as $O(3)$ vectors 1^- and scalars 0^+ in a chosen rest frame. Then we analyze products of these vectors involving ϵ pseudotensors in order to determine the highest representation of $O(3)$ present and to determine its parity. This method is effective, much simpler, and more general than the use of helicity amplitudes employed by Dorren *et al.*[8] We also avoid any need to work out Gram determinants with this method.

The problem of incorporating the ω-A_2 trajectory as an abnormally coupled trajectory to 3-π and the remaining 5-π in the 8-π amplitude was then analyzed in detail. There we showed, using our tensorial technique, how our proposed form for the N-point superstructures could be applied to achieve the desired result.

From these considerations, we are able to formulate a minimality hypothesis. In $\pi\pi\rightarrow\pi\pi$ [13] and $\pi\pi\rightarrow\pi\omega$,[14] it was possible to obtain the desired leading behavior in all channels by writing effectively only one term for each of the reactions. The minimality hypothesis then claimed that these single terms gave the complete amplitude. By contrast, for the four-point amplitudes with two spinning particles, no such minimal hypothesis can be formulated. However, for the N-point, all-spin-0 amplitude, we may take as an acceptable minimal hypothesis that we should only include the minimum number of terms necessary to include all possible subgraphs with minimum starting points on all trajectories, which have the necessary property of defined normalities on all leading trajectories.

Accordingly, by careful examination of Table I, we see that the unit α factors for both kinematic superstructures are superfluous. The minimality hypothesis leads us to reject these forms, especially as they have a higher number of nonleading trajectories. Then, in fact, we obtain the correct minimal forms for $\pi\pi\rightarrow\pi\pi$ when $\alpha(46)$, $\alpha(57)$, or $\alpha(68)=0$. In any case, the correct minimal form for $\pi\pi\rightarrow\pi\omega$ is obtained when either $\alpha(46)$ or $\alpha(68)=0$, and $\alpha(13)=1$. At least in this examples the hypothesis of minimality applied to $2n$-π amplitude, is consistent with minimality for a smaller $2n'<2n$-π amplitudes.

Since changing the set of B_N functions used in defining an amplitude changes drastically the degeneracy structure of the amplitude, we feel that such a hypothesis of minimality is critically necessary to a discussion of the factorization and level structure. We do not, however, discuss this here, since we have not included specifically all the forms of tree diagrams with two ω-A_2 trajectories and no ω-A_2 trajectories. Moreover, from the discussion given by Canning,[11] we may expect some conflict between minimality, duality, and factorization.

[13] C. Lovelace, Phys. Letters **28B**, 264 (1968).
[14] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

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Wave Equations with Compact and Noncompact Spectrum

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We present a unified approach to compact and noncompact wave equations based on the algebra of $O(6,C)$. The Bhabha and Nambu equations emerge as the simplest possibility for describing multiplets of relativistic particles. Dynamical quantities such as mass, spin, and magnetic moment are evaluated in terms of the spectra of relevant operators.

INTRODUCTION

AMONG the most interesting phenomena of the physics of strongly interacting particles is the fact that hadrons and their resonances seem to fall into more or less well-defined groups and families. Some of these are described, on a phenomenological level, by internal symmetry groups such as $SU(2)$ and $SU(3)$. Other approaches yield families or trajectories of particles with different spins. One would welcome, perhaps, a description of such supermultiplets in terms of fields and wave equations. Attempts in this direction are the finite-component wave equation of Bhabha[1] and the infinite-component equation of Nambu.[2] While neither of these equations is perhaps very physical, it seems, nontheless, to be rewarding to explore them as models of an eventually more complete theory. After all, the Dirac equation, which is physically quite relevant, is indeed the Bhabha equation of lowest order.

[1] H. Bhabha, Rev. Mod. Phys. **17**, 200 (1945); **21**, 451 (1949). See also A. Aurilia and H. Umezawa, Phys. Rev. **182**, 1682 (1969).
[2] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 368 (1966). See also Y. Nambu, Phys. Rev. **160**, 1171 (1967).

We should like to present here an approach which treats the Bhabha and Nambu equations in a unified way. Dynamical quantities will be evaluated in terms of the spectra of operators of the algebra of $O(6,C)$. We thus obtain results of great generality and—as regards the two types of wave equations—a pleasing reciprocity. In Sec. I, we explore the algebra of $O(6)$ and its complex extension $O(4,2)$. A particular family of representations is defined by means of a very useful realization. Generators are expressed as $(\bar{a}\Gamma_R a)$, thus reducing all calculations to manipulation of boson operators and Dirac matrices. From this specific realization we derive two auxiliary relations which are generalizations of $\gamma_\mu^2=1$ and $\gamma_5=-\gamma_1\gamma_2\gamma_3\gamma_4$. The two families of representations—compact and noncompact—turn out to be the basis, respectively, for the Bhabha and the Nambu equations. In fact, since both have exactly the same $O(4)$ subgroups, these two cases are the simplest algebras combining representations of $O(4)$ into supermultiplets.[3] The difference between the two depends only on which Casimir operator is incorporated as a generator of the larger algebra.

The wave equations which utilize the two sets of representations are investigated in Sec. II. By using the algebraic relations of Sec. I, we can give very concise derivations of expressions for the transformation of wave functions, for the parity operator and for mass and spin. Section III contains the covariant expressions for the current and angular-momentum and energy-momentum tensor. These results enable us to calculate the magnetic moment for any representation in either the Bhabha or the Nambu case. Ramifications of our results are investigated in a number of appendices. We derive the completeness relation for Dirac matrices from the orthogonality of the group members and explore the relation of our $O(6,C)$ representation to those of $U(4,C)$. We also consider generalizations of our Lagrangian to include terms such as those used in Nambu's work.

I. ALGEBRA OF $O(6,C)$

A. Algebra of Dirac Matrices

It is well known that one can form 16 independent quantities which are products of Dirac matrices. These 16 matrices are a realization of the generators of an $O(6)$ algebra. We can show this explicitly by introducing

$$L_{AB}'=[\gamma_A,\gamma_B]/4i\,,\quad L_{A6}'=\tfrac{1}{2}\gamma_A\,, \tag{1.1}$$

where A, $B=1, \ldots, 5$ and $L_{AB}=-L_{BA}$. The L_{AB}' are Hermitian and normalized to $\frac{1}{2}$. One finds that these quantities obey the commutation relation

$$[L_{AB}',L_{CD}']=i(\delta_{AC}L_{BD}'+\delta_{BD}L_{AC}'-\delta_{AD}L_{BC}'-\delta_{BC}L_{AD}')\,,$$

with A, $B=1, \ldots, 6$. The L_{AB}', together with the unit matrix, are therefore generators of an $O(6)$ algebra.[4] One has, furthermore, the anticommutation relation

$$\{L_{AE}',L_{ED}'\}=-\tfrac{1}{2}\delta_{AD}\quad (\text{no sum on } E),$$

which defines the Dirac algebra to be the fundamental representation of $O(6)$.

We can also use the Dirac matrices to find a realization of the generators of $O(4,2)$. If we introduce[5]

$$L_{AB}=(g_{AR})^{1/2}L_{RS}'(g_{SB})^{1/2} \tag{1.2}$$

with the metric $g=(+\,+\,+\,+\,-\,-)$, then

$$[L_{AB},L_{CD}]=i(g_{AC}L_{BD}+g_{BD}L_{AC}-g_{AD}L_{BC}-g_{BC}L_{AD})\,, \tag{1.3}$$

so that we have the noncompact algebra of $O(4,2)$. We note, however, that the L_{AB} are not Hermitian. The anticommutator is now

$$\{L_{AE},L^{ED}\}=-\tfrac{1}{2}\delta_{AD}\quad (\text{no sum on } E). \tag{1.4}$$

Indices are raised by means of the metric as usual.

