It is interesting to point out also that the well-known correlation between the $e^+e^- \to \pi^+\pi^-(I=1)$ total cross section and the spectral density of the currents is also identically obeyed in our formalism if the two discussed unitarity relations are satisfied.

Recently Schnitzer,[6] using the Ward-identity method, has developed a program of unitarization of current-algebra amplitudes. The approach described in this paper is equivalent to his "unitarized tree approximation" in the same sense as the usual chiral Lagrangian method is equivalent to the usual current-algebra method, although the relation is not obvious because he admitted the existence of an s-wave dipion resonance.

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Internal Symmetry Bootstrapped from the Veneziano Representation*

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Starting with only charge conservation, we obtain $SU(2)$ symmetry as a result of imposing unitarity constraints on a set of Veneziano amplitudes.

AMONG the many nice features of the Veneziano formula,[1] the duality property is perhaps the most attractive one from a bootstrap viewpoint. During the recent attempts to include internal symmetry and spin in the Veneziano model, several suggestive links with the quark models have emerged. Harari and Rosner[2] have given simple graphical rules, in terms of quarks, for determining the exotic channels and exhibiting the duality nature. In this connection, an earlier prescription given by Chan and Paton[3] for the internal symmetry factors can be thought of as a mathematical formulation of the duality diagrams of nonstrange quarks. Bootstrap models based on mathematical quarks providing both internal-symmetry factors as well as spin factors have been considered by Mandelstam[4] and by Bardakci and Halpern.[5]

In this paper we attack the internal-symmetry problem from a different angle. Unlike the previous authors, we do not assume any internal symmetry to begin with. It is hoped that internal symmetry for the external particles would emerge as a natural consequence of the general conditions imposed on the model. Indeed, as we shall present in the following example, $SU(2)$ symmetry is obtained for a set of Veneziano amplitudes when the unitarity restriction is utilized. While a unitarity equation cannot be explicitly used in a narrow-resonance approximation, we rely heavily on the power of factorization.

Owing to its nonlinear nature, the unitarity also puts a constraint on the normalization of amplitudes. For any two amplitudes satisfying the same elastic unitarity equation, it is evident that they must be equal if they are proportional. When two Veneziano amplitudes are proportional to each other, we shall assume that they are equal if the exact amplitudes they approximated are of the above type.

If some approximate unitarization scheme would allow us to determine uniquely the over-all normalization of a given Veneziano amplitude, then the above assumption will not be needed. For example, Igi and Storrow[6] recently have done an interesting calculation with the leading-term Veneziano $\pi\pi$ amplitude[7] of $I=1$. By using the unitarity sum rule, they were able to determine the over-all normalization for the amplitudes.

The system we considered consists of three spinless mesons (of either parity) of equal mass, designated by ψ^+, ψ^-, and ψ^0. Their antiparticles form the same set. The superscripts stand for some additive, conserved quantum numbers; for convenience we call them charge. In order to write down the Veneziano amplitudes for ψ-ψ scattering, we have to specify the trajectories in various channels. For this purpose we use duality diagrams[2] as a guide.

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We introduce four quarklike particles, $\mathcal{P}$, $\mathfrak{N}$, and their antiparticles $\bar{\mathcal{P}}$, $\bar{\mathfrak{N}}$. But unlike quarks, we *do not assume* any internal symmetry between $\mathcal{P}$ and $\mathfrak{N}$. The charge assignment of $\mathcal{P}$ can be arbitrary, and the charge of $\mathfrak{N}$ is one unit lower. To draw the duality diagrams, ψ^+ is associated with the pair of lines ($\mathcal{P}\bar{\mathfrak{N}}$), ψ^- with ($\bar{\mathcal{P}}\mathfrak{N}$), and ψ^0 with a combination ($\mathcal{P}\bar{\mathcal{P}}$) and ($\mathfrak{N}\bar{\mathfrak{N}}$). Following Ref. 2, we only allow the "planar" diagrams. For each $\psi\psi\to\psi\psi$ process, we can draw a definite set of duality diagrams. Taking $\psi^+\psi^0 - \psi^+\psi^0$ as an example, we have six distinct diagrams, as shown in Fig. 1. The solid (dashed) line stands for $\mathcal{P}(\mathfrak{N})$. Incoming (outgoing) particles are drawn at the lower (upper) corners of each diagram. Let α_1, α_2, and α_3 denote respectively the leading trajectory formed by ($\mathcal{P}\bar{\mathfrak{N}}$), ($\mathcal{P}\bar{\mathcal{P}}$), and ($\mathfrak{N}\bar{\mathfrak{N}}$). Diagrams 1(a) –1(d), in respective order, are associated with the Veneziano terms $A(s_1,t_3)$, $A(s_1,t_2)$, $A(t_3,u_1)$, and $A(t_2,u_1)$. Diagram 1(e) has the same contribution as 1(f), and is associated with $A(s_1,u_1)$. We use the shorthand notation

$$A(x_i,y_j)\equiv\Gamma(1-\alpha_i(x))\Gamma(1-\alpha_j(y))/\Gamma(1-\alpha_i(x)-\alpha_j(y)),$$

with $i,j=1, 2, 3$, and $x, y=s, t, u$.

