac{J(r)K(r)}{er^2}-\hat{\mathbf{x}}_i\hat{\mathbf{x}}_a\frac{d}{dr}\frac{J(r)}{er}\ , \tag{2.6}$$

$$B_i^a(\mathbf{x})=-(\delta_{ia}-\hat{\mathbf{x}}_i\hat{\mathbf{x}}_a)\frac{1}{er}\frac{dK(r)}{dr}-\hat{\mathbf{x}}_i\hat{\mathbf{x}}_a\frac{K^2(r)-1}{er^2}\ . \tag{2.7}$$

Since we are interested in the interaction of the fermions with the electric field of the dyon, $J(r)$ must be taken to be a dynamical variable. In the gauge in which we are working, this simply amounts to replacing (2.4) by[7,8]

$$\frac{J(r)}{r}\sim-\dot{\varphi}+\frac{e^2}{4\pi r}Q_{\text{tot}}\ \ (r\to\infty)\ , \tag{2.8}$$

where φ is the dyon degree of freedom, and Q_{tot} is the total charge of the system in units of e (the fermions have charge $\pm e/2$). On the other hand, if we ignore the back reaction of the electric field on the magnetic field,[16] $K(r)$ may be taken to be a fixed c number. Similarly, we also ignore the Higgs fields.

Then, up to a c number, the field energy is given by

$$\mathscr{E}=\tfrac{1}{2}\int d\mathbf{x}\sum_{i,a}(E_i^a)^2 = \int d\mathbf{x}\left[\frac{J(r)K(r)}{er^2}\right]^2+\tfrac{1}{2}\int d\mathbf{x}\left[\nabla\frac{J(r)}{er}\right]^2 . \quad (2.9)$$

Partial integration and Gauss's law

$$r^2J''(r)-2J(r)K^2(r)=-\frac{e^2}{4\pi}r\rho(r) \quad (2.10)$$

lead to

$$\mathscr{E}=\tfrac{1}{2}\dot{\varphi}Q_{\text{tot}}+\tfrac{1}{2}\int_0^\infty \frac{dr}{r}J(r)\rho(r) , \quad (2.11)$$

where $\rho(r)$ is the (one-dimensional) charge density of the fermions including the vacuum contribution.

To solve for $J(r)$ in terms of $\rho(r)$ and φ, we introduce the solutions $\mathscr{J}(r)$ and $\overline{\mathscr{J}}(r)$ of the homogeneous equation

$$r\frac{d^2}{dr^2}r\mathscr{J}(r)-2\mathscr{J}(r)K^2(r)=0 \quad (2.12)$$

which have the behavior

$$\mathscr{J}(r)=O(r)\quad (r\to 0) , \qquad \mathscr{J}(r)\sim 1-\frac{Ie^2}{4\pi r}\quad (r\to\infty) , \quad (2.13)$$

$$\overline{\mathscr{J}}(r)=O\left[\frac{1}{r^2}\right]\quad (r\to 0) , \qquad \overline{\mathscr{J}}(r)\sim\frac{1}{r}\quad (r\to\infty) , \quad (2.14)$$

with I a positive constant of order r_0/e^2. In the Prasad-Sommerfield limit[13]

$$\mathscr{J}(r)=\coth m_W r-\frac{1}{m_W r},\quad \overline{\mathscr{J}}(r)=\frac{1}{r}\coth m_W r . \quad (2.15)$$

The solutions $\mathscr{J},\overline{\mathscr{J}}$ are related by the Wronskian

$$\mathscr{J}(r)\overline{\mathscr{J}}\,'(r)-\overline{\mathscr{J}}(r)\mathscr{J}'(r)=-\frac{1}{r^2} \quad (2.16)$$

which integrates to

$$\overline{\mathscr{J}}(r)=\mathscr{J}(r)\int_r^\infty \frac{ds}{s^2}\frac{1}{\mathscr{J}^2(s)} . \quad (2.17)$$

Similarly, we find from Gauss's law (2.10) that

$$Q_{\text{tot}}-I\dot{\varphi}=\int_0^\infty dr\,\mathscr{J}(r)\rho(r) . \quad (2.18)$$

Identifying $I\dot{\varphi}$ as the charge associated with the dyon degree of freedom, we see that (2.18) is just the law of charge conservation.

We may now construct the Green's function

$$\mathscr{G}(r,r')=\mathscr{J}(r)\theta(r'-r)\overline{\mathscr{J}}(r')+\overline{\mathscr{J}}(r)\theta(r-r')\mathscr{J}(r') \quad (2.19)$$

and solve Gauss's law (2.10) as

$$\frac{J(r)}{r}=-\mathscr{J}(r)\dot{\varphi}+\frac{e^2}{4\pi}\int_0^\infty dr'\mathscr{G}(r,r')\rho(r') . \quad (2.20)$$

Substitution into (2.11) gives

$$\mathscr{E}=\mathscr{H}_C+\mathscr{H}'_C , \quad (2.21)$$

$$\mathscr{H}_C=\frac{I}{2}\dot{\varphi}^2=\frac{1}{2I}\left[\frac{n}{2}-\frac{\bar{\theta}}{2\pi}-Q_{\mathscr{J}}\right]^2 , \quad (2.22)$$

$$\mathscr{H}'_C=\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r') = \frac{e^2}{8\pi}\int_0^\infty\frac{dr}{r^2}\left[\frac{Q_{\mathscr{J}}(r)}{\mathscr{J}(r)}\right]^2 , \quad (2.23)$$

where we have introduced

$$Q_{\mathscr{J}}(r)=\int_0^r dr'\mathscr{J}(r')\rho(r'),\quad Q_{\mathscr{J}}=Q_{\mathscr{J}}(\infty) \quad (2.24)$$

and set

$$Q_{\text{tot}}=\frac{n}{2}-\frac{\bar{\theta}}{2\pi} \quad (2.25)$$

valid for massive fermions.[17]