It is clear that Eqs. (1.3) and (1.4) hold also for the compact case with $g_{AB}=\delta_{AB}$ as the metric. We can thus use the operators L_{AB} without having to specify whether we are dealing with a unitary representation of $O(6)$ or a nonunitary representation of $O(4,2)$.

B. Oscillatorlike Representations of $O(6,C)$

The features which make the Dirac algebra suitable for use in a wave equation are actually independent of the representation and follow directly from the $O(6)$ algebra. It is expedient, therefore, to look for other representations (or families of representations) of $O(6)$ to describe particles of arbitrary spin.

We can choose 16 linearly independent operators Γ_R among the L_{AB} plus the unity, which are normalized to $\frac{1}{2}$ and which are a realization of generators of an $O(6,C)$ algebra. Thus

$$[\Gamma_R,\Gamma_S]=g_{RST}\Gamma_T$$

for suitable g_{RST}. Suppose now we introduce four-component boson operators a_μ with

$$[a_\mu,\bar{a}_\nu]=g_{\mu\nu}. \tag{1.5}$$

For the compact case the metric $g_{\mu\nu}=\delta_{\mu\nu}$ and for the noncompact case we take $g=(+\,+\,-\,-)$. Note that $g_{\mu\nu}$ is not directly related to the metric g_{AB} of $O(6,C)$. The matrix g can, of course, be expressed in terms of Dirac matrices. In fact, for our realization[6]

$$\begin{aligned} g&=1 \quad (\text{compact case})\\ &=\gamma_5 \quad (\text{noncompact case}). \end{aligned} \tag{1.6}$$

[3] I. Gyuk and H. Umezawa, Phys. Rev. Letters 22, 972 (1969).

[4] The Dirac matrices are discussed in the more restricted framework of $O(5)$ in A. MacFarlane, Commun. Math. Phys. 2, 133 (1966).

[5] We use the convention $(+1)^{1/2}=+1$, $(-1)^{1/2}=+i$.

[6] See Appendix C3, Eqs. (C1).

Clearly the a_μ are not the customary boson operators for the noncompact case. The notation will, however, result in a particularly succinct description of both $O(6)$ and $O(6,C)$ representations. To return to the usual boson operators, one need only make the substitution $(a_3,a_4) \leftrightarrow (\bar{a}_3,\bar{a}_4)$ at the end of a calculation.

We now investigate commutators for the quantities $\bar{a}(\Gamma_R g)a$ with g defined as above:

$$[\bar{a}(\Gamma_R g)a,\bar{a}(\Gamma_S g)a]=\bar{a}[\Gamma_R,\Gamma_S]ga=g_{RST}\bar{a}(\Gamma_T g)a.$$

The operators $\bar{a}(L_{AB}g)a$ have, therefore, exactly the same commutation relations as the L_{AB} themselves and form the generators of representations of $O(6,C)$. We shall define

$$M_{AB}=\bar{a}(L_{AB}g)a\,,\quad E=\bar{a}(g/2)a\,, \tag{1.7}$$

so that

$$[M_{AB},M_{CD}]=i(g_{AC}M_{BD}+g_{BD}M_{AC}-g_{AD}M_{BC}-g_{BC}M_{AD})\,. \tag{1.8}$$

Unlike the L_{AB}, the operators M_{AB} are Hermitian:

$$M_{AB}{}^\dagger=M_{AB}\,. \tag{1.9}$$

Because of their realization in terms of boson operators, we call the representations generated by the M_{AB} the oscillatorlike representations.

C. Auxiliary Relations for Generators

We now wish to find anticommutation relations similar to Eq. (1.4) for the operators M_{AB}. To this end we use the Fierz-Pauli transformation

$$F_{ij}G_{kl}=(\Gamma_R g)_{kj}(F\Gamma_R gG)_{il}\,. \tag{1.10}$$

F and G are arbitrary 4×4 matrices, and we sum over all 16 Γ_R, noting that the matrices $\Gamma_R g$ are essentially the compact set L_{AB}'. Using this relation, one obtains

$$\begin{aligned}\{M_{AE},M^{ED}\} &= (\bar{a}\Gamma_R ga)[\bar{a}(L_{AE}g\Gamma_R gL^{ED}g+L^{ED}g\Gamma_R gL_{AE}g)a] \\ &\quad -[\bar{a}\Gamma_R gg(L_{AE}g\Gamma_R gL^{ED}g+L^{ED}g\Gamma_R gL_{AE}g)a] \\ &\quad +\mathrm{Tr}(g\Gamma_R g)[\bar{a}(L_{AE}g\Gamma_R gL^{ED}g+L^{ED}g\Gamma_R gL_{AE}g)a] \\ &\equiv A_1+A_2+A_3\,.\end{aligned}$$

To evaluate A_1, we sum over all $\Gamma_R=L_{BC}$ and add the term in E. We also notice that

$$gL_{BC}g=L^{BC}; \tag{1.11}$$

thus

$$A_1=\tfrac{1}{2}M_{BC}[\bar{a}(L_{AE}L^{BC}L^{ED}+L^{ED}L^{BC}L_{AE})ga]+\tfrac{1}{2}E[\bar{a}\{L_{AE},L^{ED}\}ga]\,.$$

Using commutation and anticommutation relations for the L_{AB}, we obtain

$$(L_{AE}L^{BC}L^{ED}+L^{ED}L^{BC}L_{AE}) =-\tfrac{1}{2}\delta_{AD}L^{BC}+\delta_{AC}L^{BD}+g_{BD}L_A{}^C-\delta_{AB}L^{CD}-g_{CD}L_A{}^B\,.$$

With this relation and Eq. (1.4), we have for A_1

$$A_1=\tfrac{1}{8}\delta_{AD}\{M_{BC},M^{CB}\}-\{M_{AE},M^{ED}\}-\tfrac{5}{2}\delta_{AD}E^2.$$

The expression for A_2, using Eq. (1.11), becomes

$$A_2=-\bar{a}[(\Gamma_R L_{AE}\Gamma^R)L^{ED}g+(\Gamma_R L^{ED}\Gamma^R)L_{AE}g]a\,.$$

We now note that

$$\Gamma_R L_{AB}\Gamma^R=0\,, \tag{1.12}$$

as one can show from either Eq. (1.4) or Eq. (1.10). We have then $A_2=0$. Since $\mathrm{Tr}\Gamma^R$ is nonzero only for the unit matrix, we see that

$$A_3=\bar{a}\{L_{AE},L^{ED}\}ga=-5\delta_{AD}E\,.$$

Collecting terms, we obtain for the anticommutator

$$\{M_{AE},M^{ED}\}=2\delta_{AD}(1-N^2)\,, \tag{1.13}$$

where we have set $N=E+1$. This relation serves to define the family of representations of $O(6,C)$ given explicitly by Eq. (1.7).[7]

Further relations for the generators of $O(6,C)$ can be obtained by considering the anticommutator between E and M_{AB}. By a calculation analogous to the previous one, we have

$$\begin{aligned}2EM_{AB}&=\{E,M_{AB}\} \\ &=\tfrac{1}{4}M^{CD}(\bar{a}\{L_{AB},L_{CD}\}ga)+\tfrac{1}{4}E(\bar{a}\{I,L_{AB}\}ga) \\ &\quad -\tfrac{1}{2}(\bar{a}\{(\Gamma_R\Gamma^R)L_{AB}g+(\Gamma_R L_{AB}\Gamma^R)g\}a) \\ &\quad +\tfrac{1}{2}\bar{a}\{I,L_{AB}\}ga\,.\end{aligned}$$

The general anticommutator for the Dirac matrices L_{AB} is given by

$$\{L_{AB},L_{CD}\}=+\epsilon_{ABCDEF}L^{EF}+\tfrac{1}{2}(g_{AC}g_{BD}-g_{AD}g_{BC})I\,, \tag{1.14}$$

with the Levi-Civita symbol including a factor of -1 for the noncompact case by definition. Using this relation, we find, upon collecting terms,

$$NM_{AB}=+\tfrac{1}{8}\epsilon_{ABCDEF}M^{CD}M^{EF}\,. \tag{1.15}$$

This relation represents a generalization of the basic[8] $\gamma_5=-\gamma_1\gamma_2\gamma_3\gamma_4$.

