Deep-Inelastic e-p Structure Functions in a Ladder Model with Spin-$\frac{1}{2}$ Nucleons*

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We find the asymptotic behavior of the inelastic electron-proton structure functions W_1 and W_2 in the limit $|q^2|\to\infty$ with $\omega=2q\cdot p/|q^2|$ fixed (q is the electron four-momentum transfer and p the target nucleon four-momentum) using a ladder model for the virtual Compton amplitude. Neutral vector mesons are exchanged between the (spin-$\frac{1}{2}$) proton legs. In addition, this model (with slight modifications) gives the asymptotic behavior for a ladder model in which pseudoscalar mesons are exchanged, and the asymptotic behavior for a truss-bridge-graph model in a mixed ϕ^3-ϕ^4 field theory. In all of these models W_1 and νW_2 do not scale because of factors of $\ln|q^2|$ (with the exception of νW_2 in the truss-bridge model). In addition, we find that the structure functions are essentially Bessel functions with the argument $\ln|q^2|\ln\omega$ in the $|q^2|\to\infty$ limit for fixed $\omega\gg1$.

I. INTRODUCTION

THE inelastic structure functions W_1 and W_2, which contain important information on the dynamics of electron-proton scattering, have been investigated in a variety of field-theoretic models. These have all been ladder models for the virtual Compton amplitude, differing mainly in the spins they assign to the photon, nucleon, and particles exchanged on the rungs of the ladders. Abarbanel, Goldberger, and Treiman[1] studied a ladder model with all particles spinless and found an asymptotic behavior of $|q^2|^{-1}f(\omega)$, with $\omega=2q\cdot p/|q^2|$ the scaling variable, q the electron four-momentum transfer, and p the target nucleon four-momentum. Altarelli and Rubinstein[2] investigated a model with spin-1 photons (and the remaining particles spinless) and found that νW_2 ($\nu=q\cdot p/M$) scaled, while $W_1\sim|q^2|^{-1}\ln|q^2|\,f(\omega)$ did not scale. These calculations suggested that spin plays an essential role in determining the scaling properties of the inelastic structure functions when calculated in ladder models.

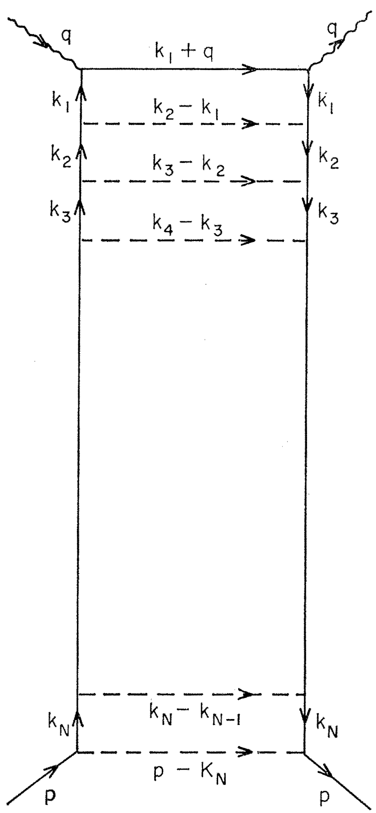

FIG. 1. (N+1)-rung-ladder graph for the virtual Compton amplitude. The solid lines are nucleon lines.

Motivated by this observation, we investigated a ladder model with spin-1 photons and spin-$\frac{1}{2}$ nucleons in which the rungs of the ladder consisted of neutral vector mesons in the Feynman gauge. Our approach was to solve a recurrence relation for the spinor polynomial associated with the N-rung ladder (see Fig. 1) and then investigate the asymptotic behavior corresponding to each term in the spinor polynomial using the Mellin-transform technique. It became apparent that we were calculating the behavior of a ladder model with neutral pseudoscalar mesons exchanged on the rungs—the difference was a factor of 2^{N-1} for the ladder with N rungs. In addition, we found that the calculation resulted in the leading behavior being given by terms which (except for a trivial factor) were studied by Bjorken and Wu[3] in a ϕ^3-ϕ^4 theory truss-bridge-graph model. Consequently, we investigated a truss-bridge-graph model with a spin-1 photon and spinless mesons in a mixed ϕ^3-ϕ^4 theory.