As a check on our results, we note that $\mathscr{H}'_C=0$ in the absence of fermions $\rho(r)=0$, and

$$Q_{\text{tot}}=I\dot{\varphi} , \quad (2.26)$$

$$\mathscr{H}_C=\frac{1}{2I}Q_{\text{tot}}{}^2 \quad (2.27)$$

as expected.[7,8] We may also mention that dropping $\mathscr{H}'_C$ for $\rho(r)\neq 0$ would correspond to the approximation

$$eA_0(\mathbf{x},t)\simeq-\tfrac{1}{2}(\hat{\mathbf{x}}\cdot\boldsymbol{\tau})\mathscr{J}(r)\dot{\varphi} \quad (2.28)$$

used by Besson,[18] the author,[5] and Polchinski.[10] As in ordinary electrodynamics, the total Hamiltonian $\mathscr{H}$ is the sum of the electrostatic energy (2.21) and the kinetic energy of the fermions[19] in the magnetic part of the field

$$\mathscr{H}_F=\int_0^\infty dr\,\chi^\dagger(r)H\chi(r) , \quad (2.29)$$

$$H=-i\gamma_5\tau_3\frac{d}{dr}+\beta M-\gamma_5\tau_2\frac{K(r)}{r} . \quad (2.30)$$

The physical significance of $\mathscr{H}_C$ may be understood as follows. From (2.8) and (2.18) we find that a transfer of charge to the fermions will be accompanied by a lowering of the potential difference between $r=0$ and $r=\infty$. Therefore, the constant I may be interpreted as the capacitance of the dyon,[20] and $\mathscr{H}_C$ is just what we would expect for the energy of a capacitor.

On the other hand, $\mathscr{H}'_C$ is easily seen to be the self-energy of the fermions. The fact that there is no cross term between $I\dot{\varphi}$ and the fermion charge density $\rho(r)$ also indicates that it is quite wrong to think of $I\dot{\varphi}$ as a localized charge. This point is discussed in more detail in Ref. 21; here, we shall only remark that although the situation may seem surprising at first, it is actually quite natural for a gauge theory.

For the Julia-Zee solution, one finds that the Higgs field does not contribute to Gauss's law (2.10). Therefore, as far as the electric charge is concerned, it arises solely from the self-interaction of the gauge fields, and cannot be associated with a unique charge distribution. The situation is quite similar to the case of gravity, where the energy-momentum pseudotensor t^{μ}_{ν} of the gravitational field is also nonlocalizable. In fact, the analogy goes further,[22] since the total energy of a gravitational system

$$\int d^3x(t^0_0+\Theta^0_0) \tag{2.31}$$

does not contain any cross term between t^0_0 and the matter energy density Θ^0_0. In particular, the Hamiltonians (2.21) and (2.29) correctly account for the interaction of the dyon degree of freedom with the fermions, as shown in the next section.

III. CANONICAL DERIVATION

For a canonical treatment, let us first work in the gauge of the preceding section. The gauge field part of the Lagrangian is given by

$$\begin{aligned}\tfrac{1}{2}\int d\mathbf{x}\sum_{i,a}[(E^a_i)^2-(B^a_i)^2]+\frac{e^2\theta}{8\pi^2}\int d\mathbf{x}\sum_{i,a}E^a_iB^a_i\\ =\int d\mathbf{x}\left[\frac{J(r)K(r)}{er^2}\right]^2\\ +\tfrac{1}{2}\int d\mathbf{x}\left[\nabla\frac{J(r)}{er}\right]^2-\frac{\theta}{2\pi}\lim_{R\to\infty}\frac{J(R)}{R}\,,\end{aligned} \tag{3.1}$$

where we have again ignored a c number. For the fermion part of the Lagrangian we retain only the lowest partial wave $\chi(r)$ to obtain

$$\int_0^\infty dr\,\chi^\dagger\left[i\frac{\partial}{\partial t}-H-\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi\,, \tag{3.2}$$

where H is defined as in (2.30).

Variation of the action with respect to J (which is essentially A_0) leads to Gauss's law (2.10) with

$$\rho=\chi^\dagger\frac{\tau_3}{2}\chi\,. \tag{3.3}$$

Similarly, variation with respect to $\chi^\dagger$ gives

$$i\frac{\partial}{\partial t}\chi=\left[H+\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi\,. \tag{3.4}$$

However, there is a possible ambiguity with the equations of motion (2.10) and (3.4) since one may add a solution $\mathscr{J}(r)$ of the homogeneous equation (2.12) to Gauss's law (2.10). To resolve the ambiguity, the natural procedure is to consider the surface term left after partial integration

$$\lim_{R\to\infty}\left[\frac{4\pi R^2}{e^2}\frac{d}{dR}\frac{J(R)}{R}-\frac{\theta}{2\pi}\right]\frac{\delta J(R)}{R}\,. \tag{3.5}$$

If the term is to vanish as required by the action principle, we must have either

$$\lim_{R\to\infty}\frac{\delta J(R)}{R}=0 \tag{3.6}$$

or

$$\lim_{R\to\infty}\frac{4\pi R^2}{e^2}\frac{d}{dR}\frac{J(R)}{R}=\frac{\theta}{2\pi}\,. \tag{3.7}$$

The first boundary condition (BC) would be the one chosen by Ref. 23, according to their general rule that fields are not to be varied at the boundaries (p. 1187 of their second paper). Unfortunately, however, it is incompatible with the natural dynamical evolution of the system, since if we start from the Julia-Zee solution at $t=0$, the emission of light charged fermions[9,10,15] implies that the difference in the electrostatic potential between $r=0$ and $r=\infty$ must decrease at later times. In other words, one must make sure that the physical situation is that of fixed charge, not fixed potential.