D. Subgroups of $O(6,C)$

Among the various subgroups of $O(6,C)$ the most interesting ones are $O(3)$ and $O(4)$. The generators for these, are simply the sets $\{M_{ij}\}$ and $\{M_{\mu\nu}\}$ with i, j, $=1, \ldots, 3$ and μ, $\nu=1, \ldots, 4$. The commutation relations are those of Eq. (1.8) with suitable range of subscripts.

The generators of $O(3)$ form a vector $\mathbf{J}$ in the usual way. The sets $\{M_{i4}\}$, $\{M_{i5}\}$, and $\{M_{i6}\}$ can also be

[7] Equation (1.13) is derived for the noncompact case and with $N=0$ by A. Böhm, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XB.

[8] We choose this convention rather than $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. Nothing is essentially altered by either choice.

regarded as vectors **K**, **L**, and **M** because of the commutators

$$[J_i M_{ia}]=i\epsilon_{ijk}M_{ka}$$

for $a>3$. The operators M_{45}, M_{46}, and M_{56} all commute with **J** and are thus scalars. Commutators between these scalars close and, in fact, they form the generators of $O(3)$ or $O(2,1)$ for compact or noncompact $O(6,C)$, respectively. We may again make a vector out of the set of scalars, defining $\{I_i\}=\{M_{45},M_{46},M_{56}\}$. We now turn to the auxiliary relation (1.13). Taking $A=D$ and summing $A=1, \ldots, 3$ and $A=4, \ldots, 6$, respectively, we see that

$$2J^2+K^2+L^2+M^2=3(N^2-1),$$
$$2I^2+K^2+L^2+M^2=3(N^2-1),$$

and immediately

$$I^2=J^2. \tag{1.16}$$

Thus the representation generated by **I** will either be the same as that generated by **J** or else its complex extension.

To consider the subgroup $O(4)$, let us introduce the quantities N_c and N_n to denote the operator N for corresponding compact and noncompact representations. We note then that $M_{56}=I_3=N_n$ for the compact case and $M_{56}=I_3=-N_c$ for the noncompact case. To calculate the Casimir operators of $O(4)$, the auxiliary relation (1.13) yields

$$K^2+(I^2-I_3^2)=N^2-1.$$

Using $I^2=J^2$, we obtain for both $O(6)$ and $O(4,2)$

$$J^2+K^2=N_c^2+N_n^2-1. \tag{1.17}$$

Next we use Eq. (1.15) with $AB=56$:

$$NI_3=\mp\mathbf{J}\cdot\mathbf{K}$$

(plus sign for compact case). Again for both groups

$$\mathbf{J}\cdot\mathbf{K}=+N_cN_n. \tag{1.18}$$

Since relations (1.17) and (1.18) are invariant under $n\leftrightarrow c$ transition, the Casimir operators J^2+K^2 and $\mathbf{J}\cdot\mathbf{K}$, and consequently also the representations of the $O(4)$ subgroup, will be the same for both $O(4,2)$ and $O(6)$.

This complete reciprocity of $O(6)$ and $O(4,2)$ with respect to their $O(4)$ subgroup provides us with a concise criterion for choosing the oscillatorlike representations of $O(6,C)$. Suppose we wish to combine multiplets of $O(4)$ into supermultiplets. We must then find a larger group which contains $O(4)$ as a subgroup but which also includes among its generators a Casimir operator of $O(4)$. This is precisely the case for the oscillatorlike representations.[3] Our $O(6)$ and $O(4,2)$ representations result directly from using $M_{56}=N_n$ or $M_{56}=-N_c$. Since $O(4)$ is isomorphic to the direct product, $O(3)\times O(3)$, of the $O(3)$ groups generated, respectively, by $\mathbf{J}_\pm=\frac{1}{2}(\mathbf{J}\pm\mathbf{K})$, we can classify its representations as $D(\phi_+,\phi_-)$, with

$$(J_\pm)^2=\phi_\pm(\phi_\pm+1).$$

One finds then that

$$\begin{aligned}\phi_++\phi_-&=N_c-1,\\ \phi_+-\phi_-&=N_n.\end{aligned} \tag{1.19}$$

Clearly the inclusion of N_c as a generator necessitates a noncompact group and vice versa.

Finally we note that the sets $(\mathbf{L},I_1)$ and $(\mathbf{M},I_2)$ will form vectors under the $O(4)$ subgroup, while I_3 will, of course, be a scalar.

II. FREE PARTICLE

A. Lagrangian and Wave Functions

Having constructed an algebra, we shall now set up an appropriate formalism for the calculation of dynamical quantities. Introducing the wave function Ψ and its adjoint $\bar{\Psi}$, we shall use as our free-field Lagrangian

$$L=\int \mathfrak{L}dx^4,\quad \mathfrak{L}=-\bar{\Psi}(M_\mu\partial_\mu+m)\Psi. \tag{2.1}$$

M_μ is the operator $M_{\mu6}$ of $O(6,C)$ while m is a constant. From variations with respect to Ψ and $\bar{\Psi}$,

$$\delta L/\delta\bar{\Psi}=0,\quad \delta L/\delta\Psi=0,$$

we obtain as equations of motion for free fields

$$\begin{aligned}(M_\mu\partial_\mu+m)\Psi&=0,\\ \bar{\Psi}(M_\mu\overleftarrow{\partial}_\mu-m)&=0.\end{aligned} \tag{2.2}$$

If we introduce the operator p_μ by

$$p_\mu=-i\partial_\mu,$$

the wave equation (2.2) takes the form

$$M_\mu p_\mu\Psi=im\Psi. \tag{2.2$'$}$$

The requirement of Hermiticity for the Lagrangian L yields a relation between the adjoint wave function $\bar{\Psi}$ and the conjugate $\Psi^\dagger$ by means of the Hermitian operator η:

$$\bar{\Psi}=\Psi^\dagger\eta,\quad \eta=\eta^\dagger. \tag{2.3}$$

Owing to the fact that ∂_μ is not Hermitian,[9]

$$\partial_\mu{}^*=(\vec{\partial},-\partial_4),$$

η cannot commute with all the M_μ but obeys

$$\eta\begin{bmatrix}\mathbf{M}\\ M_4\end{bmatrix}\eta^{-1}=\begin{bmatrix}-\mathbf{M}\\ M_4\end{bmatrix}. \tag{2.4}$$

Besides assuring the Hermiticity of L, Eqs. (2.3) and (2.4) also imply that the wave equations for Ψ and $\bar{\Psi}$

[9] Since we are not interested in the noncompactness in time, we use $O(4)$ instead of the Lorentz group. While this allows all Γ_R to be Hermitian, $x_4=ix_0$ is not.

are conjugates of each other. Note that η can be interpreted as the parity operator since it produces inversion of the space components of a four-vector. The necessity of introducing the operator η to define the adjoint wave function stems from our use of a unitary representation of the group $O(4)$ rather than $O(3,1)$.

