Path-integral derivation of the heat kernel on SU(N) space

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We express the heat kernel as a lattice path integral and obtain multiple classical solutions from the field equations. We take the continuum limit by first expanding the lattice path integral about the classical solutions, and then performing a mass renormalization. We then exactly evaluate the heat kernel using algebraic properties of the structure constants, and prove the exactness of the semiclassical approximation.

I. INTRODUCTION

The heat kernel on SU(N) group space has been extensively studied for its mathematical properties, and also has applications in lattice gauge theory (see Ref. 1 and references cited therein). Menotti and Onofri[1] have used differential equations to evaluate the heat kernel. We briefly report on a path-integral derivation of this result using the integral approach. The main feature of our derivation is the use of a lattice cutoff and the expansion of the lattice path integral about countably infinite classical solutions. We also perform a nontrivial mass renormalization to obtain the final result.

The path-integral approach to the heat kernel has also been discussed by Marinov and Terentev[2] and earlier by Dowker.[2] Although both of these references use the Lagrangian approach to evaluate the heat kernel, they both stop far short of actually performing the path integration. Instead, both of them first simply write down the semiclassical approximation for the heat kernel, and then verify, using essentially the Schrödinger differential equation, that the semiclassical approximation is an exact solution. This approach in fact is the same, in spirit, as the differential approach of Menotti and Onofri.[1] Our derivation is directly based on evaluating the path integral, and, in particular, will prove the exactness of the semiclassical approximation.

II. DEFINITIONS

The Hamiltonian for two-dimensional lattice gauge theory is given by[3]

$$H=\frac{g^2}{2}E_aE_a \ , \tag{1}$$

where E_a is the chromoelectric field operator. It can be shown that

$$H=-\frac{g^2}{2}\nabla^2 \ , \tag{2}$$

where ∇^2 is the Laplace-Beltrami operator on the SU(N) group space.[4]

The heat kernel or the evolution kernel is defined by

$$K(T)=\langle U|e^{-TH}|V\rangle \tag{3}$$

where bra and ket vectors are coordinate eigenstates on $L_2(\mathrm{SU}(N))$. Dividing the interval $(0,T)$ into M intervals with lattice spacing M/T, and using the completeness equation $(M-1)$ times, the lattice path integral for $K(T)$ is given by

$$K(T)=\prod_{n=1}^{M-1}\int dU_n\left[\prod_{n=0}^{M-1}\langle U_{n+1}|e^{-(T/M)H}|U_n\rangle\right] \ , \tag{4$'$}$$

or

$$K(T)=\lim_{M\to\infty}\prod_{n=1}^{M-1}\int dU_n e^S \tag{4}$$

[dU_n is the invariant SU(N) measure].

For U_n being in the fundamental representation, we have

$$\lim_{M\to\infty}\langle U_{n+1}|e^{-(T/M)H}|U_n\rangle \cong \exp\left[\frac{M}{g^2T}\mathrm{Tr}(U_{n+1}^\dagger U_n+U_n^\dagger U_{n+1})\right] \ ,$$

and hence from (4) and (4$'$), we have

$$S=\frac{M}{g^2T}\sum_{n=0}^{M-1}\mathrm{Tr}(U_nU_{n+1}^\dagger+U_{n+1}U_n^\dagger) \ . \tag{5}$$

The boundary conditions, after the change of variables $U_n\to VU_n$, are

$$U_0=1, \quad U_M=V^\dagger U\equiv Re^{iA_IX_I}R^\dagger \ , \tag{6}$$

where X_I, $I=1,2,\ldots,N-1$ are the diagonal generators of SU(N), and R is an element of SU(N); we use $\mathrm{Tr}(X_IX_J)=\frac{1}{2}\delta_{IJ}$.

Note that the nonlinear action S is the chiral action in one dimension as well as the Wilson action[5] for the two-dimensional lattice gauge theory in the axial gauge and with appropriate boundary conditions.