We found that the leading behavior of the contribution of the (N+1) rung ladder to the inelastic structure functions W_1 and νW_2, in the $|q^2|\to\infty$ fixed ω limit, has the form [Eq. (3.4)]

$$(\ln|q^2|)^N f_N(\omega) \tag{1.1}$$

in the neutral-vector-meson and pseudoscalar-meson models. We also found that the truss-bridge-graph model gave results which were intermediate between the ϕ^3 ladder-model calculation of Altarelli and Rubinstein and the models with spin-$\frac{1}{2}$ nucleons. The con-

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[1] H. D. I. Abarbanel, M. L. Goldberger, and S. B. Treiman, Phys. Rev. Letters 22, 500 (1969).

[2] G. Altarelli and H. R. Rubinstein, Phys. Rev. 187, 2111 (1969).

[3] J. D. Bjorken and T. T. Wu, Phys. Rev. 130, 2566 (1963).

tribution of the $(N+1)$-rung truss graph (Fig. 3) to W_1 had the form of Eq. (1.1) while the contribution to νW_2 scaled [see Eqs. (4.4) and (4.6)].

After finding the $|q^2|\to\infty$ (with ω fixed) asymptotic behavior of the structure functions, we investigated the $\omega\gg1$ region (which is essentially the Regge limit) and found the leading ω behavior of the functions $f_N(\omega)$. As a result, the graphs are easily summed and the leading behavior as $|q^2|\to\infty$ for fixed ω, $\omega\gg1$, is as follows.

(i) neutral-vector-meson model:

$$MW_1=\left(\frac{g^2\ln|q^2|}{32\pi^2\ln\omega}\right)^{1/2} I_1\left(\frac{g}{2\pi}(2\ln|q^2|\ \ln\omega)^{1/2}\right), \quad (1.2)$$

$$\nu W_2=\frac{1}{\omega}\left(\frac{g^2\ln|q^2|}{8\pi^2\ln\omega}\right)^{1/2} I_1\left(\frac{g}{2\pi}(2\ln|q^2|\ \ln\omega)^{1/2}\right); \quad (1.3)$$

(ii) γ_5 pseudoscalar-meson model:

$$MW_1=\left(\frac{g^2\ln|q^2|}{64\pi^2\ln\omega}\right)^{1/2} I_1\left(\frac{g}{2\pi}(\ln|q^2|\ \ln\omega)^{1/2}\right), \quad (1.4)$$

$$\nu W_2=\frac{1}{\omega}\left(\frac{g^2\ln|q^2|}{16\pi^2\ln\omega}\right)^{1/2} I_1\left(\frac{g}{2\pi}(\ln|q^2|\ \ln\omega)^{1/2}\right); \quad (1.5)$$

(iii) Truss-bridge model:

$$W_1=\frac{g^2}{q\cdot p}\left(\frac{\ln|q^2|}{G\ln\omega}\right)^{1/2} I_1(2(G\ln|q^2|\ \ln\omega)^{1/2}), \quad (1.6)$$

$$\nu W_2=(2g^2/\omega^2)\sinh(G^{1/2}\ln\omega), \quad (1.7)$$

with $I_1(z)$ a Bessel function,[4] g the coupling constant of the ϕ^3 term, and G the coupling constant of the ϕ^4 term in the interaction Lagrangian for case (iii). (The mass is set equal to 1 in this case.)