Next, we require the Lagrangian to be a scalar under Lorentz transformations. The divergence ∂_μ is a four-vector and transforms as

$$\partial_\mu' = A_{\mu\nu}\partial_\nu , \tag{2.5}$$

while for the wave functions Ψ and $\bar{\Psi}$ we set

$$\Psi' = \Lambda\Psi , \quad \bar{\Psi}' = \bar{\Psi}\Lambda . \tag{2.6}$$

The connection between the known $A_{\mu\nu}$ and the operator Λ is given by

$$\Lambda^{-1}M_\mu\Lambda = A_{\mu\nu}M_\nu . \tag{2.7}$$

For infinitesimal transformations, we can set

$$\begin{aligned} A_{\mu\nu} &= \delta_{\mu\nu} + \omega_{\mu\nu} , \\ \Lambda &= 1 + \tfrac{1}{2} i\omega_{\lambda\sigma}M_{\lambda\sigma} . \end{aligned} \tag{2.8}$$

Substitution in the previous relation shows this to be the correct form, provided the $M_{\lambda\sigma}$ are the $O(4)$ generators of our $O(6,C)$ algebra.

B. Explicit Form of Parity Operator

To find the specific form of the parity operator, consider the "rotation" induced among the generators by the operator $e^{i\phi R}$. This gives an automorphism for the algebra since similarity transformations leave commutators invariant. The result of such a rotation on some operator S is given by

$$S' = e^{i\phi R} S e^{-i\phi R} = \sum_{n=0}^{\infty} (i\phi)^n [R,S]_n / n! ,$$

where

$$[R,S]_n = [R,[R,[\cdots[R,S]\cdots]]]_{n\text{brackets}} .$$

Now if R and S happen to be noncommuting generators of $O(6,C)$, say M_{AB} and M_{CD}, then

$$[M_{AB},[M_{AB},M_{CD}]] = g_{AA}g_{BB}M_{CD} . \tag{2.9}$$

We can therefore evaluate the multiple commutators of the series expansion to obtain

$$\begin{aligned} M_{CD}' = M_{CD} \cos(g^{1/2}\phi) \\ + (-g)^{1/2}[M_{AB},M_{CD}] \sin(g^{1/2}\phi) . \end{aligned} \tag{2.10}$$

Here, g stands for $g_{AA}\ g_{BB}$ and equals ± 1. We now assume the space inversion operator η to be of the form $e^{i\phi R}$. To effect the transformation $M_i \to -M_i$, we can take any generator for R, provided it does not commute with the set M_i and provided we choose $\phi = \pm g^{1/2}\pi$. Since, on the other hand, R must commute with M_4, we arrive at the form

$$\eta \sim \exp(\pm i\pi g^{1/2} M_4) .$$

For the compact case, $g=1$ and thus $\phi=\pi$. For $O(4,2)$, however, $g=-1$ and thus the angle of rotation is complex.

Next, we shall make η a Hermitian operator. In the noncompact case the exponent is real, equaling πM_4. Hence η is already Hermitian, or $\eta^\dagger = \eta$. In the compact case, however, we have $\eta^\dagger = \eta$ only when the eigenvalues of M_4 are integral. When the spectrum is half-integral, we find that $\eta^\dagger = -\eta$. This can be remedied by using instead of M_4, the operator $(E - M_4)$. Since E and M_4 are either both integral or both half-integral, $(E-M_4)$ is always integral. Thus

$$\eta = \exp[\pi(-g)^{1/2}(E-M_4)] , \tag{2.11}$$

$$\eta^\dagger = \eta . \tag{2.12}$$

Note that in the compact case the operator η is unitary as well as Hermitian. It is therefore also normalized, $\eta^2 = 1$.

Our relation for the parity operator holds for all relevant representations of $O(6,C)$. However, for specific compact representations the exponential will become a polynomial due to constraints on the operator M_4. Thus for the Dirac algebra with $E=\frac{1}{2}$, we have $(M_4)^2 = \frac{1}{4}$ and hence

$$\eta = \exp[i\pi(\tfrac{1}{2} - M_4)] = 2M_4 = \gamma_4 ,$$

yielding the customary result. Similarly, for the Kemmer algebra we have $E=1$ and $(M_4)^3 = M_4$. The series expansion η becomes

$$\eta = \exp[i\pi(1-M_4)] = 2(M_4)^2 - 1 = 2(\beta_4)^2 - 1 = \eta_4 ,$$

again giving a familiar result.

C. Intrinsic Properties

We now investigate the properties of particles described by the Lagrangian (2.1) and the corresponding free-field equations of motion (2.2). Specifically, mass and spin spectra are already implied by these equations. The mass $\mathfrak{M}$ is defined in terms of the momentum p_μ as

$$\mathfrak{M}^2 = -p_\mu p_\mu . \tag{2.13}$$

To evaluate the spectrum of $\mathfrak{M}$, we introduce first the three-vector $\hat{p}_i$ by means of an imaginary angle ϕ:

$$\begin{aligned} p_i &= i\mathfrak{M} \sin\phi\, \hat{p}_i , \\ 1 &= \hat{p}_i\hat{p}_i . \end{aligned} \tag{2.14}$$

We then note the double commutator

$$[K_i\hat{p}_i,[K_i\hat{p}_i,M_4]] = M_4 .$$

This relation, which corresponds to Eq. (2.9), allows us to perform a rotation by means of the operator $\exp(i\phi K\cdot\hat{p})$ on M_4, with the result

$$\begin{aligned} \exp(i\phi K\cdot\hat{p})(i\mathfrak{M}M_4)\exp(-i\phi K\cdot\hat{p}) \\ = i\mathfrak{M}M_4\cos\phi + i\mathfrak{M}M_i\hat{p}_i\sin\phi \\ = M_\mu p_\mu . \end{aligned} \tag{2.15}$$

Since the eigenvalues of the operators $\exp(\pm i\phi K\cdot\hat{p})$ are inverses of each other, it follows that[10]

$$i\langle\mathfrak{M}M_4\rangle=\langle M_\mu p_\mu\rangle.$$

The term on the right-hand side of this relation is defined by the wave equation (2.2′) as equal to im. We obtain then for the mass

$$\langle\mathfrak{M}\rangle=m/\langle M_4\rangle. \tag{2.16}$$

Clearly, the operator $\exp(i\phi K\cdot\hat{p})$ is precisely the "booster" which relates the moving frame to the rest frame of the particle. Our approach is, however, quite different from the usual calculation, since we employ purely algebraic relations among operators instead of the rest-frame wave function.