III. CLASSICAL SOLUTIONS AND THE CONTINUUM LIMIT

Obtaining the classical field equations from $\delta S=0$, we have for the lattice

$$\mathrm{Tr}X_a(U_{n+1}^\dagger U_n-U_n^\dagger U_{n+1}+U_{n-1}^\dagger U_n-U_n^\dagger U_{n-1})=0 \ . \tag{7}$$

Solving the field equations for the lattice, we have the exact multiple classical solutions [which are paths in SU(N)

space] given by $(n=0,1,2,\ldots,M)$

$$e_n(l)=R\exp\left[i\frac{n}{M}(A_I+2\pi l_I)X_I\right]R^{\dagger}\ , \tag{8}$$

$$l_I=0,\ \pm1,\ \pm2,\ \ldots,\ \pm(M-1)\ . \tag{8'}$$

Note that *all* the classical paths labeled by integers $l=(l_1,\ldots,l_{N-1})$ satisfy the boundary conditions given in (6), i.e.,

$$e_0(l)=1,\quad e_M(l)=V^{\dagger}U\ . \tag{9}$$

In Ref. 2, the multiple classical solutions are obtained indirectly, since the field equations are written for the Lie algebra rather than for the group elements themselves, as in Eq. (7).

The multiplicity of classical solutions essentially results from the compactness of the SU(N) degree of freedom; l is the number of times the classical path winds around the maximal Abelian toral subspace of SU(N).

If one naively takes the $M\to\infty$ continuum limit of (4), one obtains an intractable nonlinear action and an ill-defined expression due to a quadratic masslike divergence arising from the measure dU_n. We instead take the continuum limit by first expanding the lattice path integral about the multiple classical solutions. We show that there is zero-mass renormalization order by order in g^2T about each of these classical solutions, a result similar to that obtained in other nonlinear lattice field theories;[6] we also show that about these classical solutions the nonlinear terms in the action which do not contribute to mass renormalization are all down by a factor of at least $1/M$. See the Appendix for details of the $M\to\infty$. We obtain the continuum limit (ignoring overall normalization terms) given by Eqs. (A13) and (A14), as

$$K(T)=\prod_{I=1}^{N-1}\sum_{l_I=-\infty}^{+\infty}e^{S_{\rm cl}(l)}\prod_{t=0}^{T}\prod_a\int_{-\infty}^{+\infty}d\xi_a(t)\exp\left[\int_0^T L(l)\,dt\right],\quad \xi_a(0)=\xi_a(T)=0\ , \tag{10}$$

where $S_{\rm cl}(l)$ is the classical action for the lth classical solution

$$S_{\rm cl}(l)=-\frac{1}{2g^2T}\sum_{I=1}^{N-1}(A_I+2\pi l_I)^2\ , \tag{11}$$

and the effective Lagrangian about the lth classical solution is $(\dot{\xi}_a=\partial\xi_a/\partial t)$

$$L(l)=-\frac{1}{2g^2}\left[\dot{\xi}_a\dot{\xi}_a+\frac{1}{T}(A_I+2\pi l_I)C_{Iab}\xi_a\dot{\xi}_b\right]\ . \tag{12}$$

The field ξ_a, $a=1,2,\ldots,N^2-1$, is an SU(N) vector, and C_{Iab} are the structure constants of SU(N). Note that L is velocity dependent and it contains total time T explicitly, as well as in the boundary conditions, namely, $\xi_a(0)=\xi_a(T)=0$.

IV. PATH INTEGRATION

We evaluate the nonzero values of C_{Iab}. Using the double-index notation $\pm(KL)$, $K,L=1,2,\ldots,N(N-1)$, with $K<L$ for the off-diagonal generators of SU(N), we obtain for $a=(KL)$ the following:[4]

$$C_{Iab}=[\lambda_I(K)-\lambda_I(L)]\delta_{a,-b}\ , \tag{13}$$

where $\lambda_I(K)$ are the weight vectors of SU(N). Using (12) and (13), we have

$$L(l)=-\frac{1}{2g^2}\left[\sum_{I=1}^{N-1}\dot{\xi}_I{}^2+\sum_{K<L}[(\dot{\xi}_{(KL)})^2+(\dot{\xi}_{-(KL)})^2]+\frac{4}{T}\sum_{K<L}(\phi_K+2\pi m_K-\phi_L-2\pi m_L)\xi_{(KL)}\dot{\xi}_{-(KL)}\right]\ , \tag{14}$$

where

$$(\phi_K+2\pi m_K)=\tfrac{1}{2}\sum_{I=1}^{N-1}\lambda_I(K)(A_I+2\pi l_I)\ . \tag{15}$$