The inelastic structure functions are the imaginary parts of the virtual Compton amplitude in the forward direction:

$$W_j=(1/\pi)\,\mathrm{Im}T_j, \quad j=1,2$$

where the virtual Compton amplitude is

$$T_{\mu\nu}=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)T_1 +\frac{1}{M^2}\left(p_\mu-\frac{q\cdot p}{q^2}q_\mu\right)\left(p_\nu-\frac{q\cdot p}{q^2}q_\nu\right)T_2. \quad (1.8)$$

The structure functions are normalized so that the cross section for inelastic electron-proton scattering is given by[5]

$$\frac{d\sigma}{d\Omega dE'}=\frac{\alpha^2}{4E^2\sin^4(\frac12\theta)} \times[2W_1\sin^2(\tfrac12\theta)+W_2\cos^2(\tfrac12\theta)], \quad (1.9)$$

where E and E' are the initial and final electron energies, respectively, and θ is the electron scattering angle (all three variables are measured in the laboratory frame). Since the electromagnetic current is conserved, we concentrate our attention on the $p_\mu p_\nu$ and $g_{\mu\nu}$ parts of the spinor traces in subsequent sections. Their coefficients are T_2 and T_1, respectively. The contribution to T_2 of the $(N+1)$-rung diagram is denoted T_{2N} and the contribution to T_1 is denoted T_{1N}.

In Sec. II we describe the vector-meson model we investigated. Section III A contains the calculation of the asymptotic behavior of terms in the spinor polynomial which lead to Eqs. (1.2) and (1.3). Section III B shows that the remaining terms in the spinor polynomial do not contribute to the leading asymptotic behavior of the N-rung graph. In Sec. IV we evaluate the asymptotic behavior of truss-bridge graphs.

After our work was completed we became aware of Chang and Fishbane's[6] calculation of W_2 in a ladder model with spin-$\frac12$ nucleons and pseudoscalar mesons. They use infinite-momentum techniques to evaluate the leading behavior of graphs. If one keeps only the leading ω behavior (corresponding to keeping only the leading pole in the Mellin-transform variable, τ^{-m} in their notation), then their results agree with our Eq. (1.5).

II. LADDER MODEL WITH SPIN-$\frac12$ NUCLEONS

The model we use to calculate the forward Compton amplitude for massive photons scattering off spinor nucleons assumes that the amplitude is a sum of ladder graphs in which neutral vector mesons are exchanged between the external nucleon legs (Fig. 1). We further assume that the vector-meson–nucleon interaction Hamiltonian density is given by $:g\bar\psi\gamma_\mu\psi B^\mu:$, with B^μ the neutral-vector-meson field, and we use the vector-meson propagator in the Feynman gauge, i.e.,

$$-ig_{\mu\nu}/(k^2-M^2).$$

Since this neutral-vector-meson–nucleon theory has a conserved current, the effect of the $k_\mu k_\nu$ part of the vector-meson propagator is expected to be canceled when all graphs are summed.

We now proceed to evaluate the integral correspond-

[4] Bateman Manuscript Project, *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill, New York, 1954), p. 371.

[5] S. D. Drell and J. D. Walecka, Ann. Phys. (N. Y.) **28**, 18 (1964).

[6] S. J. Chang and P. M. Fishbane, Phys. Rev. D **2**, 1084 (1970).

ing to Fig. 1 with the nucleon spins averaged:

$$T_{N\mu\nu}=\frac{-1}{4M}\left(\frac{-ig^2}{(2\pi)^4}\right)^N\int\left(\prod^N d^4k_i\right)\\ \times\mathrm{Tr}[\gamma_\mu(\boldsymbol{k}_1+\boldsymbol{q}+M)\gamma_\nu P_N]/\left(\prod_{j=1}^N(k^2{}_j-M^2)^2\right)\\ \times\left(\prod_{i=1}^{N-1}[(k_{i+1}-k_i)^2-m^2]\right)[(p-k_N)^2-m^2]\\ \times[(q+k_1)^2-M^2],\quad(2.1)$$

with P_N given in Eq. (A1), m the neutral-vector-meson mass, and M the nucleon mass. The denominator factors can be exponentiated through use of the identity

$$\frac{1}{L^2-M^2+i\epsilon}=-i\int_0^\infty dz\, e^{iz(L^2-M^2+i\epsilon)}.\quad(2.2)$$

