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Effective Lagrangians, Field Algebra, and Vector-Meson Dominance with Unstable Particles*

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The finite-width mass distribution of the vector and axial-vector mesons is introduced in the phenomenological Lagrangian model. Tree-graph amplitudes becoming complex, unitarity constraints may be imposed. The $SU_2\times SU_2$ algebra of fields and field-current identities are discussed. The well-known relation between the Schwinger term and the integral over the spectral density is identically obeyed in our tree-graph approach. A tentative unitary solution gives a rather broad A_1 resonance. A discussion of the $\pi\pi$ scattering amplitude and of the pion form factor in this approach is given.

INTRODUCTION

In the usual formulation of the so-called effective Lagrangian models, the vector and axial-vector particles are considered as stable. Recent experiments with colliding electron-positron beams in the ρ-mass region point out the importance of finite-width effects.[1,2] The purpose of this paper is to show a simple algebraic way to include the finite width of resonances in the formalism of the effective Lagrangians. Although the primary motivation of such an improvement was a pragmatic one, the content of this paper is mostly formal, and the few numerical results which were obtained are due to some theoretical speculations allowed by the new formalism.

In our approach an unstable particle is characterized by a mass distribution which acts as a weight in the phenomenological Lagrangian. In Sec. I the vector-meson dominance of the isoscalar electromagnetic current is considered in this framework. In Secs. II and III the field algebras of SU_2 and chiral $SU_2\times SU_2$, respectively, are discussed. It is shown that the equality between the Schwinger term and the spectral-function integral is an identity even in the tree-graph approximation. Speculations about a possible linear interpolation formula for the $\rho\gamma$ coupling constant between the threshold and the resonance give rise to $\Gamma_{A_1}/\Gamma_\rho=1.5$.

A discussion of the $\pi\pi$ scattering amplitude and of the pion form factor and their unitarity constraints is given in Sec. IV.

I. ISOSCALAR VECTOR MESONS

Let us consider an isoscalar massive vector field φ_μ. The corresponding Yang-Mills Lagrangian is

$$-\tfrac{1}{4}\varphi_{\mu\nu}{}^2-\tfrac{1}{2}M^2\varphi_\mu{}^2+L'(u_\alpha,D_\mu u_\alpha;\,\varphi_{\mu\nu}),\qquad(1.1)$$

where L' is built up from the other fields u_α, their covariant derivatives

$$D_\mu u_\alpha=(\partial_\mu-\tfrac{1}{2}i\gamma Y_\alpha\varphi_\mu)u_\alpha\qquad(1.2a)$$

(Y_α being the hypercharge of the particle described by the field u_α), and the 4-rotor

$$\varphi_{\mu\nu}=\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu.\qquad(1.2b)$$

Such a Lagrangian gives rise to the relation

$$i\int d\mathbf{x}\,J_4{}^Y(x)=Y,\qquad(1.3a)$$

where

$$J_\mu{}^Y(x)=(M^2/\gamma)\varphi_\mu(x)\qquad(1.3b)$$

and $\partial_\mu J_\mu{}^Y(x)=0$; that is, we have the vector-meson dominance of the hypercharge current.

If we consider the introduction of the electromagnetism by the replacement

$$\varphi_\mu\rightarrow\varphi_\mu+(e/\gamma)\mathcal{A}_\mu\qquad(1.4)$$

in the Lagrangian (except for the mass term), then we will have the well-known current-field identity

$$J_\mu{}^{\mathrm{e.m.}\,Y}(x)=(M^2/\gamma)\varphi_\mu(x)\qquad(1.5)$$

for the isoscalar part of the hadronic electromagnetic current.

Since the isoscalar vector mesons supposed to dominate the electromagnetic e.m. current (the ω and ϕ) are unstable, all our previous discussion must be looked at with much circumspection.

In the following we want to take into account the width of these vector mesons. Our approach will be phenomenological, in the spirit of the effective Lagrangain models. All the mathematical subtleties and perhaps inconsistencies of the consequent quantum field theory will thus be skipped.

Instead of the field φ_μ, we will introduce a continuous set of fields $\varphi_\mu{}^{(m)}$ with an associated mass distribution $\kappa(m^2)$.

The generalization of the Lagrangian (1.1) is then

$$L=\int dm^2\,\kappa(m^2)L_{(m)},\qquad(1.6a)$$

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[1] G. Gounaris and J. Sakurai, Phys. Rev. Letters 21, 244 (1968).