Crossing symmetry requires that diagrams 1(a) and 1(c), as well as 1(b) and 1(d), carry the same weight. Thus we can write

$$A^{+0\to+0}(s,t,u)=\beta_1[A(s_1,t_2)+A(u_1,t_2)] +\beta_2[A(s_1,t_3)+A(u_1,t_3)]+\beta_3A(s_1,u_1). \quad (1)$$

In the same manner we obtain

$$A^{+-\to+-}(s,t,u)=\beta_4[A(s_2,t_3)+A(s_3,t_2)], \quad (2)$$

$$A^{00\to00}(s,t,u)=\beta_5[A(s_2,t_2)+A(s_2,u_2)+A(t_2,u_2)] +\beta_6[A(s_3,t_3)+A(s_3,u_3)+A(t_3,u_3)]. \quad (3)$$

All the other amplitudes can be obtained from the above three by crossing, e.g.,

$$A^{+-\to00}(s,t,u)=A^{00\to+-}(s,t,u)=\beta_1[A(t_1,s_2)+A(u_1,s_2)] +\beta_2[A(t_1,s_3)+A(u_1,s_3)]+\beta_3A(t_1,u_1). \quad (1')$$

The full amplitudes for ψ-ψ scattering form a 9×9 symmetric matrix. In the region where s, t, and u are real, so are all the Veneziano amplitudes. Thus, we can diagonalize the S matrix by a *real orthogonal* matrix (in general depending on energy and scattering angle) in that region. In the subspace of unit charge, the S matrix will be diagonalized by the states $(|+0\rangle\pm|0+\rangle)/\sqrt{2}$, which are even and odd under exchange of the two particles. The diagonal amplitudes are denoted by $A_\pm{}^{+0}=A^{+0\to+0}(s,t,u)\pm A^{+0\to0+}(s,t,u)$. In the subspace of zero charge, there are two even states $(|+-\rangle-+\rangle)/\sqrt{2}$ and $|00\rangle$, which are connected by scattering. The antisymmetric state $(1+-\rangle+|-+\rangle)/\sqrt{2}$ has the diagonal amplitude given by $A_-{}^{+-}=A^{+-\to+-}(s,t,u)-A^{+-\to-+}(s,t,u)$. Thus we are left with a 2×2 matrix A_{ij} to be diagonalized, where $A_{11}\equiv A^{+-\to+-}+A^{+-\to-+}$, $A_{12}\equiv\sqrt{2}A^{+-\to00}$, and $A_{22}\equiv A^{00\to00}$.

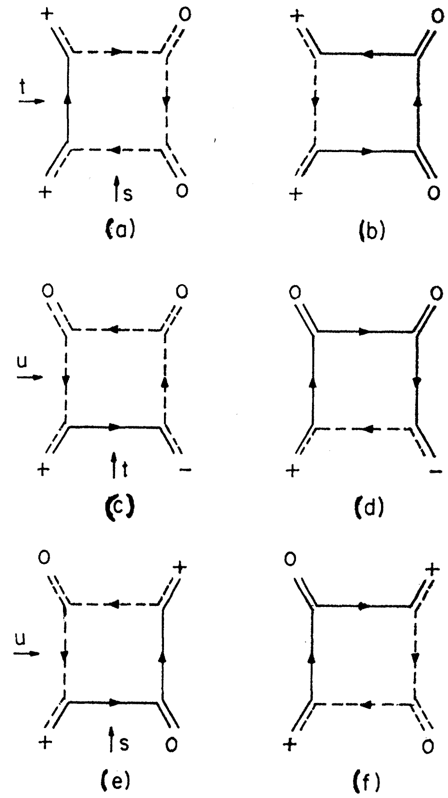

FIG. 1. Duality diagrams for $\psi^+\psi^0\to\psi^+\psi^0$.

Let $R_{ij}(n)$ be the residues of A_{ij} at the pole $s=s_n$; they can be written as

$$R_{ij}(n)\equiv\sum_{l=0}^{n} r_{ij}{}^l(n)P_l(\cos\theta).$$

To impose factorization, we demand that

$$r_{11}{}^l(n)r_{22}{}^l(n)=[r_{12}{}^l(n)]^2 \quad (4)$$

be satisfied at all l and n.