The Feynman parameter associated with each line is shown in Fig. 2. In addition we introduce the momenta $A_0, A_1, \ldots, A_N$, as shown in Fig. 2, into the exponential

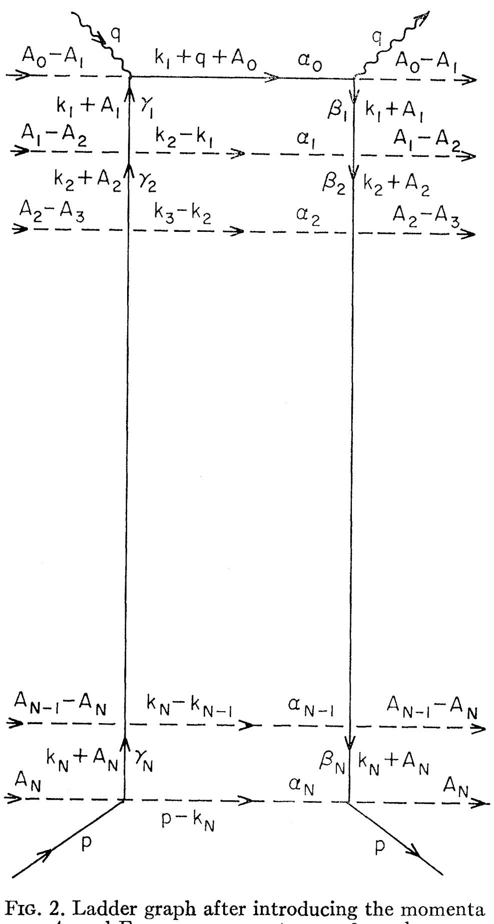

FIG. 2. Ladder graph after introducing the momenta A_i and Feynman parameters α_i, β_i, and γ_i.

only:

$$T_{N\mu\nu}=\frac{i}{4M}\left(\frac{g^2}{(2\pi)^4}\right)^N\int_0^\infty d\Omega\int\prod^N d^4k_j\,\mathcal{P}_N\\ \times\exp\{i\alpha_0[(k_1+q+A_0)^2-M^2]+i\alpha_N[(p-k_N)^2-m^2]\\ +i\sum_{r=1}^N(\beta_r+\gamma_r)[(k_r+A_r)^2-M^2]\\ +i\sum_{r=1}^{N-1}\alpha_r[(k_{r+1}-k_r)^2-m^2]\},\quad(2.3)$$

where $\mathcal{P}_N=\mathrm{Tr}[\gamma_\mu(\boldsymbol{q}+\boldsymbol{k}_1+M)\gamma_\nu P_N]$ is unchanged from Eq. (2.1) and $d\Omega=d\alpha_0 d\alpha_1\cdots d\alpha_N d\beta_1\cdots d\beta_N d\gamma_1\cdots d\gamma_N$. We now replace loop momenta in $\mathcal{P}_N$ with derivatives with respect to the introduced momenta A_i. Thus $\boldsymbol{q}+\boldsymbol{k}_1$ is replaced with $(2i\alpha_0)^{-1}\gamma^\mu(\partial/\partial A_0{}^\mu)$. Similarly, in each monomial term in P_N, $k_{j\mu}$ is replaced with

$$[2i(\beta_j+\gamma_j)]^{-1}(\partial/\partial A_j{}^\mu).$$

If a given loop momentum appears quadratically in any term, then the following replacement is made:

$$k_{r\mu}k_{r\nu}\to\frac{1}{[2i(\beta_r+\gamma_r)]^2}\\ \times\left[\frac{\partial^2}{\partial A_r{}^\mu\partial A_r{}^\nu}-2i(\beta_r+\gamma_r)g_{\mu\nu}\right].\quad(2.4)$$

Setting all $A_i=0$ after applying these derivatives would give the original expression for $\mathcal{P}_N$.