[2] G. Gounaris, Phys. Rev. 181, 2066 (1969).

with

$$L_{(m)}=-\tfrac{1}{4}\varphi_{\mu\nu}{}^{(m)2}-\tfrac{1}{2}m^2\varphi_\mu{}^{(m)2} +L_{(m')}(u_\alpha,D_\mu{}^{(m)}u_\alpha,\varphi_{\mu\nu}{}^{(m)}),$$

$$D_\mu{}^{(m)}u_\alpha=[\partial_\mu-\tfrac{1}{2}i\gamma(m^2)Y_\alpha\varphi_\mu{}^{(m)}]u_\alpha, \tag{1.6b}$$

$$\varphi_{\mu\nu}{}^{(m)}=\partial_\mu\varphi_\nu{}^{(m)}-\partial_\nu\varphi_\mu{}^{(m)}, \tag{1.6c}$$

and

$$\int dm^2\,\kappa(m^2)=1. \tag{1.6d}$$

(The correctly normalized fields are $[\kappa(m^2)]^{1/2}\varphi_\mu{}^{(m)}(x)$.) As is easy to see, we now have instead of (1.3b)

$$J_\mu{}^Y(x)=\int dm^2\,\kappa(m^2)\frac{m^2}{\gamma(m^2)}\varphi_\mu{}^{(m)} \tag{1.7}$$

and, introducing the electromagnetic interactions by the same recipe,

$$\varphi_\mu{}^{(m)}\to\varphi_\mu{}^{(m)}+[e/\gamma(m^2)]\mathcal{A}_\mu, \tag{1.8}$$

we have the identity

$$J_\mu{}^{Y\text{e.m.}}(x)=J_\mu{}^Y(x). \tag{1.9}$$

The free propagator of the fields $\varphi_\mu{}^{(m)}$ being

$$\varphi_\mu{}^{(m)\cdot}(p)\,\varphi_\nu{}^{(m')\cdot}(p') =\frac{\delta(m^2-m'^2)}{\kappa(m^2)}\frac{i}{(2\pi)^4}\frac{\delta_{\mu\nu}+p_\mu p_\nu/m^2}{p^2+m^2}\delta(p+p'), \tag{1.10}$$

we have in the tree-graph approach the spectral representation

$$\langle 0|T\{J_\mu{}^Y(p)J_\nu{}^Y(p')\}|0\rangle=\frac{i}{(2\pi)^4}\delta(p+p') \times\int dm^2\,\kappa(m^2)\frac{m^4}{\gamma(m^2)^2}\frac{\delta_{\mu\nu}+p_\mu p_\nu/m^2}{p^2+m^2}, \tag{1.11}$$

with the spectral density

$$\rho(m^2)=\kappa(m^2)m^4/\gamma(m^2)^2.$$

In the narrow-width limit, $\kappa(m^2)\to\delta(m^2-M^2)$, we recover the usual expressions.

II. ϱ DOMINANCE

Now we shall consider the field algebra of SU_2, that is, the ρ-meson dominance with fields of distributed mass. The corresponding Lagrangian built up along the same analogy with the stable-particle theory is

$$L=\int dm^2\,\kappa(m^2)L_{(m)}, \tag{2.1}$$

with

$$L_{(m)}=-\tfrac{1}{4}(\mathbf{f}_{\mu\nu}{}^{(m)})^2-\tfrac{1}{2}m^2(\boldsymbol{\varrho}_\mu{}^{(m)})^2 +L_{(m)}'(u_\alpha,\hat{D}_\mu{}^{(m)}u_\alpha;\mathbf{f}_{\mu\nu}{}^{(m)}), \tag{2.2a}$$

$$\hat{D}_\mu{}^{(m)}u_\alpha=\partial u_\alpha+\tfrac{1}{2}ig(m^2)\rho_\mu{}^{(m)}\tau_{\alpha\beta}u_\beta, \tag{2.2b}$$

$$\mathbf{f}_{\mu\nu}{}^{(m)}=\partial_\mu\boldsymbol{\varrho}_\nu{}^{(m)}-\partial_\nu\boldsymbol{\varrho}_\mu{}^{(m)}-g(m^2)\boldsymbol{\varrho}_\mu{}^{(m)}\times\boldsymbol{\varrho}_\nu{}^{(m)}, \tag{2.2c}$$

and

$$\int dm^2\,\kappa(m^2)=1. \tag{2.2d}$$

Now we have the current-field identities for the isovector current components:

$$\mathbf{J}_\mu=\int dm^2\,\kappa(m^2)\frac{m^2}{g(m^2)}\boldsymbol{\varrho}_\mu{}^{(m)}, \tag{2.3}$$