We see that the SU(N) case splits into $N(N-1)/2$ separate path integrations, each of which is quadratic and involves only the variables $\xi_{(KL)}$ and $\xi_{-(KL)}$. Performing the path integrations[7] we obtain

$$K(T)=\prod_{I=1}^{N-1}\sum_{m_I=-\infty}^{+\infty}\prod_{K<L}\left[\frac{\phi_K-\phi_L+2\pi m_K-2\pi m_L}{\sin\frac{1}{2}(\phi_K-\phi_L+2\pi m_K-2\pi m_L)}\right]\times\exp\left[-\frac{2}{g^2T}\sum_{I=1}^{N-1}(\phi_I+2\pi m_I)^2\right]\ , \tag{16}$$

where

$$e^{iA_IX_I}=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},\ldots,e^{i\phi_N})\ , \tag{17}$$

with

$$\sum_{I=1}^{N}\phi_I=0\ . \tag{18}$$

The result we have obtained agrees with Menotti and Onofri.[1]

V. CONCLUSIONS

Our derivation used the algebraic properties of the group as well as the multiple classical solutions. To take the continuum limit we had to perform a mass renormalization due to the nonlinear nature of SU(N) degrees of freedom. We also found that taking a naive continuum limit makes the theory ill defined and intractable; and in fact it is only by solving the theory about the classical solutions that we can reduce the action to a quadratic form. We have hence derived the reason why the semiclassical approximation is exact for the SU(N) heat kernel.

The continuum Lagrangian can easily be generalized to include fermions,[8] and can also be applied to higher dimensions. Note that the Lagrangian given in (12) also appears in proving the Atiyah-Singer theorem using supersymmetric quantum mechanics.

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APPENDIX: MASS RENORMALIZATION AND THE CONTINUUM LIMIT

We briefly discuss the main features of the $M\to\infty$ limit. If one naively takes the $M\to\infty$ limit for the path integral given by (4), then defining $\partial_t U=(M/T)(U_{n+1}-U_n)$, we obtain

$$K(T)=\int dU \exp\left[-\frac{1}{2g^2}\int_0^T dt\, \mathrm{Tr}(\partial_t U^\dagger \partial_t U)\right],$$
$$U(0)=V,\quad U(T)=U\ , \tag{A1}$$

where

$$\int dU=\lim_{M\to\infty}\prod_{n=1}^{M-1}\int dU_n\ . \tag{A2}$$

The expressions (A1) and (A2) are the ones derived by Marinov and Terentev,[2] and, as they stand, are ill defined and meaningless. The action in (A1) is highly nonlinear and there is no apparent reason, as is recognized in Ref. 2, why the semiclassical approxomation should be exact for $K(T)$. The more serious problem is the divergences arising from the nonlinear pieces of S, as well as from the measure dU_n. Using the canonical coordinates for $U_n=\exp(iB_n^a X_a)$, where X_a are the generators, we have

$$dU=\lim_{M\to\infty}\prod_{n=1}^{M-1}dU_n$$
$$\simeq \lim_{M\to\infty}\prod_{na}dB_n^a\exp\left[-\frac{N}{24}\sum_{na}(B_n^a)^2+O(B^4)\right] \tag{A3}$$
$$=\lim_{M\to\infty}\prod_{t=0}^{T}\prod_{a}\int dB_t^a\exp\left[-\frac{N}{24}\frac{M}{T}\int_0^T B^2 dt\right]\ . \tag{A4}$$

The measure has a linear divergence in M for the coefficient of the masslike B^2 term arising from the invariant measure. Since there is no mass term in the action given by Eq. (5), the only way that a finite path-integral expression can exist for the heat kernel $K(T)$ is for the divergent nonlinear terms from the action to exactly cancel this divergence, and in effect give zero-mass renormalization.