Since $\mathcal{P}_N$ is no longer a function of loop momenta, the integral over the N loop momenta is now the same as one would encounter in, for example, a ϕ^3-theory calculation. Following the usual procedure,[7] the loop integrals can be evaluated and the result is

$$T_{N\mu\nu}=\frac{i}{4M}\left(\frac{g^2}{16\pi^2 i}\right)^N\\ \times\int_0^\infty\frac{d\Omega}{C^2}(\mathcal{P}_N e^{iD(A)/C})_{A_0=A_1=\cdots=A_N=0},\quad(2.5)$$

where C and $D(A)$ are Feynman-parameter polynomials given in Eqs. (B4) and (B6).

III. VECTOR-MESON-MODEL CALCULATION

A. Leading Terms

In Appendix A we show that P_N may be written

$$P_N=P_N{}^a+P_N{}^b\boldsymbol{p}+\sum_{j=1}^N\tilde{P}_N{}^j\boldsymbol{k}_j,$$

[7] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., New York, 1966), pp. 151–154.

where $\tilde{P}_N{}^i$ is a scalar function of momenta for $i=a$, b, 1, 2, ..., N. Before replacing the loop momenta with derivatives with respect to the A_i, $\mathcal{P}_N$ would have been written

$$\tfrac{1}{4}\mathcal{P}_N=g_{\mu\nu}[MP_N{}^a-(q+k_1)\cdot(pP_N{}^b+\sum_{j=1}^N k_j\tilde{P}_N{}^j)]+(q+k_1)_\mu(p_\nu P_N{}^b+\sum_{j=1}^N k_{j\nu}\tilde{P}_N{}^j)+(q+k_1)_\nu(p_\mu P_N{}^b+\sum_{j=1}^N k_{j\mu}\tilde{P}_N{}^j).$$

Our first task will be to find the asymptotic behavior of the term in the Feynman integral corresponding to $(q+k_1)_\mu p_\nu P_N{}^b$, which is in many respects the easiest to evaluate and the best for illustrating our method. In Appendix A we show that

$$P_N{}^b=2^N\prod_{i=1}^N(k_i{}^2-M^2),$$

and if we had inserted this expression in Eq. (2.1), we would have been able to cancel N factors in the denominator and thus have not had to introduce N Feynman parameters $\gamma_1,\gamma_2,\ldots,\gamma_N$. Taking advantage of this, we drop the $\prod(k_i{}^2-M^2)$ factor in $P_N{}^b$, set $\gamma_j=0$ for all j (dropping the γ_j integrations), and introduce a factor of i^N to compensate for the unnecessary factors of $-i$ introduced by (2.2). Thus the contribution of the $(q+k_1)_\mu p_\nu P_N{}^b$ term to $T_{N\mu\nu}$ becomes [using Eq. (B6)]

$$\mathcal{H}_{N\mu\nu}=\frac{ip_\nu}{M}\left(\frac{g^2}{8\pi^2}\right)^N\int_0^\infty\frac{d\Omega}{C^2}\left(\frac{1}{2i\alpha_0}\frac{\partial}{\partial A_0{}^\mu}e^{iD(A)/C}\right)_{A_0=A_1=\cdots=0}=\frac{ip_\nu}{M}\left(\frac{g^2}{8\pi^2}\right)^N\int_0^\infty\frac{d\Omega}{C^2}\frac{p_\mu d_{1N}+q_\mu C'}{C}e^{iD/C},\qquad(3.1)$$

where it is understood that all γ_j are set equal to zero in the expressions for D and C which are given in Eqs. (B4) and (B13):

$$D=D(0)=\alpha_0q^2(\rho d_{1N}+C')+JC,$$

with $C'=C|_{\alpha_0=0}$, $d_{ij}=\alpha_i\alpha_{i+1}\cdots\alpha_j$, and $\rho=2q\cdot p/q^2$ ($\omega=|\rho|$). We keep the $p_\mu p_\nu$ part of $\mathcal{H}_{N\mu\nu}$, since it will eventually lead to a contribution to T_2:

$$\mathcal{H}_N=\frac{i}{M}\left(\frac{g^2}{8\pi^2}\right)^N\int_0^\infty\frac{d\Omega}{C^3}d_{1N}e^{iD/C}.\qquad(3.2)$$

The use of the Mellin-transform technique[7] will enable us to find the leading asymptotic behavior of $\mathcal{H}_N$. We first determine the leading $|q^2|$ behavior of $\mathcal{H}_N$ for fixed ρ.