This leaves us with the other possibility (3.7), in which we are restricted to the sector with total charge

$$Q_{\rm tot}=-\lim_{R\to\infty}\frac{4\pi R^2}{e}\frac{d}{dR}\frac{J(R)}{R}=-\frac{\theta}{2\pi}\,. \tag{3.8}$$

Taking (2.13) into account, we may now uniquely solve Gauss's law (2.10) as

$$\begin{aligned}\frac{J(r)}{r}=\frac{\mathscr{J}(r)}{I}\left[\frac{\theta}{2\pi}+Q_{\mathscr{F}}\right]\\ +\frac{e^2}{4\pi}\int_0^\infty dr'\mathscr{G}(r,r')\rho(r')\,,\end{aligned} \tag{3.9}$$

where $Q_{\mathscr{F}}$ and $\mathscr{G}(r,r')$ are defined as in (2.24) and (2.19). Since the original Lagrangian does not contain $\dot{J}$, we may substitute (3.9) back to obtain the reduced Lagrangian

$$\begin{aligned}\int_0^\infty dr\,\chi^\dagger\left[i\frac{\partial}{\partial t}-H\right]\chi-\frac{1}{2I}\left[\frac{\theta}{2\pi}+Q_{\mathscr{F}}\right]^2\\ -\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r')\,.\end{aligned} \tag{3.10}$$

From (3.10), we may trivially recover the Hamiltonians (2.21) and (2.29) given previously.

However, the treatment above, although straightforward, does not shed light on the possible properties of $Q_{\rm tot}$ as a dynamical variable. To treat all charge sectors at once, it is necessary to modify the action itself by adding the surface term[8]

$$\int dt\,q\left[\lim_{R\to\infty}\frac{J(R)}{R}+\dot{\varphi}\right], \tag{3.11}$$

where φ is the dyon degree of freedom as before. Variation with respect to the Lagrange multiplier q leads to

$$\lim_{R\to\infty}\frac{J(R)}{R}=-\dot{\varphi} \tag{3.12}$$

and we recover (2.18) and (2.20).

Substituting (2.20) back again, we obtain the reduced Lagrangian

$$\mathscr{L}=\frac{I}{2}\dot{\varphi}^2+\frac{\theta}{2\pi}\dot{\varphi}+\int_0^\infty dr\,\chi^\dagger\left[i\frac{\partial}{\partial t}-H+\mathscr{J}(r)\dot{\varphi}\frac{\tau_3}{2}\right]\chi$$
$$-\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r')\ . \tag{3.13}$$

Canonical quantization is now standard. The conjugate momentum of φ is given by

$$p_\varphi\equiv\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}=I\dot{\varphi}+\frac{\theta}{2\pi}+\int_0^\infty dr\,\mathscr{J}(r)\rho(r)\ . \tag{3.14}$$

Comparison with (2.18) gives

$$p_\varphi=Q_{\rm tot}+\frac{\theta}{2\pi} \tag{3.15}$$

so it is just the total charge up to a constant. Since φ is a cyclic variable ($\partial\mathscr{L}/\partial\varphi=0$), we find that p_φ and hence $Q_{\rm tot}$ is a constant of motion. This is to be expected,[8] since the equations of motion which follow from the surface terms (3.5) and (3.11) are (3.12) and

$$-Q_{\rm tot}-\frac{\theta}{2\pi}+q=0,\ \ \dot{q}=0\ . \tag{3.16}$$

The Hamiltonian is given by

$$\mathscr{H}=p_\varphi\dot{\varphi}+\int_0^\infty dr\,\chi^\dagger i\frac{\partial}{\partial t}\chi-\mathscr{L}$$
$$=\int_0^\infty dr\,\chi^\dagger H\chi+\frac{1}{2I}\left[p_\varphi-\frac{\theta}{2\pi}-Q_{\mathscr{J}}\right]^2$$
$$+\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r')\ . \tag{3.17}$$

Since p_φ is a constant of motion, it may be taken to be a c number, and we recover our previous results.

We emphasize that a correct treatment of the dyon degree of freedom automatically leads to charge conservation, independently of any detailed dynamical calculation. Indeed if total charge were not conserved (or spontaneously broken as some authors have suggested), there would be no electromagnetism, and hence no monopoles in the first place. We also note that an effective BC as in Ref. 23 on the gauge-invariant fermion field χ

$$\chi^+(0)=-e^{-ig\varphi}\chi^-(0) \tag{3.18}$$

does not commute with p_φ and is incompatible with charge conservation. (See the appendix for a more detailed discussion of this point.)

It is also instructive to consider a gauge in which A_0 still has the form

$$eA_0(\mathbf{x},t)=(\hat{\mathbf{x}}\cdot\boldsymbol{\tau})\frac{\hat{J}(r)}{2r} \tag{3.19}$$

but $\hat{J}(r)$ now obeys the BC

$$\lim_{r\to 0}\frac{\hat{J}(r)}{r}=\lim_{r\to\infty}\frac{\hat{J}(r)}{r}=0\ . \tag{3.20}$$

Inspection of the Julia-Zee solution under this condition indicates that one must allow the spatial components to rotate in charge space

$$e\,\mathbf{A}(\mathbf{x},t)=G(\hat{\mathbf{x}}\times\boldsymbol{\tau})\frac{K(r)-1}{2r}G^\dagger-iG\nabla G^\dagger\ , \tag{3.21}$$

$$G=\exp\left[\frac{i}{2}\hat{\mathbf{x}}\cdot\boldsymbol{\tau}\mathscr{J}(r)\varphi\right]\ , \tag{3.22}$$

where the nature of φ as a collective coordinate is now evident. In particular, we note that

$$\varphi\to\varphi+4\pi,\ \ \hat{\chi}\to e^{-2\pi i\tau_3\mathscr{J}(r)}\hat{\chi} \tag{3.23}$$

corresponds to a "large" (i.e., proper but topologically nontrivial) gauge transformation.[24]

The gauge field part of the Lagrangian is the same as before, provided we replace $J(r)$ in (3.1) by $-r\mathscr{J}(r)\dot{\varphi}+\hat{J}(r)$. On the other hand, the fermionic piece of the Lagrangian reads

$$\int_0^\infty dr\,\hat{\chi}^\dagger\left[i\frac{\partial}{\partial t}-\hat{H}-\mathscr{J}'(r)\varphi\frac{\gamma_5}{2}-\frac{\hat{J}(r)}{r}\frac{\tau_3}{2}\right]\hat{\chi}\ , \tag{3.24}$$

$$\hat{H}=-i\gamma_5\tau_3\frac{d}{dr}+\beta M-\gamma_5\tau_2 e^{i\mathscr{J}(r)\varphi\tau_3}\frac{K(r)}{r}\ . \tag{3.25}$$