obtained from the field equations. (Note that $\partial_\mu\mathbf{J}_\mu=0$, but $\partial_\mu\rho_\mu{}^{(m)}\neq0$.) These currents obey the following equal-time commutation relations (obtained from the canonical quantization rules):

$$\begin{aligned}
&[J_4{}^i(\mathbf{x},0),J_4{}^j(0)]=\epsilon_{ijk}J_4{}^k(0)\delta(\mathbf{x}),\\
&[J_4{}^i(\mathbf{x},0),J_a{}^j(0)]=\epsilon_{ijk}J_a{}^k(0)\delta(\mathbf{x})\\
&\qquad+\delta_{ij}\partial_a\delta(\mathbf{x})\int dm^2\,\kappa(m^2)\frac{m^2}{g(m^2)^2},\\
&[J_a{}^i(\mathbf{x},0),J_b{}^j(0)]=0,
\end{aligned} \tag{2.4}$$

where $a,\,b=1,\,2,\,3$.

On the other hand, we have the spectral representation

$$\langle 0|T\{J_\mu{}^i(p)J_\nu{}^j(p')\}|0\rangle=\delta_{ij}\frac{i}{(2\pi)^4}\delta(p+p') \times\int dm^2\,\kappa(m^2)\frac{m^4}{g(m^2)^2}\frac{\delta_{\mu\nu}+p_\mu p_\nu/m^2}{p^2+m^2} \tag{2.5}$$

(valid in the tree-graph approach).

We see that the well-known relation between the Schwinger term and the spectral density,

$$S=\int\frac{dm^2\rho(m^2)}{m^2},$$

is identically obeyed by our formulas.

The ρ-meson dominance of the electromagnetic current is achieved through the introduction of electromagnetic interactions by the replacement

$$\rho_\mu{}^{(m)0}\to\rho_\mu{}^{(m)0}+[e/g(m^2)]\mathcal{A}_\mu.$$

III. $SU_2\times SU_2$ FIELD ALGEBRA

It is interesting to apply our approach to the field algebra of chiral $SU_2\times SU_2$.

Let us first consider the symmetric case. The ρ and A_1 mesons are assigned to the $(1,0)+(0,1)$ representation of chiral $SU_2\times SU_2$. We will consider a Lagrangian involving the interactions of these particles with the zero-mass pions in the nonlinear σ model (see Ref. 3). With unstable vector and axial-vector mesons described by the fields $\rho_\mu^{(m)}$ and $a_\mu^{(m)}$ and the zero-mass pion described by the field ϕ, our chiral-invariant Lagrangian is

$$L=\int dm^2\,\kappa(m^2)L_{(m)}\,, \tag{3.1}$$

$$\begin{aligned}L_{(m)}=&-\tfrac{1}{4}(\mathbf{F}_{\mu\nu}^{(m)})^2-\tfrac{1}{4}(\mathbf{G}_{\mu\nu}^{(m)})^2+\tfrac{1}{2}m^2(\boldsymbol{\rho}_\mu^{(m)2}+\mathbf{a}_\mu^{(m)2})\\&-\tfrac{1}{2}(\Delta_\mu^{(m)}\boldsymbol{\phi})^2-\tfrac{1}{2}(\Delta_\mu^{(m)}\tilde{\sigma})^2\\&+\chi[\tfrac{1}{2}\mathbf{F}_{\mu\nu}^{(m)}\cdot(\Delta_\mu^{(m)}\boldsymbol{\phi}\times\Delta_\nu^{(m)}\boldsymbol{\phi})\\&\qquad+\mathbf{G}_{\mu\nu}^{(m)}\cdot\Delta_\mu^{(m)}\boldsymbol{\phi}\Delta_\nu^{(m)}\tilde{\sigma}]\,,\end{aligned} \tag{3.2a}$$

where

$$\mathbf{F}_{\mu\nu}^{(m)}=\mathbf{f}_{\mu\nu}^{(m)}-g(m^2)\mathbf{a}_\mu^{(m)}\times\mathbf{a}_\nu^{(m)}\,, \tag{3.2b}$$

$$\begin{aligned}\mathbf{G}_{\mu\nu}^{(m)}&=\hat{D}_\mu^{(m)}\mathbf{a}_\nu^{(m)}-\hat{D}_\nu^{(m)}\mathbf{a}_\mu^{(m)}\,,\\\hat{D}_\mu^{(m)}\mathbf{a}_\nu^{(m)}&=[\partial_\mu-g(m^2)\boldsymbol{\rho}_\mu^{(m)}\times]\mathbf{a}_\nu^{(m)}\,,\end{aligned} \tag{3.2c}$$