If one defines the covariant Pauli-Lubanski operator W_μ by

$$W_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M_{\nu\lambda}p_\sigma, \tag{2.17}$$

then the value of the spin s^2 is given by

$$s^2=-W_\mu W_\mu/\mathfrak{M}^2. \tag{2.18}$$

Using the definitions of J_i and K_i in terms of $M_{\mu\nu}$, we find

$$W_\mu W_\mu=J^2p_4^2+(J\cdot p)^2+(K\times p)^2 -[(J\times K)-(K\times J)]\cdot pp_4. \tag{2.19}$$

To evaluate the spectrum of this quantity, we shall again use transformations by means of $\exp(i\phi K\cdot\hat{p})$. We note the double commutator

$$[K\cdot\hat{p},[K\cdot\hat{p},(\mathbf{J}-(J\cdot\hat{p})p)]]=\mathbf{J}-(J\cdot\hat{p})\hat{p}.$$

This allows us to use Eq. (2.10) for the transformation of $\{\mathbf{J}-(J\cdot\hat{p})\hat{p}\}$. The helicity vector $(J\cdot\hat{p})\hat{p}$, which is orthogonal to this, commutes with $K\cdot\hat{p}$. For $\mathbf{J}$ itself we obtain therefore

$$\exp(i\phi K\cdot\hat{p})\begin{bmatrix}\mathbf{J}-(J\cdot\hat{p})\hat{p}\\ (J\cdot\hat{p})\hat{p}\end{bmatrix}\exp(-i\phi K\cdot\hat{p}) =\begin{bmatrix}\{\mathbf{J}-(J\cdot\hat{p})\hat{p}\}\cos\phi-(K\cdot\hat{p})\sin\phi\\ (J\cdot\hat{p})\hat{p}\end{bmatrix}.$$

Squaring this and using the definition (2.19), we have for the scalar J^2

$$\exp(i\phi K\cdot\hat{p})J^2\exp(-i\phi K\cdot\hat{p})=-W_\mu W_\mu/\mathfrak{M}^2. \tag{2.20}$$

By the same argument as in the case of $M_\mu p_\mu$, we obtain thus for the spectrum of s^2

$$\langle s^2\rangle=\langle J^2\rangle. \tag{2.21}$$

D. Spectra for Mass and Spin

We must now evaluate the spectrum of the quantities M_4 and $\mathbf{J}$. The operator $\mathbf{J}$ will be characterized by the $O(3)$ representation $D(\phi_J)$ and the value of, say, J_3.

[10] We shall use the notation $\langle T\rangle$ for the expectation value $\bar{\Psi}T\Psi$.

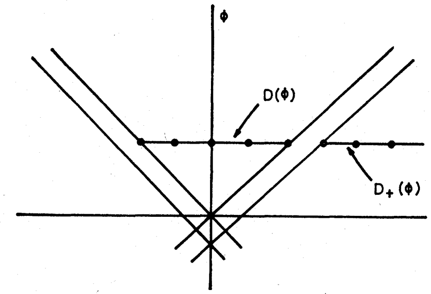

FIG. 1. Compact and noncompact representations of $O(3,C)$.

Eigenvalues of $M_4=I_2$ are defined by the representation $D(\phi_I)$. Since Eq. (1.16) yields $J^2=I^2$, we have also that

$$\phi_J=\phi_I\equiv\phi, \tag{2.22}$$

ϕ having been introduced as the common value of ϕ_J and ϕ_I. The spectrum of both J_3 and I_2 follows immediately[11] as

$$-\phi\leqslant J_3\leqslant+\phi,\quad -\phi\leqslant I_2\leqslant+\phi\quad\text{for } O(6), \tag{2.23}$$

$$-\phi\leqslant J_3\leqslant+\phi,\quad (\phi+1)\leqslant I_2\quad\text{for } O(4,2). \tag{2.24}$$

For the noncompact case, we have taken the representation $D_+(\phi)$ as the relevant complex extension of $D(\phi)$. Note that the spectrum of $M_5=I_3$, if we choose this operator diagonal, is the same as that of I_2. The representations are shown in Fig. 1.

Next, we recall that the $O(3)$ algebra generated by the J_i is a subalgebra of the $O(4)$ generated by $M_{\mu\nu}$. For the representations $D(\phi_+,\phi_-)$ of this $O(4)$ we obtain, using the expressions for ϕ_+ and ϕ_- from Eq. (1.19) and the proper identification of N_c and N_n,

$$D(\tfrac{1}{2}(E+M_5),\tfrac{1}{2}(E-M_5))\quad\text{for } O(6),$$
$$D(-\tfrac{1}{2}(M_5-E),-\tfrac{1}{2}(M_5+E+2))\quad\text{for } O(4,2).$$

However, in the case of $O(4,2)$ we notice that $\phi_\pm$ are negative and we shall switch, therefore, to the equivalent representations $\phi_\pm'=-(\phi_\pm+1)$, so that we have

$$D(\tfrac{1}{2}(M_5-E-2),\tfrac{1}{2}(M_5+E))\quad\text{for } O(4,2).$$

Now the representations $D(\phi)$ contained in $D(\phi_+,\phi_-)$ are given by the inequality $|\phi_+-\phi_-|\leqslant\phi\leqslant|\phi_++\phi_-|$, each possible representation occurring exactly once. Accordingly,

$$|M_5|\leqslant\phi\leqslant E\quad\text{for } O(6),$$
$$(E+1)\leqslant\phi\leqslant|M_5-1|\quad\text{for } O(4,2).$$

Since the inequalities involving M_5 are satisfied automatically because of Eqs. (2.23) and (2.24), they do not form any limitation on ϕ and can be deleted. We obtain

[11] In relations such as $(\phi+1)\leqslant I_2$ we mean, of course, that the values of the spectrum of I_2 are greater or equal than $(\phi+1)$. When no danger of confusion exists, we shall freely use the same symbol both for the operator and for its eigenvalue in a specific state.

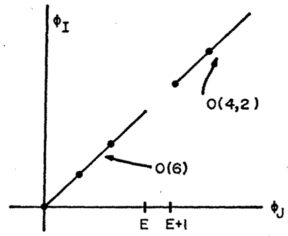

FIG. 2. Spectrum of ϕ_I and ϕ_J in $O(6)$ and $O(4,2)$.

simply

$$\phi \leqslant E \quad \text{for } O(6), \tag{2.25}$$

$$\geqslant E+1 \quad \text{for } O(4,2). \tag{2.26}$$

These relations are depicted in Fig. 2.

III. ELECTROMAGNETIC INTERACTION

A. Field Invariants

We shall now calculate the various conserved quantities arising by Noether's theorem from invariance of the Lagrangian under translation, rotation, and gauge transformation. In order to find covariant expressions for the current and energy-momentum and angular-momentum tensor, we introduce a symmetrized Lagrangian density

$$\mathcal{L}=\bar{\Psi}(M_\mu \overleftrightarrow{\partial}_\mu - m)\Psi. \tag{3.1}$$

We are using $\overleftrightarrow{\partial}_\mu = \frac{1}{2}(\overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu)$. The symmetrized Lagrangian density is, of course, equivalent to that of Eq. (2.1). One can now introduce the current as

$$j_\mu = -i\left(\frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}}\Psi - \bar{\Psi}\frac{\partial \mathcal{L}}{\partial \bar{\Psi}_{,\mu}}\right),$$

with the notation $\Psi_{,\mu}=\partial\Psi/\partial x_\mu$. One obtains then from our Lagrangian

$$j_\mu = i\bar{\Psi}M_\mu\Psi. \tag{3.2}$$

This quantity is a four-vector in $O(3,1)$. We note that the current is conserved:

$$\partial_\mu j_\mu = 0. \tag{3.3}$$

The fourth component of the current, j_4, can be interpreted as a probability density.[12] Its integral over all space (a covariant quantity) is used to define the norm of the wave functions:

$$\int j_4 dx^3 = \int \bar{\Psi}M_4\Psi dx^3 = 1. \tag{3.4}$$

The energy-momentum tensor is defined as

$$T_{\mu\nu} = \left(\frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}}\Psi_\nu + \bar{\Psi}_\nu \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_{,\mu}}\right) - \delta_{\mu\nu}\mathcal{L}.$$

[12] j_0 is a Hermitian quantity and must thus be used rather than j_4. Similarly, the physical spin and momentum density are given by $S_{\mu\nu}^0$ and $J_{0\mu}$.