Variation with respect to $\hat{J}(r)$ again leads to Gauss's law

$$r^2\hat{J}''-2\hat{J}K^2=-\frac{e^2}{4\pi}r\rho \tag{3.26}$$

this time without any worries at $r=\infty$. Also, (3.26) is uniquely solved as

$$\frac{\hat{J}(r)}{r}=\int_0^\infty dr'\mathscr{G}(r,r')\rho(r') \tag{3.27}$$

leading to the reduced Lagrangian

$$\hat{\mathscr{L}}=\frac{I}{2}\dot{\varphi}^2+\frac{\theta}{2\pi}\dot{\varphi}+\int_0^\infty dr\,\hat{\chi}^\dagger\left[i\frac{\partial}{\partial t}-\hat{H}-\mathscr{J}'(r)\varphi\frac{\gamma_5}{2}\right]\hat{\chi}$$
$$-\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r')\ . \tag{3.28}$$

The canonical treatment is again standard. The conjugate of φ is now

$$\hat{p}_\varphi\equiv\frac{\partial\hat{\mathscr{L}}}{\partial\dot{\varphi}}=I\dot{\varphi}+\frac{\theta}{2\pi} \tag{3.29}$$

and the Hamiltonian is

$$\hat{\mathscr{H}}=\frac{1}{2I}\left[\hat{p}_\varphi-\frac{\theta}{2\pi}\right]^2+\int_0^\infty dr\,\hat{\chi}^\dagger\left[\hat{H}+\mathscr{J}'(r)\varphi\frac{\gamma_5}{2}\right]\hat{\chi}$$
$$+\frac{e^2}{8\pi^2}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathscr{G}(r,r')\rho(r')\ . \tag{3.30}$$

We find that $\hat{p}_\varphi$ is no longer conserved; however, we still have

$$\dot{\Omega}=i[\hat{\mathscr{H}},\Omega]=0\ , \tag{3.31}$$

$$\Omega\equiv\hat{p}_\varphi+\int_0^\infty dr\,\mathscr{J}(r)\rho(r)\ . \tag{3.32}$$

The significance of (3.31) is twofold. On one hand, it implies that the Hamiltonian is invariant under the "large" gauge transformation (3.23) generated by $e^{4\pi i\Omega}$. On the other hand, it guarantees that the total charge

$$Q_{\text{tot}}=\Omega-\frac{\theta}{2\pi} \tag{3.33}$$

is conserved. In particular, since we have put θ in the Lagrangian, we may require physical states to be invariant under "large" transformations[24]

$$e^{4\pi i\Omega}|\ \rangle=|\ \rangle \tag{3.34}$$

which leads to the Witten condition[25–27]

$$Q_{\text{tot}}|\ \rangle=\left[\frac{n}{2}-\frac{\theta}{2\pi}\right]|\ \rangle,\quad n=\text{integer}\ . \tag{3.35}$$

It is also evident that the two Hamiltonians (3.17) and (3.30) lead to the same physics since $(\Omega,e^{i\mathcal{J}\varphi\tau_3/2}\hat{\chi})$ in the present gauge corresponds to (p_φ,χ) in the previous gauge.

For completeness, let us also briefly mention what happens in the Weyl gauge where

$$eA_0(\mathbf{x},t)=0\ , \tag{3.36}$$

$$e\,\mathbf{A}(\mathbf{x})=G(\hat{\mathbf{x}}\times\tau)\frac{K(r)-1}{2r}G^\dagger-iG\nabla G^\dagger\ , \tag{3.37}$$

$$G=\exp\left[-\frac{i}{2}\hat{\mathbf{x}}\cdot\tau\lambda(r,t)\right]\ . \tag{3.38}$$

(The details may be found in Goldstein.[9]) The Lagrangian is now given by

$$\begin{aligned}\frac{2\pi}{e^2}\int_0^\infty dr[r^2(\dot{\lambda}')^2+2K^2\dot{\lambda}^2]-\frac{\theta}{2\pi}\dot{\lambda}(\infty)&\\+\int_0^\infty dr\,\chi^\dagger\Bigg[i\frac{\partial}{\partial t}+i\gamma_5\tau_3\frac{\partial}{\partial r}-\beta M&\\+\gamma_5\tau_2e^{i\lambda\tau_3}\frac{K(r)}{r}+\lambda'\frac{\gamma_5}{2}\Bigg]\chi\ .&\end{aligned} \tag{3.39}$$

Again, we may investigate the surface term

$$\lim_{R\to\infty}\left[-\frac{4\pi R^2}{e^2}\ddot{\lambda}'(R)+\chi^\dagger(R)\frac{\gamma_5}{2}\chi(R)\right]\delta\lambda(R)\ . \tag{3.40}$$

The condition $\delta\lambda(\infty)=0$ is incompatible with the dynamics as before, whereas free variation with respect to $\delta\lambda(\infty)$ leads to

$$\lim_{R\to\infty}\left[-\frac{4\pi R^2}{e^2}\ddot{\lambda}'(R)+\chi^\dagger(R)\frac{\gamma_5}{2}\chi(R)\right]=0\ . \tag{3.41}$$

Unlike (3.7), we find that (3.41) is also unacceptable since it implies that the total charge is not conserved,

$$\dot{Q}_{\text{tot}}=-\lim_{R\to\infty}\frac{4\pi R^2}{e^2}\ddot{\lambda}'(R)=-\lim_{R\to\infty}\chi^\dagger(R)\frac{\gamma_5}{2}\chi(R)\ . \tag{3.42}$$

Furthermore, the situation cannot be remedied by a surface term either of the form

$$\int dt\,q[\lambda(\infty)+\varphi] \tag{3.43}$$

or

$$\int dt\,q[\dot{\lambda}(\infty)+\dot{\varphi}]\ . \tag{3.44}$$

The only way out is to split λ by hand as

$$\lambda(r)=-\mathcal{J}(r)\varphi+\hat{\lambda}(r)\ , \tag{3.45}$$

$$\hat{\lambda}(0)=\hat{\lambda}(\infty)=0\ . \tag{3.46}$$

Some work may be saved at this point if we make the (proper) gauge transformation

$$\chi\to e^{i\hat{\lambda}\tau_3/2}\chi\ . \tag{3.47}$$

The situation is then the same as the second gauge except that the basic dynamical variable is $\hat{\lambda}$ rather than $\hat{\mathcal{J}}=r\hat{\lambda}$. In particular, we obtain only the time derivative of Gauss's law

$$\dot{\Pi}_{\hat{\lambda}}=0 \tag{3.48}$$

rather than Gauss's law itself,

$$\Pi_{\hat{\lambda}}\equiv\frac{4\pi}{e^2}\left[-r\frac{d^2}{dr^2}r\dot{\hat{\lambda}}+2K^2\dot{\hat{\lambda}}\right]-\rho=0\ . \tag{3.49}$$