$$\begin{aligned}\Delta_\mu^{(m)}\boldsymbol{\phi}&=\hat{D}_\mu^{(m)}\boldsymbol{\phi}+g(m^2)\tilde{\sigma}\mathbf{a}_\mu^{(m)}\,,\\\hat{D}_\mu^{(m)}\boldsymbol{\phi}&=[\partial_\mu-g(m^2)\boldsymbol{\rho}_\mu^{(m)}\times]\boldsymbol{\phi}\,,\end{aligned} \tag{3.2d}$$

$$\begin{aligned}\Delta_\mu^{(m)}\tilde{\sigma}&=\partial_\mu\tilde{\sigma}-g(m^2)\mathbf{a}_\mu^{(m)}\cdot\boldsymbol{\phi}\,,\\\tilde{\sigma}&=(\sigma_0^2-\boldsymbol{\phi}^2)^{1/2}.\end{aligned} \tag{3.2e}$$

The conserved vector and axial-vector currents are

$$\mathbf{J}_\mu=\int dm^2\,\kappa(m^2)\frac{m^2}{g(m^2)}\boldsymbol{\rho}_\mu^{(m)}\,, \tag{3.3a}$$

$$\mathbf{J}_{5\mu}=\int dm^2\,\kappa(m^2)\frac{m^2}{g(m^2)}\mathbf{a}_\mu^{(m)}\,, \tag{3.3b}$$

obeying the commutation relations

$$\begin{aligned}&[J_4^i(\mathbf{x},0),J_{5,\mu}^j(0)]=\epsilon_{ijk}J_{5\mu}^k(0)\delta(\mathbf{x})\,,\\&[J_a^i(\mathbf{x},0),J_{5,b}^j(0)]=0\,,\quad[J_{5,a}^i(\mathbf{x},0),J_{5,b}^j(0)]=0\,,\\&[J_{5,4}^i(\mathbf{x},0),J_{5,4}^j(0)]=\epsilon_{ijk}J_4^k(0)\delta(\mathbf{x})\,,\\&[J_{5,4}^i(\mathbf{x},0),J_{5,a}^j(0)]=\epsilon_{ijk}J_a^k(0)\delta(\mathbf{x})\\&\qquad\qquad+\delta_{ij}\partial_a\delta(\mathbf{x})\int dm^2\,\kappa(m^2)\frac{m^2}{g(m^2)}\,,\end{aligned} \tag{3.4}$$

and those of Sec. II.

Again the equality between the Schwinger term and the spectral density is trivially obtained since the mass distributions of the ρ and A_1 are identical in the chiral-symmetric limit.

Let us now introduce the symmetry breaking

$$L^{\text{break}}=m_\pi^2 f\tilde{\sigma}\,.$$

This modification of the Lagrangian will not affect the commutation relations of the currents but it gives rise to the partial conservation of the axial-vector current.

On the other hand, to get rid of the cross terms of the form $\mathbf{a}_\mu^{(m)}\hat{D}_\mu\boldsymbol{\phi}$ (which lead to mixing between the axial-vector meson and the pion as soon as the symmetry is broken), a redefinition of the physical fields is necessary in the usual manner.[3] The physical π and A_1 mesons will be described by the fields $\boldsymbol{\pi}$ and $\mathbf{A}_\mu^{(m)}$ defined through

$$\boldsymbol{\phi}=Z^{1/2}\boldsymbol{\pi}\,, \tag{3.5a}$$

$$\mathbf{a}_\mu^{(m)}=\mathbf{A}_\mu^{(m)}-\frac{[m_A^2(m^2)-m^2]^{1/2}}{m_A^2(m^2)}Z^{1/2}\hat{D}_\mu^{(m)}\boldsymbol{\pi}\,, \tag{3.5b}$$

where

$$Z^{-1}=\int dm^2\,\kappa(m^2)\frac{m^2}{m_A^2(m^2)}$$

and

$$m_A^2(m^2)=m^2+\sigma_0^2g^2(m^2) \tag{3.6}$$

is the mass of the $\mathbf{A}_\mu^{(m)}$ field.