Since the value of the Lagrangian is zero, this becomes

$$T_{\mu\nu} = \bar{\Psi}M_\mu \overleftrightarrow{\partial}_\nu \Psi. \tag{3.5}$$

We note again the conservation law

$$\partial_\mu T_{\mu\nu} = 0. \tag{3.6}$$

The angular-momentum tensor consists of two parts. Orbital angular momentum is given by

$$M_{\mu\nu}{}^\lambda = x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu}, \tag{3.7}$$

while the spin is

$$S_{\mu\nu}{}^\lambda = -\left(\frac{\partial \mathcal{L}}{\partial \Psi_{,\lambda}} A_{\mu\nu}\Psi + \bar{\Psi}A_{\mu\nu}{}^\dagger \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_{,\lambda}}\right).$$

The operator A is defined from the infinitesimal transformation

$$\Psi' = (I + \tfrac{1}{2}A_{\mu\nu}\omega_{\mu\nu})\Psi.$$

By comparison with Eq. (2.8), we see that

$$A_{\mu\nu} = iM_{\mu\nu},$$

and thus

$$S_{\mu\nu}{}^\lambda = \mathrm{f}\bar{\Psi}\{M_\lambda, M_{\mu\nu}\}\Psi. \tag{3.8}$$

The total angular momentum is conserved:

$$\partial_\lambda S_{\mu\nu}{}^\lambda = -\partial_\lambda M_{\mu\nu}{}^\lambda.$$

By using the explicit form of $M_{\mu\nu}{}^\lambda$ and the conservation law for $T_{\mu\nu}$, we obtain from this the useful relation

$$\partial_\lambda S_{\mu\nu}{}^\lambda = T_{\mu\nu} - T_{\nu\mu}. \tag{3.9}$$

B. Magnetic Moment

We shall now introduce minimal electromagnetic coupling by means of the substitution

$$\overleftrightarrow{\partial}_\mu \to \overleftrightarrow{\partial}_\mu + ieA_\mu$$

into the Lagrangian of Eq. (3.1). We obtain then for the interaction Lagrangian $\mathcal{L}'$

$$\mathcal{L}' = ej_\mu A_\mu. \tag{3.10}$$

If the potential A_μ gives rise to a constant electromagnetic field $F_{\mu\nu}{}^0$ then

$$A_\mu = \tfrac{1}{2}x_\nu F_{\nu\mu}{}^0.$$

Now, introducing the quantity $m_{\mu\nu}$ as

$$m_{\mu\nu} = \tfrac{1}{2}e(j_\mu x_\nu - j_\nu x_\mu), \tag{3.11}$$

we see that

$$\mathcal{L}' = \tfrac{1}{2}m_{\mu\nu}F_{\nu\mu}{}^0. \tag{3.12}$$

The quantity $m_{\mu\nu}$—or rather its spatial part—is thus the magnetic moment density. The magnetic moment itself is given by

$$\mathfrak{M}_{ij} = \tfrac{1}{2}e\int (j_i x_j - j_j x_i)dx^3. \tag{3.13}$$

To evaluate this, we note that $T_{4\nu}$ represents a momentum density. In the rest frame, its space components will therefore be zero. Furthermore, since the eigenvalue of $\overleftrightarrow{\partial}_4$ is $\mathfrak{M}$, the quantity $T_{\mu 4}$ is just proportional to the current. In the rest frame, therefore, Eq. (3.9) becomes

$$\partial_\lambda S_{i4}{}^\lambda = -i\mathfrak{M} j_i \quad \text{(rest frame)}.$$

We can use this result in Eq. (3.13), and upon applying partial integration, we obtain

$$\mathfrak{M}_{ij} = \frac{ie}{2\mathfrak{M}} \int [S_{i4}(\partial_\lambda x_j) - S_{j4}{}^\lambda(\partial_\lambda x_i)] dx^3$$
$$= \frac{ie}{2\mathfrak{M}} \int (S_{i4}{}^j - S_{j4}{}^i) dx^3. \tag{3.14}$$

Note that thus far the calculation has been completely independent of the choice of either $O(6)$ or $O(4,C)$. Unfortunately, the quantity $S_{i4}{}^j - S_{j4}{}^i$ is not directly the spin density $S_{ij}{}^4$, to which, however, it can be related by means of the auxiliary relation (1.15) with $A, B = k$, 5. We find then[13]

$$S_{i4}{}^j - S_{j4}{}^i = S_{ij}{}^4 + i\epsilon_{ijk} N \bar{\Psi} L_k \Psi. \tag{3.15}$$

This expression is still not a pure spin density. Since L_3 commutes with both J_3 and M_4, we can, in fact, diagonalize this quantity. The magnetic moment need not, therefore, be proportional to the spin, but we have in general[12]

$$\mathfrak{M}_{ij} = \frac{-e}{2\mathfrak{M}} \int S_{ij}{}^0 dx^3 - \frac{ie}{2\mathfrak{M}} \epsilon_{ijk} \int \bar{\Psi} L_k \Psi dx^3. \tag{3.16}$$

Although L_3 can be diagonalized simultaneously with J_3 and M_4, it does not commute with J^2. A component of the magnetic moment cannot, therefore, be evaluated for a particle in a definite spin state.[14] This seems to indicate that, in fact, interaction with a photon will induce transitions to different spins. A similar situation might well occur in other theories involving spin multiplets.

As a special case we consider the Dirac algebra. Here $N = \frac{3}{2}$ and we obtain, for this representation only, that $\mathbf{L} = -2M_4\mathbf{J}$. Therefore,

$$i\epsilon_{ijk} N \bar{\Psi} L_k \Psi = -3S_{ij}{}^0 \tag{3.17}$$

and the magnetic moment becomes

$$\mathfrak{M}_{ij} = 2\frac{e}{2\mathfrak{M}} \int S_{ij}{}^0 dx^3, \tag{3.18}$$

thus giving the correct g factor $g = 2$. Another interesting result follows by considering only that part of $\mathbf{L}$ which is proportional to $\mathbf{J}$. Again, using auxiliary relation (1.15), now with $A, B = 4$, 6, we find

$$NM_4 = -J \cdot L$$

and

$$(N\mathbf{L}) \cdot \mathbf{J} = (-N^2 M_4 \mathbf{J}/J^2) \cdot \mathbf{J}. \tag{3.19}$$

Thus if we consider only the projection of the magnetic moment operator on the spin, we have[15]

$$\mathfrak{M}_{ij}' = g\frac{e}{2\mathfrak{M}} \int S_{ij}{}^0 dx^3,$$
$$g = (N^2/J^2 - 1). \tag{3.20}$$

APPENDIX A: ORTHOGONALITY RELATIONS FOR Γ MATRICES

Recognition of the quantities Γ^R as infinitesimal generators of $O(6)$ can yield illuminating derivations of Dirac matrix results. We shall consider here the grand orthogonality relation for the matrix elements of an n-dimensional representation:

$$\int D_{\mu\nu}(\omega) D_{\sigma\lambda}{}^*(\omega) d\omega = \delta_{\mu\sigma\lambda\nu\lambda} \int \frac{d\omega}{n}. \tag{A1}$$