The canonical treatment is straightforward, and we find that the resulting Hamiltonian

$$\begin{aligned}\frac{1}{2I}\left[\hat{p}_\varphi-\frac{\theta}{2\pi}\right]^2+\int_0^\infty dr\,\hat{\chi}^\dagger\left[\hat{H}+\mathcal{J}'(r)\varphi\frac{\gamma_5}{2}\right]\hat{\chi}\ ,&\\+\frac{e^2}{8\pi}\int_0^\infty dr\int_0^\infty dr'[\Pi_{\hat{\lambda}}(r)+\rho(r)]&\\\times\mathcal{G}(r,r')[\Pi_{\hat{\lambda}}(r')+\rho(r')]&\end{aligned} \tag{3.50}$$

is equivalent to the previous one (3.30) in the sector where Gauss's law (3.49) holds.

Up to now, all our reasoning has been canonical. For (3.17) or (3.30) to be acceptable in quantum theory, there are two further problems to be considered. One is that of operator ordering and ultraviolet divergences. Fortunately, this problem is easily resolved, since we have reduced the theory to $1+1$ dimensions. The solution is most easily given in the first gauge, where we have (2.5). The charge density (3.3) should be replaced by the charge-symmetric form

$$\rho(r)=\lim_{r'\to r}\tfrac{1}{2}\left[\chi^\dagger(r'),\frac{\tau_3}{2}\chi(r)\right] \tag{3.51}$$

whereas $\mathcal{H}_F$ should be normal ordered with respect to a suitable ground state.

The other problem involving quantum effects is the chiral anomaly, on which a detailed discussion will be given in another paper.[3] However, the essential feature is not hard to see. The commutation relation $[p_\varphi,\varphi]=-i$ admits the shift

$$p_\varphi\to p_\varphi+\tilde{\omega} \tag{3.52}$$

corresponding to the redefinition

$$\theta\to\theta+2\pi\tilde{\omega}\ . \tag{3.53}$$

The freedom will not affect the form of the Hamiltonian so long as it is written in terms of $Q_{\rm tot}$; however, we are faced with the question of the appropriate value of $Q_{\rm tot}$, given the coefficient of $\mathbf{E}\cdot\mathbf{B}$ and the chiral phase of the fermion mass term. In particular, we should expect a discontinuity at $M=0$.[26]

IV. BOSONIZATION

In this section, we adopt the conventions of Ref. 2, except that we work in the first gauge to avoid complications associated with Schwinger terms.[10] Taking the $\bar\gamma$ matrices as

$$\bar\gamma^0=-\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \bar\gamma^1=-i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$
$$\bar\gamma^5=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{4.1}$$

we have the kinetic term

$$\mathscr{H}_F=-i\int_{r_0}^{\infty}dr\sum_{\pm}\bar\chi_\pm\bar\gamma^1\frac{d}{dr}\chi_\pm -m\int_{r_0}^{\infty}dr(\bar\chi_+\chi_-+\text{H.c.}) \tag{4.2}$$

the fermion charge density

$$\rho=-\tfrac{1}{2}(\bar\chi_+\bar\gamma^1\chi_+-\bar\chi_-\bar\gamma^1\chi_-) \tag{4.3}$$

and fermion number

$$F=\int_{r_0}^{\infty}dr(\bar\chi_+\bar\gamma^0\chi_++\bar\chi_-\bar\gamma^0\chi_-)\,. \tag{4.4}$$

To bosonize the theory, we follow Callan, and take the fermion-boson correspondence in the interaction picture as

$$\tilde\chi(r,t)=\left[\frac{c\mu}{2\pi}\right]^{1/2}\begin{bmatrix}:\exp\left[i\sqrt{\pi}\left[\phi_\pm(r,t)-\int_{r_0}^{r}ds\,\dot\phi_\pm(s,t)\right]\right]:\\ i:\exp\left[i\sqrt{\pi}\left[\phi_\pm(r,t)+\int_{r_0}^{r}ds\,\dot\phi_\pm(s,t)\right]\right]:\end{bmatrix}, \tag{4.5}$$

where $\phi_\pm(r,t)$ are free scalar fields on the half line $r>r_0$ with an infinitesimal mass μ, and $c=\ln\Gamma'(1)$. The ϕ's satisfy the BC

$$\phi'_\pm(r_0,t)=0 \tag{4.6}$$

and the $\tilde\chi$'s are found to be free massless fields

$$\left[\bar\gamma^0\frac{\partial}{\partial t}+\bar\gamma^1\frac{\partial}{\partial r}\right]\tilde\chi(r,t)=0 \tag{4.7}$$

subject to the BC (Ref. 28)

$$\bar\gamma^0\tilde\chi_\pm(r_0,t)=-\tilde\chi_\pm(r_0,t)\,. \tag{4.8}$$

However, with Callan's prescription (4.5), we find that the different chiral components $\tilde\chi_+$ and $\tilde\chi_-$ commute rather than anticommute. To obtain proper Fermi fields, it is necessary to perform in addition a Klein transformation[29] such as

$$\chi_+=\tilde\chi_+,\quad \chi_-=(-1)^F\tilde\chi_-\,. \tag{4.9}$$

There are many other choices besides (4.9) which give the same propagators; a simple example would be the generalization

$$\chi_+=\tilde\chi_+,\quad \chi_-=e^{i(\pi-\alpha)F}\tilde\chi_-e^{i\alpha F}\,. \tag{4.10}$$