To take the continuum limit, we make the change of variables

$$U_n=e_n(l)\exp(i\xi_n^a X_a)\equiv e_n(l)V_n\ , \tag{A5}$$

with

$$\xi_0^a=\xi_M^a=0,\quad dU_n=dV_n\ . \tag{A6}$$

We then have, for the action about the lth classical path,

$$S(l)=\frac{M}{2g^2T}\sum_n \mathrm{Tr}\{e_1^+(l)V_nV_{n+1}^\dagger+V_{n+1}V_n^\dagger e_1(l)\}\ . \tag{A7}$$

We expand the path integral given in (4) about the stationary classical paths in the same spirit as the multi-instanton expansion for the Yang-Mills field. We hence have the expansion

$$K(T)=\sum_l\prod_n\int dV_n\exp[S(l)]+O(e^{-M})\ . \tag{A8}$$

Using Eq. (8) for the classical paths and the Campbell-Hausdorff formula, we have [C_{abc} are the SU(N) structure constants]

$$S(l)=-\frac{M}{2g^2T}\sum_n\left[\sum_a(\delta\xi_n^a)^2-\frac{1}{12}\sum_a(C_{abc}\xi_n^b\delta\xi_n^c)^2\right.$$
$$\left.+\frac{1}{M}(A_l+2\pi l_l)C_{lab}\xi_n^a\delta\xi_n^b+O((\xi^2\delta\xi)^2)\right]$$
$$-\frac{1}{2g^2TM}\sum_n\sum_l(A_l+2\pi l_l)^2+O(1/M)\ , \tag{A9}$$

where $\delta\xi_n^a\equiv\xi_{n+1}^a-\xi_n^a$, and

$$\prod_n dV_n=\prod_{na}d\xi_n^a\exp\left[-\frac{N}{24}\sum_{na}(\xi_n^a)^2+O(\xi^4)\right]\ . \tag{A10}$$

Since the coupling of the ξ_n^a field to the classical solutions is down by a factor of $1/M$, only the pure ξ part of the action will contribute to possible divergences in M. Calculating the propagator of the ξ field to one-loop order, we have, for the renormalized mass (as an expansion in g^2T)

$$\delta m=-\frac{N}{6}g^2T+\frac{4Ng^2T}{3M^2}\left[\frac{M^2}{8}+\frac{M}{4}+\frac{3}{8}\right]$$
$$+O((g^2T)^2) \tag{A11}$$
$$=\frac{Ng^2T}{3M}\left[1+\frac{3}{2M}\right]+O((g^2T)^2)\sim O(1/M)\ , \tag{A12}$$

where the first term in (A11) comes from the measure given in (A10) and the second term arises from a single contraction of the quartic term in the action (A9). We see that, due to a nontrivial cancellation of the terms independent of M, $\delta m\to 0$ as $M\to\infty$.

Hence the coefficient of the divergent masslike term arising from the measure is exactly zero as $M\to\infty$.

This cancellation of the quadratic piece of the measure term seems to be a general feature of massless SU(N) lattice field theories.[6] By power counting, we see that the other nonlinear terms which do not contribute to δm all vanish as $M\to\infty$, leaving an action which is only quadratic in ξ_n^a. Also, the range of ξ_n^a extends from the group manifold to the real line as $M\to\infty$, and, using

$$\dot\xi_a(t)=\lim_{M\to\infty}(M/T)\delta\xi_n^a\ ,$$

we obtain from (A9), (A10), and (A12)

$$S(l) = -\frac{1}{2g^2}\int_0^T dt\left\{\dot{\xi}_a\dot{\xi}_a + \frac{C_{lab}}{T}(A_l + 2\pi l_l)\xi_a\dot{\xi}_b\right\} - \frac{1}{2g^2 T}\sum_{l=1}^{N-1}(A_l + 2\pi l_l)^2 \ , \tag{A13}$$

and

$$K(T) = \sum_l \prod_{t=0}^{T}\prod_a \int_{-\infty}^{+\infty} d\xi^a(t)\exp[S(l)] \ , \tag{A14}$$

$$\xi^a(0) = \xi^a(T) = 0 \ ,$$

where the sum in l is over all the classical solutions.

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