Integration is over the entire group which, for the fundamental representation of $O(6)$, is described by the 15 parameters ω_R. In terms of the generators Γ_R,

$$D(\omega) = \exp(\omega_R \Gamma^R), \quad R = 1, \ldots, 15. \tag{A2}$$

Expanded, the exponential takes the form

$$D(\omega) = I \sin(\tfrac{1}{4}\omega_R\omega_R)^{1/2} + 2i[\omega_S \Gamma^S/(\omega_T\omega_T)^{1/2}] \cos(\tfrac{1}{4}\omega_R\omega_R)^{1/2}. \tag{A3}$$

Use was made in the expansion of the result

$$(\omega_R \Gamma^R)^2 = \tfrac{1}{2}\{\Gamma^R, \Gamma^S\}\omega_R\omega_S = \tfrac{1}{4}\delta_{RS}\omega_R\omega_S = \tfrac{1}{4}\omega_R\omega_R,$$

which is obtained with the aid of the anticommutator of Eq. (1.14). Actually we do not need the explicit expression of Eq. (A3), but only the general form of $D(\omega)$ and $D^*(\omega)$. Using the Hermiticity of the Γ^R,

$$D(\omega) = Ia(\omega) + 2i\Gamma^R b_R(\omega),$$
$$D^*(\omega) = \tilde{I}a(\omega) - 2i\tilde{\Gamma}^R b_R(\omega). \tag{A4}$$

When this is substituted into the orthogonality relation (A1), we obtain

$$I_{\mu\nu} I_{\lambda\sigma} A + 2i(I_{\mu\nu}\Gamma_{\lambda\sigma}{}^R - \Gamma_{\mu\nu}{}^R I_{\lambda\sigma}) A_R + 4\Gamma_{\mu\nu}{}^R \Gamma_{\lambda\sigma}{}^S B_{RS} = \delta_{\mu\sigma}\delta_{\nu\lambda} C. \tag{A5}$$

A, A_R, B_{RS}, and C are integral expressions the direct evaluation of which is rather tedious. Instead, multiplying by $\Gamma_{\mu\nu}{}^A$ and $\Gamma_{\sigma\lambda}{}^B$, we have

$$4(\Gamma^A\Gamma^R)_{\mu\mu}(\Gamma^S\Gamma^B)_{\lambda\lambda} B_{RS} = (\Gamma^A\Gamma^B)_{\sigma\sigma} C,$$

[13] Equation (3.15) is again true for both $O(6)$ and $O(4,2)$, ϵ_{ijk} being here defined the same for both algebras.

[14] Consequently all we can say about the spectrum of either $|L_3|$ or $|J_3|$ is that it will be less than or equal to E for $O(6)$ and greater than or equal to N for $O(4,2)$.

[15] Result (3.20) is given in Ref. 2. $\mathfrak{M}_{ij}'$ does not, however, yield the value of $\mathfrak{M}_{ij}$ in general.

all 15 Γ^R being traceless. Furthermore,

$$B_{AB}=\tfrac{1}{4}\delta_{AB}C.$$

Using this result, we set $\nu=\lambda$ in Eq. (A5) and sum. Then

$$A+(15/4)C=4C.$$

Finally setting $\mu=\nu$ in Eq. (A5), we find, by comparing with the previous relation, that the A_R must be zero. This essentially evaluates the integral expressions and the orthogonality relation becomes

$$\Gamma_{\mu\nu}{}^R\Gamma_{\lambda\sigma}{}^R=\delta_{\mu\sigma}\delta_{\nu\lambda},\qquad R=0,\ldots,15. \tag{A6}$$

Γ^0 has been introduced as $\frac{1}{2}I$. We recognize Eq. (A6) as the completeness relation for the Γ^R. In passing, we note that the derivation holds also if the Γ^R are replaced by the generators of $O(3)$ the Pauli matrices σ^R. As a useful corollary we obtain from Eq. (A6), by multiplying with arbitrary $F_{\alpha\lambda}$ and $G_{\sigma\beta}$, the Fierz-Pauli transformation

$$F_{\alpha\nu}G_{\mu\beta}=\Gamma_{\mu\nu}{}^R(F\Gamma^RG)_{\alpha\beta}. \tag{A7}$$

APPENDIX B: OSCILLATORLIKE REPRESENTATIONS IN $U(4,C)$

Since the algebras of $O(6,C)$ and $U(4,C)$ are isomorphic, it is of interest to find the representations of $U(4,C)$ corresponding to those of $O(6,C)$ which we have considered. To this end, we note that the 16 independent Dirac matrices (including unity) form a complete set of 4×4 matrices. The two sets of 16 operators

$$\bar{a}(\Gamma_Rg)a \quad\text{and}\quad \bar{a}_\mu a_\nu$$

will therefore also be equivalent. Accordingly we introduce

$$A_{\mu\nu}=\bar{a}_\mu a_\nu \tag{B1}$$

and obtain the commutators

$$[A_{\mu\nu},A_{\sigma\lambda}]=A_{\mu\lambda}g_{\nu\sigma}-A_{\sigma\nu}g_{\mu\lambda}, \tag{B2}$$

which show that the $A_{\mu\nu}$ are a realization of the generators of $U(4,C)$.

We recognize the operators $A_{\mu\nu}$, in the compact case, as the generators of the invariance group of a four-dimensional harmonic oscillator. The representations must then be those which correspond to the symmetric Young tableaux. For the noncompact case, we have just the relevant complex extensions. Now the number of boxes of a Young tableau for a given representation is equal to the linear Casimir operator. Thus

$$C=g_{\nu\mu}A_{\mu\nu}=\bar{a}ga=2E. \tag{B3}$$

To elucidate the role of the quantity E, we shall make the subduction of $U(4)$ with respect to $U(2)\times U(2)$. This yields

$$(2E)=(2E,2E)+(2E-2,\,2E-2)+(2E-4,\,2E-4)+\cdots, \tag{B4}$$

representations of $U(4)$ and $U(2)$ being labeled by the number of boxes in the corresponding symmetric Young tableaux. Since the two commuting $U(2)$ algebras can be interpreted as those generated by J_i and I_i, we see that $J^2=I^2$ as in Eq. (1.16). We also see immediately that E is the maximum spin contained in the (compact) representation and that all lower spins are contained in exactly one irreducible representation of $U(2)\times U(2)$. This is borne out by the dimensionality of the representations, which for symmetric tableaux of $U(4)$ is equal to

$$\tfrac{1}{6}(2E+1)(2E+2)(2E+3).$$

Alternatively, we find for this quantity from Eq. (B4)

$$(2E+1)^2+(2E-1)^2+(2E-3)^2+\cdots,$$

which is equal to the previous relation. Thus the Dirac algebra with $E=\frac{1}{2}$ corresponds to the fundamental Young tableau of a single box and contains $2^2=4$ states. The Kemmer algebra, $E=1$, corresponds to the symmetric tableau with two boxes and contains $3^2+1^2=10$ states.

Finally, we note the bilinear auxiliary relation

$$\{A_{\mu\nu},A_{\sigma\lambda}\}=\{A_{\mu\lambda},A_{\sigma\nu}\}+g_{\sigma\nu}A_{\mu\lambda}+g_{\mu\lambda}A_{\sigma\nu}-g_{\sigma\lambda}A_{\mu\nu}-g_{\mu\nu}A_{\sigma\lambda}, \tag{B5}$$

which follows directly from our realization. There are 36 independent relations contained in this equation, corresponding therefore to Eqs. (1.13) and (1.15) which contain, respectively, 21 and 15 relations.