The bosonized version of[14]

$$\bar\chi_\pm\bar\gamma^\mu\chi_\pm=\frac{-1}{\sqrt{\pi}}\partial^\mu\phi_\pm\,, \tag{4.11}$$

$$i\bar\chi_\pm\bar\gamma^\mu\partial_\mu\bar\chi_\pm=\tfrac{1}{2}\partial^\mu\phi_\pm\partial_\mu\phi_\pm \tag{4.12}$$

will remain unaffected by the additional transformation (4.10); in particular, the electric field remains the same. However, the fermion mass term will become proportional to

$$\cos(2\sqrt{\pi}\hat{Q}-\alpha)+\cos(2\sqrt{\pi}\hat\Phi-\alpha)\,, \tag{4.13}$$

where

$$\hat{Q}(r,t)=Q(r,t)+\frac{\sqrt{\pi}}{2}F = \tfrac{1}{2}[\phi_+(r,t)-\phi_-(r,t)] -\tfrac{1}{2}\int_r^{\infty}ds[\dot\phi_+(s,t)+\dot\phi_-(s,t)]\,, \tag{4.14}$$

$$\hat\Phi(r,t)=\Phi(r,t)-\frac{\sqrt{\pi}}{2}F = \tfrac{1}{2}[\phi_+(r,t)-\phi_-(r,t)] +\tfrac{1}{2}\int_r^{\infty}ds[\dot\phi_+(s,t)+\dot\phi_-(s,t)]\,. \tag{4.15}$$

Once we insert the correct mass term into the Hamiltonian, we find that another term requires amendment in Ref. 2. Callan gives the electrostatic energy as

$$\mathscr{E}=\frac{e^2}{8\pi^2}\int_{r_0}^{\infty}\frac{dr}{r^2}B^2(r,t)\,, \tag{4.16}$$

where[14]

$$\frac{1}{\sqrt{2\pi}}B(r,t)=\frac{-1}{2\sqrt{\pi}}\left[\hat{Q}(r,t)+\hat\Phi(r,t)+\frac{\theta}{\sqrt{\pi}}\right] \tag{4.17}$$

is identified as the total charge of the system within the radius r. However, combining the mass term (4.13) with the Coulomb term (4.16), we find that the form of the supposed charge distribution (4.17) depends on the unphysical parameter α.[30]

On the other hand, with our Hamiltonian, no unwanted α dependence appears after bosonization. For example, it follows from (2.20) that the total charge inside r may be taken to be[21]

$$\frac{4\pi r^2}{e}E_r \simeq Q_{\rm tot} - \int_r^\infty dr'\rho(r')$$

$$= Q_{\rm tot} + \frac{1}{\sqrt{2\pi}}[B(r,t)-B(\infty,t)] \qquad (4.18)$$

independent of α.

Actually, the problem is well known in the Schwinger model.[31] To quote the paper by Kogut and Susskind (p. 3596):

$$\text{“} \cdots = \frac{1}{2}\frac{g^2}{\pi}\int dz\,\phi^2(z)\ . \qquad (2.28)$$

This gives a mass term in the Lagrangian for a massive free boson. However, Eq. (2.28) *cannot be correct* because it no longer has invariance under $\phi \to \phi + \alpha$” (our emphasis).

However, even with the correct choice for the fermionic Hamiltonian, there is still a problem in the bosonic theory, owing to the fact that the bosonization procedure (4.5) and (4.6) is poorly adapted for taking the limit $r_0 \to 0$. As $r_0 \to 0$, we may expect that $\mathscr{H}_C$ will force all the electric charge to reside with the fermions

$$Q_{\mathscr{f}}\,|\ \rangle = Q_{\rm tot}\,|\ \rangle \qquad (4.19)$$

or in bosonic language

$$\frac{1}{\sqrt{2\pi}}[B(\infty,t)-B(r_0,t)] = \frac{n}{2} - \frac{\bar\theta}{2\pi}\ . \qquad (4.20)$$

However, rewriting (4.6) as

$$\sqrt{2}A'(r_0,t) \equiv \phi'_+(r_0,t) + \phi'_-(r_0,t) = 0\ , \qquad (4.21)$$

$$B'(r_0,t) = 0\ , \qquad (4.22)$$

we find that (4.22) is not what we would expect from (4.20) and a finite energy for the mass term $B(\infty,t)=\text{const}$. In other words, with a charge-mixing BC (4.8) for free fermions, the dyon degree of freedom will be excited, and the perturbation series in the original interaction picture (4.5) and (4.6) will diverge as $r_0 \to 0$.[32] This implies that the BC for the bosons and/or the fermions must be dynamically modified, an issue which will be discussed in full elsewhere[3] (see also the appendix).

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APPENDIX

In this appendix, we deal with some discrepancies between our work and the existing literature. The first problem we encounter is the gauge dependence of the fermion BC's. Although our treatment is in agreement with the majority of the authors, there seem to be two exceptions. One is Callan's first paper,[26] which works in the Weyl ($A_0=0$) gauge. On p. 2142 of Ref. 26 we find the equation

$$\left[\tau_3\left[\partial_t \mp i\frac{\lambda'}{2}\right] + i\tau_1\partial_r + \left[A-\frac{1}{r}\right]e^{\pm i\lambda\tau_2}\right]\chi_\pm = 0 \qquad (A1)$$

and the statement that this leads to the BC $\tau_+\chi(r_0)=0$. However, we believe the correct BC (for λ a c number) should read

$$\tau_+ e^{\pm i\lambda\tau_2/2}\chi_\pm\,|_{r=r_0} = 0\ . \qquad (A2)$$

We have also checked Besson's thesis[18] quoted by Callan on this point. We have found that, in Chap. VI of Ref. 18, Besson imposes BC's *after* transforming to the gauge (2.1) and (2.3), so this seems to be a case of misquotation.

Similarly, the BC of Balachandran and Schechter[23] [Eq. (4) of the first paper, Eq. (2.19) of the second paper] in the limit $r_0 \to 0$

$$\hat\chi^{\,+}(0) = -\hat\chi^{\,-}(0) \qquad (A3)$$

is valid only in the gauge (2.1) and (2.3) of the text. Unfortunately, Ref. 23 works in a gauge where $\mathbf{A}$ is allowed to fluctuate [Eq. (2.8) of second paper], so their use of (A3) is inappropriate. In more detail, we have the following.