The Mellin transform of $\mathcal{H}_N$ in $w=|q^2|$ is[8]

$$\tilde{\mathcal{H}}_N=\int_0^\infty dw\,w^{-\eta-1}\mathcal{H}_N=\frac{i}{M}\left(\frac{g^2}{8\pi^2}\right)^N i^\eta\Gamma(-\eta)\times\int_0^\infty\frac{d\Omega}{C^{3+\eta}}d_{1N}\alpha_0{}^\eta(\rho d_{1N}+C')^\eta e^{iJ},\quad \mathrm{Re}\eta<0.$$

$\tilde{H}_N$ is defined for $\mathrm{Re}\eta>-1$ but becomes divergent at $\eta=-1$ since the α_0 integration then diverges. There are additional divergences in $\mathcal{H}_N$ at $\eta=-1$ which can be brought out by scaling over various sets of Feynman parameters. Let

$$\begin{aligned}
&\alpha_0=x_1x_2x_3\cdots x_N\tilde{\alpha}_0,\\
&\alpha_1=x_1x_2\cdots x_N\tilde{\alpha}_1,\\
&\beta_1=x_1\cdots x_N\tilde{\beta}_1=x_1\cdots x_N(1-\tilde{\alpha}_0-\tilde{\alpha}_1),\\
&\alpha_2=x_2x_3\cdots x_N\tilde{\alpha}_2,\\
&\beta_2=x_2x_3\cdots x_N\tilde{\beta}_2\\
&\qquad=x_2x_3\cdots x_N[1-x_1(\tilde{\alpha}_0+\tilde{\alpha}_1+\tilde{\beta}_1)-\tilde{\alpha}_2],\qquad(3.3)\\
&\alpha_3=x_3x_4\cdots x_N\tilde{\alpha}_3,\\
&\quad\vdots\\
&\alpha_N=x_N\tilde{\alpha}_N,\\
&\beta_N=x_N\tilde{\beta}_N=x_N[1-x_1x_2\cdots x_{N-1}(\tilde{\alpha}_0+\tilde{\alpha}_1+\tilde{\beta}_1)\\
&\qquad-x_2x_3\cdots x_{N-1}(\tilde{\alpha}_2+\tilde{\beta}_2)\\
&\qquad-\cdots-x_{N-1}(\tilde{\alpha}_{N-1}+\tilde{\beta}_{N-1})-\tilde{\alpha}_N].
\end{aligned}$$

The Jacobian for this transformation is $x_1{}^2x_2{}^4x_3{}^6\cdots x_N{}^{2N}$ and the homogeneous polynomial C becomes $x_1x_2{}^2x_3{}^3\cdots x_N{}^N\tilde{C}$, where $\tilde{C}$ is the same function of $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ as C was of α_i and β_i. Under this transformation $\tilde{\mathcal{H}}_N$ becomes

$$\tilde{\mathcal{H}}_N=\frac{i}{M}\left(\frac{g^2}{8\pi^2}\right)^N i^\eta\Gamma(-\eta)\int_0^1 d\tilde{\Omega}\int_0^\infty\frac{dx_1dx_2\cdots dx_N}{C^{3+\eta}}\times\tilde{d}_{1N}(\alpha_0x_1x_2\cdots x_N)^\eta(\rho d_{1N}+C')^\eta e^{iJ}\delta(1-\tilde{\alpha}_0-\tilde{\alpha}_1-\tilde{\beta}_1)\times\delta(1-x_1(\tilde{\alpha}_0+\tilde{\alpha}_1+\tilde{\beta}_1)-\tilde{\alpha}_2)\cdots,$$

where $d\tilde{\Omega}$, $\tilde{d}_{1N}$, and $\tilde{C}$ have α_i and β_i replaced with $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, and each term in J contains at least one factor of x_i for some i. The Dirac δ functions $\delta(x)$ implement the constraints in the scaling [Eq. (3.3)].