Several authors[8] recently have shown that, at least for the single-term Veneziano formula, the residue of a nonleading pole factorizes in a finite number of terms. In this case factorization for each degenerate level would allow us to write $r_{ij}{}^l(n)=\mathbf{r}_i{}^l(n)\cdot\mathbf{r}_j{}^l(n)$. We emphasize, however, that condition (4) can still be satisfied provided $\mathbf{r}_i{}^l$ and $\mathbf{r}_j{}^l$ are parallel vectors. Applying condition (4) to the trajectory $\alpha_2(s)$, we are led to the conclusion $\alpha_1(t)=\alpha_2(t)=\alpha_3(t)\equiv\alpha(t)$. Thus we can rewrite A_{ij} as

$$A_{11}=a[A(s,t)+A(s,u)], \quad (5a)$$

$$A_{12}=b[A(s,t)+A(s,u)]+cA(t,u), \quad (5b)$$

$$A_{22}=d[A(s,t)+A(s,u)+A(t,u)], \quad (5c)$$

with $b^2=ad$.

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Using Eqs. (5a) and (5b), we find A_-^{+-} and A_-^{+0} have the following relation:

$$A_-^{+-}=[\sqrt{2}a/(b-c)]A_-^{+0}=a[A(s,t)-A(s,y)]. \quad (6)$$

The amplitudes A_-^{+-} and A_-^{+0}, in their exact forms, would satisfy the same type of elastic unitarity equation since they are both diagonal amplitudes. Now Eq. (6) says their Veneziano representations are proportional to each other; thus by our normalization assumption discussed earlier, we should set

$$b-c=\sqrt{2}a. \quad (7)$$

It takes a simple algebra to diagonalize A_{ij}, and the diagonal amplitudes are

$$A_{1,2}=\tfrac{1}{2}(A_{11}+A_{22})\pm\tfrac{1}{2}[4A_{12}^2+(A_{11}-A_{22})^2]^{1/2}. \quad (8)$$

Substituting Eq. (5) into Eq. (8), we find that the residues of A_1 and A_2 at poles in t will not be polynomials in $\cos\theta_t$ unless[9]

$$(d-a)/b=d/c. \quad (9)$$

Solving Eqs. (5d), (7), and (9) in terms of a, we obtain two sets of solutions,

$$d=a/2, \quad b=-c=a/\sqrt{2} \quad (10)$$

and

$$d=2a, \quad b=-\sqrt{2}a, \quad c=-2\sqrt{2}a. \quad (11)$$

Solution (11) is rejected by the positivity requirement on the residues of elastic process. For this set of solutions, d and $b+c$ are of opposite sign; thus the residues of $A^{00\to00}$ and $A^{+0\to+0}$ cannot be made positive simultaneously. With solution (10), we obtain

$$A_1=\tfrac{3}{2}a[A(s,t)+A(s,u)]-\tfrac{1}{2}aA(t,u), \quad (12)$$

$$A^{++\to++}=A_+^{+0}=A_2=aA(t,u). \quad (13)$$

[9] This is because for any two real polynomials $P(x)$ and $Q(x)$, $[P^2(x)+Q^2(x)]^{1/2}$ will not be a polynomial unless $P(x)$ and $Q(x)$ are proportional.

Thus there are only three distinct diagonal amplitudes: A_1, A_-^{+-}, and A_2. The crossing relations for these three amplitudes from the u channel to the s channel can be easily obtained from Eqs. (6), (12), and (13):

$$\begin{pmatrix} A_1(s,t,u) \\ A_-^{+-}(s,t,u) \\ A_2(s,t,u) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -1 & 5/3 \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} A_1(u,t,s) \\ A_-^{+-}(u,t,s) \\ A_2(u,t,s) \end{pmatrix}. \quad (14)$$

We recognize at once that the crossing matrix in (14) is identical to the one for the $\pi\pi$ amplitudes of isospin 0, 1, and 2. The vector coupling coefficients (to within certain phases), which are the same as in the $\pi\pi$ system, can also be readily calculated. Thus we conclude that the ψ system belongs to the triplet representation of $SU(2)$.

We may naturally raise the question of just how much our results depend on the Veneziano representation. The most crucial point is that we are able to diagonalize the ψ-ψ amplitudes by a matrix which is independent of energy and momentum transfer. This is possible because of the factorization requirement and because of the very nature of the Veneziano formula, which is built solely out of resonances. In a pure-resonance model, relations between the residues usually give rise to relations between the amplitudes. Equation (9) is a statement for the residues at poles in t, and in turn implies that $A_{11}-A_{22}$ is proportional to A_{12}; thus A_{ij} is diagonalized by a constant matrix. Once we arrive at this stage, it is not too surprising that we finally obtain the $SU(2)$ symmetry. It has been shown[10] that starting with charge conservation and *assuming* $\pi\pi$ amplitudes can be diagonalized by a constant, real, orthogonal matrix, a rigorous application of unitarity and crossing will lead to $SU(2)$ as a least-trivial symmetry.

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