(1) If the Jackiw-Rebbi formalism[33] and the Marciano-Muzinich solution[34] are to be used [Ref. 2 of the first paper in Ref. 23; Eq. (2.17) of the second paper], one must choose the gauge (2.1) and (2.3) so that $\mathbf{A}$ is a fixed c number. Then the correct charge-mixing BC in the limit $r_0 \to 0$ is given by (A3). However, $\hat\chi^{\,\pm} = \chi^\pm$ so that $\chi^+(0) = -\chi^-(0)$ rather than (3.18). Also, in this gauge, we have seen that A_0 must be varied at spatial infinity, contrary to their general rule [p. 1187 of second paper in Ref. 23].

(2) If we work in a gauge such as (3.19)–(3.22) so that $A_0 \to 0$, charge conservation requires that terms proportional to $\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\boldsymbol{\tau}) - \boldsymbol{\tau}$ must be kept in $\mathbf{A}$, even in the point limit. This is in agreement with Ref. 11, but in disagreement with the statement of Ref. 23 [preceding Eq. (2.8) of second paper]. Also, with (3.19)–(3.22), the effective BC must be imposed on the combination $e^{i\mathscr{f}\varphi\tau_3/2}\hat\chi$ (in our notation). Since $\mathscr{f}(r_0) \simeq 1$, it follows that the correct BC in the point limit involves $e^{i\varphi\tau_3/2}\hat\chi(0)$, not simply $\hat\chi(0)$.

(3) In the gauge (3.19)–(3.22), there is another problem of whether the path-ordered phase factor (2.5) should be taken from r to the origin or from r to infinity, when constructing the gauge-invariant fermion field. Both constructions are invariant under proper gauge transformations, and Ref. 23 has chosen the latter. However, only the former is invariant under the improper gauge

transformations generated by Ω, which are the ones considered by Witten.[25] Since these transformations still commute with A_0 and the Higgs fields at infinity and consequently are well defined, the choice of Ref. 23 is also inconsistent with their own criterion for admissible symmetry operators [p. 1187 of the second paper].

(4) The last consideration is whether their $\hat{\chi}$ rather than χ can be identified as the gauge-invariant fermion field. The BC (A3) then can be reconciled with our results. However, we find that the commutation relation with their charge operator still cannot.

Let us now proceed to the second problem, which is the correct form of the electrostatic energy. We have already noted that Callan's expression (4.16) leads to unacceptable results in conjunction with the correct mass term (4.13), a problem well known in the Schwinger model. Actually, however, even without going through the details of Sec. IV, we may see that something is peculiar with the results of Refs. 2 and 4 for a Higgs mass.

According to these authors [cf. Eq. (4.17)], the total electric charge q is determined by the scalar fields at infinity. Since the charge-mixing term $K(r)$ drops off exponentially, it follows that q is determined by the fermion mass term after bosonization, regardless of whether the fermion mass m is greater or smaller than the vector boson mass m_W.

However, for $m_W \lesssim m$, one should be able to apply the semiclassical approximation,[7] treating the monopole as a background field. The Jackiw-Rebbi mode[33] then obeys the charge-mixing BC, and one has $F=\pm\frac{1}{2}$ but $q=0$ (or more generally $\bar{\theta}=\theta$). Our analysis recovers this result for $m_W \lesssim m$, whereas Refs. 2 and 4 do not, which in our opinion, is already evidence against their results.[35]

Also, if one is willing to go through the details of Callan's derivation, it is not hard to identify the source of the discrepancy. Let us begin with Eq. (3.6) of the second paper[2]

$$L_{MF}^{\lambda}=\frac{1}{2e^2}\int 4\pi r^2 dr(\dot{\lambda}')^2 +\frac{\theta}{2\pi}\int dr\,\dot{\lambda}'+\int dr\frac{\lambda'}{2}(\bar{\chi}_+\bar{\gamma}_0\chi_+-\bar{\chi}_-\bar{\gamma}_0\chi_-)\ . \tag{A4}$$

Substituting the next line[14]

$$\bar{\chi}_+\bar{\gamma}_0\chi_+-\bar{\chi}_-\bar{\gamma}_0\chi_-=\frac{-1}{\sqrt{\pi}}(\dot{\phi}_+-\dot{\phi}_-) \tag{A5}$$

we find

$$L_{MF}^{\lambda}=\frac{1}{2e^2}\int 4\pi r^2 dr(\dot{\lambda}')^2 +\frac{\theta}{2\pi}\int dr\,\dot{\lambda}'-\int dr\frac{\lambda'}{2\sqrt{\pi}}(\dot{\phi}_+-\dot{\phi}_-) \tag{A6}$$

which is invariant under

$$\phi_\pm(r,t)\to\phi_\pm(r,t)+\alpha_\pm,\ \ \lambda(r,t)\to\lambda(r,t)\ . \tag{A7}$$

However, the supposed equation of motion

$$\dot{\lambda}'=-\frac{e^2}{8\pi^{3/2}r^2}\left[\phi_+-\phi_-+\frac{\theta}{\sqrt{\pi}}\right] \tag{A8}$$

and the Hamiltonian (4.16) are not invariant, so the source of the discrepancy must lie in between.