Let us look now at the spectral representation of the axial-vector current propagator. Since the free propagators of the fields π and $A_\mu^{(m)}$ are

$$\pi^{i\cdot}(p)\pi^{j\cdot}(p')=\delta_{ij}[i/(2\pi)^4](p^2+m_\pi^2)^{-1}\delta(p+p')\,, \tag{3.7a}$$

$$\begin{aligned}&A_\mu^{i(m)\cdot}(p)A_\nu^{j(m')\cdot}(p')\\&\quad=\delta_{ij}\frac{\delta(m^2-m'^2)}{\kappa(m^2)}\frac{i}{(2\pi)^4}\frac{\delta_{\mu\nu}+p_\mu p_\nu/m_A^2(m^2)}{p^2+m_A^2(m^2)}\,,\end{aligned} \tag{3.7b}$$

and we are considering the tree-graph approach, we find

$$\begin{aligned}\langle 0|T\{J_{5\mu}^i(p)J_{5\nu}^j(p')\}|0\rangle&=\delta_{ij}\frac{i\delta(p+p')}{(2\pi)^4}\\&\times\int d\lambda^2\frac{\rho_{55}^{(1)}(\lambda^2)(\delta_{\mu\nu}+p_\mu p_\nu)+\rho_{55}^{(0)}(\lambda^2)p_\mu p_\nu}{p^2+\lambda^2}\,,\end{aligned} \tag{3.8}$$

with

$$\rho_{55}^{(0)}(\lambda^2)=f_\pi^2\delta(\lambda^2-m_\pi^2)\,, \tag{3.9}$$

$$\rho_{55}^{(0)}(\lambda^2)=f_\pi^2\frac{m^4}{m_A^2(1-m^2/m_A^2)\langle m^2/m_A^2\rangle}\times\frac{dm^2}{dm_A^2}\kappa(m^2)\bigg|_{m_A^2=\lambda^2}\,, \tag{3.10}$$

where we used

$$g^2=(m_A^2-m^2)/\sigma_0^2\quad(\sigma_0=f_\pi\langle m^2/m_A^2\rangle^{-1/2}) \tag{3.11}$$

and the brackets $\langle\cdots\rangle$ mean take the average $\int dm^2\,\kappa(m^2)$.

The corresponding relation with the Schwinger term

$$S=\int d\lambda^2\left(\frac{\rho_{55}^{(1)}(\lambda^2)}{\lambda^2}+\rho_{55}^{(0)}(\lambda^2)\right), \tag{3.12}$$

as is easily seen, now gives rise again to an identity.

Now a little speculation about the possible dependence of $m_A(m)$: Since the mass distribution of the ρ

meson begins from $4m_\pi^2$, while that of the A_1 must begin from $9m_\pi^2$ and $\kappa_A(m_A^2)=(dm^2/dm_A^2)\kappa(m^2)$ is the relation between the mass distributions, we must have $m_A^2(4m_\pi^2)=9m^2$. On the other hand, from the experimental masses we have $m_A^2(\bar{m}_\rho^2)\approx 2\bar{m}_\rho^2$. A linear dependence of the form $m_A^2=2.25m^2$ gives a good interpolation between these two points. Then for the A_1 mass distribution $\kappa_A(m_A^2)$ we have

$$\kappa_A(m_A^2)=\frac{\kappa(m_A^2/2.25)}{2.25}; \tag{3.13}$$

that is, the two mass distributions are related by a scale transformation.

One can define the width of a narrow resonance through the height of the mass distribution peak,

$$\kappa(\bar{m}_\rho^2)=1/\pi\bar{m}_\rho\Gamma_\rho\,,\quad \kappa_A(\bar{m}_A^2)=1/\pi\bar{m}_A\Gamma_A\,. \tag{3.14}$$

(A simple Breit-Wigner form is supposed to hold for the mass distributions.) Thus we obtain the ratio of the widths

$$\Gamma_A/\Gamma_\rho=1.5\,. \tag{3.15}$$

The assumed mass ratio is close to Weinberg's well-known $\sqrt{2}$, being consistent with the experimental data (if we take $\bar{m}_\rho=760$ MeV it follows that $\bar{m}_{A_1}=1130$ MeV), while the ratio of the widths indicates a rather wide A_1 resonance ($\Gamma_{A_1}=165$ MeV for $\Gamma_\rho=110$ MeV) which is also consistent with some recent measurements.[4,5]

One can also compute the absolute value of the widths in our effective-Lagrangian approach through the decay diagrams. So we obtain

$$\Gamma_{\rho\to2\pi}=\frac{g^2(\bar{m}_\rho^2)}{4\pi}[1-0.28(1-\delta(\bar{m}_\rho^2))]\times52\text{ MeV}\,, \tag{3.16a}$$

$$\Gamma_{A\to\rho\pi}=\frac{g^2(\bar{m}_\rho^2)}{4\pi}[1-0.985\delta(\bar{m}_\rho^2)+0.245\delta(\bar{m}_\rho^2)^2]\times79\text{ MeV}\,, \tag{3.16b}$$

and to have the discussed ratio of the widths we must take $\delta(m_\rho^2)=0.2$. Furthermore, from $m_A^2(m^2)=m^2+f_\pi^3g^2(m^2)/\langle m^2/m_A^2\rangle$ and $m_A^2=2.25m^2$, we get our modified Kawarabayashi-Suzuki-Fayyazuddin-Riazuddin (KSFR) relation $f_\pi^2=m^2/[1.8g^2(m^2)]$, which gives $g^2(m_\rho^2)/4\pi=2.86$. Introducing these values in the preceding formulas, we obtain $\Gamma_{\rho\to2\pi}\approx100$ MeV, which gives a happy end to our numerical game.