APPENDIX C: REALIZATION OF NONCOMPACT GENERATORS BY ORDINARY BOSON OPERATORS

We have given a realization for the oscillatorlike representations of $O(4,2)$ in terms of boson operators with the commutator

$$[a_\mu,\bar{a}_\nu]=g_{\mu\nu}.$$

The generators M_{AB} are then given by

$$M_{AB}=\bar{a}(L_{AB}g)a.$$

We have also noted that the quantities $L_{AB}g$ are—up to minus signs—just the Dirac matrices. It is expedient to have expressions for the generators in terms of ordinary boson operators as well.

For convenience, we shall use a special realization for the generators of the fundamental representation of $O(6)$—that is, the Dirac matrices L_{AB}':

$$\begin{aligned}&J_i=\tfrac{1}{2}(1\times\sigma_i),\quad M_i=\tfrac{1}{2}(\sigma_1\times\sigma_i),\\&K_i=\tfrac{1}{2}(\sigma_3\times\sigma_i),\quad I_i=\tfrac{1}{2}(\sigma_i\times1),\\&L_i=-\tfrac{1}{2}(\sigma_2\times\sigma_i),\ E=\tfrac{1}{2}(1\times1).\end{aligned} \tag{C1}$$

This realization, since it represents all the L_{AB}' as Kronecker products of Pauli matrices, is well suited to the transformation of quantities involving quasiboson operators into expressions of regular boson operators. We thus apply the substitution $(a_3,a_4) \leftrightarrow (\bar{a}_3,\bar{a}_4)$ to the M_{AB}. The result, however, is not yet in very useful form. Accordingly we make the following automorphism among the generators:

$$\begin{aligned}(1)&\to\ (3)\,, & (4)&\to-(2)\,,\\ (2)&\to\ (4)\,, & (5)&\to\ (6)\,,\\ (3)&\to-(1)\,, & (6)&\to\ (5)\,.\end{aligned}$$

The notation $(3)\to-(1)$, for example, means that for the M_{AB} we replace the index 3 by the index 1 and put a minus sign in front of the resultant operator. Thus $M_{23}\to -M_{41}$ or $J_1\to K_1$. After applying this automorphism, the generators present themselves in very simple form:

$$\begin{aligned}M_{\mu\nu}&=\tfrac{1}{2}\bar{a}\sigma_{\mu\nu}a\,,\\ L_\mu&=\tfrac{1}{4}(\bar{a}\gamma_\mu C\bar{a}+aC\gamma_\mu a)\,,\\ M_\mu&=\tfrac{1}{4}i\{\bar{a}\gamma_\mu C\bar{a}-aC\gamma_\mu a\}\,,\\ M_5&=\tfrac{1}{4}\{\bar{a}1a+a1\bar{a}\}\,.\end{aligned}\tag{C2}$$

The operators a and $\bar{a}$ are now ordinary boson operators. $\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu)/i$ and C is defined by $C\gamma_\mu C=-\tilde{\gamma}_\mu$, in our case equal to $(1\times\sigma_2)$. We must introduce C because $\bar{a}\gamma_\mu C\bar{a}$ is a four-vector, whereas $\bar{a}\gamma_\mu\bar{a}$ is not. Note that with this realization the generators for the $O(4)$ subgroup are explicitly the same for both $O(6)$ and $O(4,2)$. Furthermore, we have now that $M_5=N_c$ for $O(4,2)$ and $M_5=N_n$ for $O(6)$, so that the relationship between the two sets of representations is truly reciprocal.

APPENDIX D: GENERALIZED LINEAR WAVE EQUATION

It is of interest to investigate more general forms of wave equations. We have available the four-vectors M_μ and L_μ as well as the scalars—under $O(4)$—N and M_5. If we restrict ourselves to expressions linear in the dynamical vector quantities p_μ and w_μ, we can use in our Lagrangian the combinations $M_\mu p_\mu$, $M_\mu w_\mu$, $L_\mu p_\mu$, $L_\mu w_\mu$, mN, and $m'M_5$, all of which yield scalars under $O(4)$. We consider first a term of the form

$$(M_\mu \cos\alpha+L_\mu\sin\alpha)p_\mu=M_\mu' p_\mu\,,\tag{D1}$$

with arbitrary α. Because of the commutators

$$[M_5,M_\mu]=-igL_\mu\,,$$

with $g=g_{55}=g_{66}$, we find using Eq. (2.10)

$$e^{i\alpha gM_5}M_\mu e^{-i\alpha gM_5}=M_\mu'\,.\tag{D2}$$

The spectra of M_μ and M_μ' are thus the same, making a term in $L_\mu p_\mu$ superfluous.

To treat the quantity $w_\mu M_\mu$, we use the definition for w_μ, Eq. (2.17), to obtain

$$w_\mu M_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M_{\nu\lambda}M_{\mu 6}p_\sigma\,.$$

Observing that $\epsilon_{\mu\nu\lambda\sigma}\equiv g\epsilon_{\mu\nu\lambda\sigma 56}$, we have

$$w_\mu M_\mu=\tfrac{1}{2}g^2\epsilon_{\sigma 5\nu\lambda\mu 6}M^{\nu\lambda}M^{\mu 6}p_\sigma=\tfrac{1}{8}\epsilon_{\sigma 5CDEF}M^{CD}M^{EF}p_\sigma\,.$$

With the auxiliary relation (1.15), this becomes simply

$$w_\mu M_\mu=NL_\sigma p_\sigma\,.\tag{D3}$$

The spectrum of the wave equation is therefore unaffected also by the inclusion of the terms $w_\mu M_\mu$ or $w_\mu L_\mu$.

A nontrivial modification of the Lagrangian does, however, result from the term $m'M_5$. To consider the new wave equation

$$-(iM_\mu p_\mu+m'M_5)\Psi=m\Psi\,,\tag{D4}$$

we note first that according to Eq. (2.15)

$$\begin{aligned}\exp(i\phi K\cdot\hat{p})(\mathfrak{M}M_4-m'M_5)\exp(-i\phi K\cdot\hat{p})\\ =-(iM_\mu p_\mu+m'M_5)\,,\end{aligned}\tag{D5}$$

since $K\cdot\hat{p}$ commutes with M_5. The operators M_4 and M_5 can, of course, not be diagonalized simultaneously. Suppose, however, we introduce the transformation

$$e^{i\xi L_4}M_4e^{-i\xi L_4}=M_4\cos(g^{1/2}\xi)+i(-g)^{1/2}M_5\sin(g^{1/2}\xi)\,.$$

If we let

$$\tan(g^{1/2}\xi)=g^{1/2}m'/\mathfrak{M}\tag{D6}$$

and define

$$\mathfrak{M}'=(\mathfrak{M}^2+gm'^2)^{1/2}\,,\tag{D7}$$

then

$$e^{i\xi L_4}(\mathfrak{M}'M_4)e^{-i\xi L_4}=\mathfrak{M}M_4-m'M_5\,.\tag{D8}$$

Therefore, we have

$$\langle\mathfrak{M}'M_4\rangle=m$$

and the spectrum of $\mathfrak{M}$ follows immediately as

$$\langle\mathfrak{M}\rangle=\pm[m^2/(\langle M_4\rangle^2-gm'^2)]^{1/2}\,.\tag{D9}$$

The sign of the square root is the same as that of M_4. In the noncompact case, with $g=-1$, the modified Lagrangian leads to the wave equation of Nambu. For the compact case, the Dirac and Duffin-Kemmer equations remain unaltered—except for a redefinition of m.