After a partial integration of (A6) in the time variable

$$L_{MF}^{\lambda}=\frac{1}{2e^2}\int 4\pi r^2 dr(\dot{\lambda}')^2 +\frac{\theta}{2\pi}\int dr\,\dot{\lambda}'+\int dr\frac{\dot{\lambda}'}{2\sqrt{\pi}}(\phi_+-\phi_-)\ , \tag{A10}$$

Callan varies with respect to $\dot{\lambda}'$ to obtain (A8). However, $\dot{\lambda}'$ is the radial electric field, so this is an unusual choice of variation. The usual procedure would be to vary with respect to λ itself, which leads to the time derivative of Gauss's law

$$\frac{4\pi}{e^2}(r^2\ddot{\lambda}')+\frac{1}{2\sqrt{\pi}}(\dot{\phi}'_+-\dot{\phi}'_-)=0 \tag{A11}$$

as expected for the Weyl ($A_0=0$) gauge. As shown in the text, the integration of (A11) then introduces the dyon degree of freedom. Owing to the incorrect variation with respect to $\dot{\lambda}'$, we believe Callan has misidentified $B(r_0,t)$ (which is an artifact of the bosonization procedure) with a physical quantity. This point (as well as the omission of the Klein factor) is insignificant if the fermions are massless;[21] however, the results are affected for massive fermions.[5,6]

The undesirability of varying with respect to the electric field is even more apparent in Ref. 36, where no dyon degree of freedom is involved. Corresponding to Callan's (A4) and (A6), one has [Eqs. (2.7) and (2.8)]

$$L=\int 4\pi r^2 dr[\tfrac{1}{2}\dot{b}_r^{\,2}-gb_r(\bar{\psi}_{1\uparrow}\hat{\mathbf{r}}\cdot\gamma\psi_{1\uparrow}-\bar{\psi}_{1\downarrow}\hat{\mathbf{r}}\cdot\gamma\psi_{1\downarrow} -\bar{\psi}_{2\uparrow}\hat{\mathbf{r}}\cdot\gamma\psi_{2\uparrow}+\bar{\psi}_{2\downarrow}\hat{\mathbf{r}}\cdot\gamma\psi_{2\downarrow})]\ , \tag{A12}$$

$$L=\int dr\left[2\pi r^2\dot{b}_r^{\,2}+\frac{g}{\sqrt{\pi}}b_r(\dot{\Phi}_1+\dot{Q}_1-\dot{\Phi}_2-\dot{Q}_2)\right]\ . \tag{A13}$$

Varying with respect to the electric field $-\dot{b}_r$ after partial integration gives the analog of (4.16)

$$\mathscr{E}=\frac{g^2}{8\pi^2}\int_{r_0}^{\infty}\frac{dr}{r^2}[\Phi_1(r)+Q_1(r)-\Phi_2(r)-\Phi_2(r)]^2 \tag{A14}$$

which leads to a dynamical BC

$$(\Phi_1+Q_1-\Phi_2-Q_2)\simeq 0\ \text{ at } r=r_0\ . \tag{A15}$$

However, there is no reason the electrostatic energy of the fermions in the absence of the dyon degree of freedom should lead to any modification of BC's. Indeed elementary electrostatics gives

$$\begin{aligned}-\dot{b}_r&=\frac{g}{4\pi r^2}\int_0^r dr'(\psi_{1\uparrow}^{\dagger}\psi_{1\uparrow}-\psi_{1\downarrow}^{\dagger}\psi_{1\downarrow}-\psi_{2\uparrow}^{\dagger}\psi_{2\uparrow}+\psi_{2\downarrow}^{\dagger}\psi_{2\downarrow})\\ &=\frac{g}{4\pi r^2}\int_0^r dr'\frac{1}{\sqrt{\pi}}[\Phi_1'(r)+Q_1'(r)-\Phi_2'(r)-Q_2'(r)]\end{aligned} \tag{A16}$$

and therefore the correct answer should be

$$\mathscr{E}=\int_0^\infty 2\pi r^2 dr\, \dot{b}_r{}^2$$
$$=\frac{g^2}{8\pi^2}\int_0^\infty \frac{dr}{r^2}[\Phi_1(r)+Q_1(r)-\Phi_2(r)-Q_2(r)-\Phi_1(0)$$
$$-Q_1(0)+\Phi_2(0)+Q_2(0)]^2 \qquad \text{(A17)}$$

which as expected, is invariant under $\phi\to\phi+\alpha$.

In this connection, we may also consider the following issue. Let us take the Hamiltonian

$$\mathscr{H}=\frac{1}{2}\int_0^\infty dr\left[(\dot{\phi})^2+(\phi')^2+\frac{e^2}{4\pi^2 r^2}\phi^2\right] \qquad \text{(A18)}$$

starting with the BC

$$\phi'(0)=0\ . \qquad \text{(A19)}$$

Since (A18) is quadratic, the solution is trivially found to be

$$\phi(r)\sim\sqrt{kr}\,J_\nu(kr),\quad \nu=\left[\frac{1}{4}+\frac{e^2}{4\pi^2}\right]^{1/2}, \qquad \text{(A20)}$$

which obeys the BC

$$\phi(0)=0 \qquad \text{(A21)}$$

as well as (A19). In the literature, this is sometimes taken to be evidence that the charge-mixing BC is compatible with charge conservation.

However, besides the fact that (A18) again breaks $\phi\to\phi+\alpha$, there is another reason to regard this argument as specious. Let us take the approximation (2.28) where the Hamiltonian is given by $\mathscr{H}=\mathscr{H}_F+\mathscr{H}_C$. Then the solutions are exponentials rather than Bessel functions,[5,10] and (A19) and (A21) are no longer compatible. To summarize:

(1) Overall charge conservation is only a matter of correctly introducing the dyon degree of freedom.

(2) The dynamical conservation of fermion electric charge is due to $\mathscr{H}_C$, rather than the fermion self-energy $\mathscr{H}_C$.

Finally, we may briefly comment on Refs. 6 and 9–11 which deal with related matters. References 9 and 10 essentially give the same treatment as ours, and the differences are at most minor. In particular, Goldstein[9] has also noted that it is misleading to ascribe overall charge conservation to "radiative corrections." The authors of Ref. 6 also have found disagreement with Callan, and agreement with us for a Higgs mass, without employing the s-wave approximation.

As for Ref. 11, we may happily report that this new paper is free of the objection we have raised[5] against a previous paper. In particular, the authors have properly emphasized that the fermion electric current need not be conserved by itself.

Unfortunately, however, we cannot agree with their suggestion [following Eq. (2.57)] that the dyon degree of freedom can be excited in a low-energy process. As shown by Besson,[18]

$$\langle (I\dot{\varphi})^2\rangle \gg \langle I\dot{\varphi}\rangle^2 \qquad \text{(A22)}$$

for the semiclassical ground state in the presence of light fermions, so we believe their key equation (4.47) is not well founded.

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