However, we must observe that our linear solution is likely to be forbidden by the use of the three-pion unitarity. Indeed, owing to the difference between the two-pion and three-pion phase spaces, a simple proportionality between the spectral functions of the ρ, and A mesons seems to be ruled out, at least at the threshold. Nevertheless, if the proportionality holds at the resonance region the corresponding ratio of the widths will emerge. (We would like to thank Professor B. Renner for useful comments on this point.)

IV. PION-PION SCATTERING AND PION FORM FACTOR

As an example of the described formalism we shall give here the $\pi\pi$ scattering amplitude in the tree-graph approach. With the usual notation

$$M_{\alpha\beta\gamma\delta}=\delta_{\alpha\beta}\delta_{\gamma\delta}A(s,t,u)+\delta_{\alpha\gamma}\delta_{\beta\delta}A(t,s,u)+\delta_{\alpha\delta}\delta_{\beta\gamma}A(u,t,s)\,, \tag{4.1}$$

we find

$$A(s,t,u)=\int dm^2\,\kappa(m^2)\biggl\{\frac{g^2(m^2,t)(s-u)}{m^2-t}+\frac{g^2(m^2,u)(s-t)}{m^2-u}+a(m^2)+b(m^2)s+c(m^2)[(t-4m_\pi^2)^2+(u-4m_\pi^2)^2-2(s-4m_\pi^2)^2]\biggr\}-\frac{m_\pi^2}{f_\pi^2}, \tag{4.2}$$

where

$$a(m^2)=\frac{4m_\pi^2}{m_A^2}\frac{m^2/m_A^2}{\langle m^2/m_A^2\rangle^2}g^2(m_2)\,, \tag{4.3a}$$

$$b(m^2)=-\frac{1}{m_A^2}\frac{m^2/m_A^2}{\langle m^2/m_A^2\rangle^2}\frac{3m^2/m_A^2-1}{1-m^2/m_A^2}g^2(m^2)\,, \tag{4.3b}$$

$$c(m^2)=-\frac{1-4\delta(m^2)m^2/m_A^2}{4m_A^2}\times\frac{(1-m^2/m_A^2)^2}{\langle m^2/m_A^2\rangle^2}g^2(m^2)\,, \tag{4.3c}$$

$$g^2(m^2,s)=g^2(m^2)\biggl[1-\frac{s}{2m_A^2}\times\frac{1-m^2/m_A^2}{\langle m^2/m_A^2\rangle}(1-\delta(m^2))\biggr]\,, \tag{4.3d}$$

$$\delta(m^2)=-\frac{\chi(m^2)}{g(m^2)}\frac{m^2/m_A^2}{1-m^2/m_A^2}m_A^2\,. \tag{4.3e}$$

It is obvious that our tree-graph amplitude has acquired an imaginary part in the $I=1$, $J=1$ partial wave. Thus one can put the problem of satisfying unitarity requirements in this partial wave.

Let us define

$$T_1^1(s)=\frac{1}{64\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\int_{-1}^{1}dz\,zT_1^s(s,t)\times[T_1^s=A(t,s,u)-A(u,t,s)] \tag{4.4}$$

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[3] S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. **41**, 531 (1969).

[4] G. Alexander, A. Firestone, and G. Goldhaber, Phys. Rev. **183**, 1169 (1969).

to have the simple form of the partial-wave elastic unitarity

$$\mathrm{Im}T_1^1(s)=|T_1^1(s)|^2 \quad (4m_\pi^2<s<16m_\pi^2). \quad (4.5)$$

One can remark here that unitarity is automatically satisfied in the tree-graph approach in the lowest order in the coupling constant. But this unitarity considers unstable particles in the complete set of states. The unitarity we consider is the physical one and it can fix the otherwise undetermined spectral functions.

The explicit form of our partial-wave amplitude is

$$T_1^1(s)=\frac{s(1-4m_\pi^2/s)^{3/2}}{48\pi}\int_{4m_\pi^2}^{\infty} dm^2\,\kappa(m^2)\left[\frac{g^2(m^2,s)}{m^2-s-i\epsilon}+\frac{3}{4}\int_{-1}^{1}dz\frac{z(3+z+2m_\pi^2/\nu)g^2(m^2,\,-2\nu(1-z))}{m^2+2\nu(1-z)}+\tfrac{1}{2}b(m^2)-2c(m^2)(s-4m_\pi^2)(8m_\pi^2-s)\right]. \quad (4.6)$$

Its imaginary part is

$$\mathrm{Im}T_1^1(s)=\pi\kappa(s)\varphi(s)\,, \quad (4.7)$$

where

$$\varphi(s)=\frac{s(1-4m_\pi^2/s)^{3/2}}{48\pi}g^2(s;s)\,.$$

On the other hand, if elastic unitarity is obeyed and we take into account the existence of the ρ meson (phase shift passing through $\frac{1}{2}\pi$ at $s=m_\rho^2$), the same imaginary part must be of the form

$$\psi(s)^2/[(s-m_\rho^2)^2+\psi(s)^2]\,.$$

This gives a convenient way to parametrize the spectral distribution $\kappa(s)$. The width of the resonance, defined as

$$\bar{\Gamma}_\rho=\frac{1}{m_\rho}\frac{d\delta_{11}(s)}{ds}\bigg|_{s=m_\rho^2},$$

is simply related to the function ψ through

$$\bar{\Gamma}_\rho=\psi(m_\rho^2)/m_\rho\,.$$

The width computed from the decay diagram at the ρ-meson mass, which coincides obviously with that defined in Sec. III through the height of the mass distribution, is

$$\Gamma_\rho=\varphi(m_\rho^2)/m_\rho\,.$$

If these two definitions are equivalent, then

$$\psi(m_\rho^2)=\varphi(m_\rho^2)\,.$$

It is useful to consider the problem of the $I=1$, $J=1$ $\pi\pi$ partial wave together with that of the pion electromagnetic form factor.

In our vector-meson-dominance approach we find

$$F_\pi(s)=\int dm^2\,\kappa(m^2)\frac{m^2}{m^2-s}\frac{g(m^2;s)}{g(m^2)}\,. \quad (4.8)$$

The normalization of the total charge $F_\pi(0)=1$ is obviously respected, since $g(m^2,0)=g(m^2)$. The unitarity condition for the form factor is

$$\mathrm{Im}F_\pi(s)=F_\pi(s)e^{-i\delta_{11}(s)}\sin\delta_{11}(s) \qquad (4m_\pi^2<s<16m_\pi^2) \quad (4.9)$$

or, equivalently,

$$\mathrm{Im}F_\pi(s)=\mathrm{Re}F_\pi(s)\tan\delta_{11}(s)\,. \quad (4.10)$$

The two unitarity equations (4.5) and (4.10) may be looked at as the basic equations for determining the unknown functions $\kappa(m^2)$, $g(m^2)$, and $\delta(m^2)$. If we also admit the linear relation between the masses, then we remain with only two of these undetermined. One might invent several ingeneous schemes to find approximate solutions of these equations but we will not discuss them here. Nevertheless, it is interesting to discuss some consequences of these equations at the threshold. Here one can apply the narrow-width approximation in calculating the real parts:

$$\mathrm{Re}T_1^1(s)\approx\frac{m_\pi^2}{12\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\left[\frac{g^2(m_\rho^2,4m_\pi^2)}{m_\rho^2-4m_\pi^2}+\frac{g^2(m_\rho^2;0)}{2m_\rho^2}+\tfrac{1}{2}b(m_\varphi^2)\right], \quad (4.11a)$$

$$\mathrm{Re}F_\pi(s)\approx\frac{m_\rho^2}{m_\rho^2-4m_\pi^2}\frac{g(m_\rho^2;4m_\pi^2)}{g(m_\rho^2)}\,. \quad (4.11b)$$

The unitary equations themselves simplify at the same limit to

$$\mathrm{Im}T_1^1\approx(\mathrm{Re}T_1^1)^2\,, \quad (4.12a)$$

$$\mathrm{Im}F_\pi\approx\mathrm{Re}F_\pi\,\mathrm{Re}T_1^1\,. \quad (4.12b)$$

Since

$$\mathrm{Im}T_1^1(s)/\mathrm{Im}F_\pi(s)=\varphi(s)g(s)/sg(s,s)$$

and

$$g^2(m^2,s)\approx g^2(m^2) \quad \text{for} \quad s/m^2\ll 1\,,$$

we obtain

$$g^2(m_\rho^2)/g(4m_\pi^2)g(4m_\pi^2,4m_\pi^2)\approx m_\rho^2/4m_\pi^2\,. \quad (4.13)$$

This is in fair agreement with the ratio

$$g^2(m_\rho^2)/g^2(4m_\pi^2)=m_\rho^2/4m_\pi^2\,, \quad (4.14)$$

which results from the linear relation between the masses.

From the above relations, one can find also

$$\frac{\psi(s)}{\varphi(s)}\bigg|_{s=4m_\pi^2}\approx\frac{m_\rho^2}{4m_\pi^2}\frac{g(4m_\pi^2)}{g(4m_\pi^2,4m_\pi^2)} \quad (4.15)$$

and thus conclude that generally $\psi(s)\neq\varphi(s)$.

It is interesting to point out also that the well-known correlation between the $e^+e^- \rightarrow \pi^+\pi^-(I=1)$ total cross section and the spectral density of the currents is also identically obeyed in our formalism if the two discussed unitarity relations are satisfied.

Recently Schnitzer,[6] using the Ward-identity method, has developed a program of unitarization of current-algebra amplitudes. The approach described in this paper is equivalent to his "unitarized tree approximation" in the same sense as the usual chiral Lagrangian method is equivalent to the usual current-algebra method, although the relation is not obvious because he admitted the existence of an s-wave dipion resonance.

[5] J. C. Fayolle, B. de Montaignac, M. Morand, Z. Strauchman, and H. Thibaut, Nucl. Phys. **B13**, 40 (1969).

[6] H. J. Schnitzer, Phys. Rev. Letters **24**, 1384 (1970).

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Internal Symmetry Bootstrapped from the Veneziano Representation*

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Starting with only charge conservation, we obtain $SU(2)$ symmetry as a result of imposing unitarity constraints on a set of Veneziano amplitudes.

AMONG the many nice features of the Veneziano formula,[1] the duality property is perhaps the most attractive one from a bootstrap viewpoint. During the recent attempts to include internal symmetry and spin in the Veneziano model, several suggestive links with the quark models have emerged. Harari and Rosner[2] have given simple graphical rules, in terms of quarks, for determining the exotic channels and exhibiting the duality nature. In this connection, an earlier prescription given by Chan and Paton[3] for the internal symmetry factors can be thought of as a mathematical formulation of the duality diagrams of nonstrange quarks. Bootstrap models based on mathematical quarks providing both internal-symmetry factors as well as spin factors have been considered by Mandelstam[4] and by Bardakci and Halpern.[5]

In this paper we attack the internal-symmetry problem from a different angle. Unlike the previous authors, we do not assume any internal symmetry to begin with. It is hoped that internal symmetry for the external particles would emerge as a natural consequence of the general conditions imposed on the model. Indeed, as we shall present in the following example, $SU(2)$ symmetry is obtained for a set of Veneziano amplitudes when the unitarity restriction is utilized. While a unitarity equation cannot be explicitly used in a narrow-resonance approximation, we rely heavily on the power of factorization.

Owing to its nonlinear nature, the unitarity also puts a constraint on the normalization of amplitudes. For any two amplitudes satisfying the same elastic unitarity equation, it is evident that they must be equal if they are proportional. When two Veneziano amplitudes are proportional to each other, we shall assume that they are equal if the exact amplitudes they approximated are of the above type.

If some approximate unitarization scheme would allow us to determine uniquely the over-all normalization of a given Veneziano amplitude, then the above assumption will not be needed. For example, Igi and Storrow[6] recently have done an interesting calculation with the leading-term Veneziano $\pi\pi$ amplitude[7] of $I=1$. By using the unitarity sum rule, they were able to determine the over-all normalization for the amplitudes.

The system we considered consists of three spinless mesons (of either parity) of equal mass, designated by ψ^+, ψ^-, and ψ^0. Their antiparticles form the same set. The superscripts stand for some additive, conserved quantum numbers; for convenience we call them charge. In order to write down the Veneziano amplitudes for ψ-ψ scattering, we have to specify the trajectories in various channels. For this purpose we use duality diagrams[2] as a guide.

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] H. Harari, Phys. Rev. Letters **22**, 562 (1969); J. Rosner, *ibid.* **22**, 689 (1969).

[3] H. M. Chan and J. Paton, Nucl. Phys. **B10**, 516 (1969).

[4] S. Mandelstam, Phys. Rev. **184**, 1625 (1969).

[5] K. Bardakci and M. B. Halpern, Phys. Rev. **183**, 1456 (1969).

[6] K. Igi and J. K. Storrow, CERN report, 1969 (unpublished).

[7] C. Lovelace, Phys. Letters **28B**, 204 (1968).