We can now partially integrate $N+1$ times and exhibit the integral's singularity at $\eta=-1$:

$$\tilde{\mathcal{H}}_N=(-1)^{N+1}\frac{i}{M}\left(\frac{g^2}{8\pi^2}\right)^N i^\eta\Gamma(-\eta)\times\int_0^1 d\tilde{\Omega}\int_0^\infty dx\frac{(\tilde{\alpha}_0x_1x_2\cdots x_N)^{\eta+1}}{(\eta+1)^{N+1}}\frac{\partial^{N+1}}{\partial\tilde{\alpha}_0\partial x_1\partial x_2\cdots\partial x_N}\times\left[\frac{d_{1N}(\rho\tilde{d}_{1N}+\tilde{C}')^\eta}{C^{3+\eta}}e^{iJ}\delta(1-\tilde{\alpha}_0-\tilde{\alpha}_1-\tilde{\beta}_1)\times\delta(1-x_1(\tilde{\alpha}_0+\tilde{\alpha}_1+\tilde{\beta}_1)-\tilde{\alpha}_2)\cdots\right],$$

[8] Reference 4, p. 312.

with $dx = dx_1 dx_2 \cdots dx_N$. Since the leading singularity is an $(N+1)$th-order pole at $\eta = -1$, the leading asymptotic behavior of $\mathcal{K}_N$ in $|q^2|$ (for fixed ρ) is given by $|q^2|^{-1}(N!)^{-1}(\ln|q^2|)^N$ times the residue of $\mathcal{K}_N$ at $\eta = -1$:

$$\mathcal{K}_N \to \frac{1}{M|q^2|N!}\left(\frac{g^2 \ln|q^2|}{8\pi^2}\right)^N \int_0^1 \frac{d\Omega' d_{1N}\delta(1-\alpha_1-\beta_1)\delta(1-\alpha_2-\beta_2)\cdots\delta(1-\alpha_N-\beta_N)}{C'^2(\rho d_{1N}+C')}, \tag{3.4}$$

where we have dropped the tilde over the Feynman parameters and let $d\Omega' = d\alpha_1 d\alpha_2 \cdots d\alpha_N d\beta_1 d\beta_2 \cdots d\beta_N$. Equation (3.4) displays the leading $|q^2|$ behavior of $\mathcal{K}_N$ for fixed ρ.

We now find the leading ρ behavior of the remaining Feynman-parameter integral. This corresponds to going to the Regge limit. The Mellin transform of Eq. (3.4) in $w = \rho$ gives[9]

$$\frac{1}{M|q^2|N!}\left(\frac{g^2 \ln|q^2|}{8\pi^2}\right)^N (-\pi \csc\pi\eta) \times \int_0^1 \frac{d\Omega'\, d_{1N}{}^{1+\eta}\delta(1-\alpha_1-\beta_1)\cdots\delta(1-\alpha_N-\beta_N)}{C'^{3+\eta}}. \tag{3.5}$$

There is a pole of order 1 at $\eta = -1$ due to the $\csc\pi\eta$ factor and a pole of order $N+1$ at $\eta = -2$ due to the $\csc\pi\eta$ factor and the poles which appear upon partially integrating with respect to $\alpha_1, \alpha_2, \ldots, \alpha_N$ in the same manner as above. The result, after evaluating the residues at the various poles, is

$$\mathcal{K}_N \to \frac{1}{M|q^2|\rho N!}\left(\frac{g^2 \ln|q^2|}{8\pi^2}\right)^N \times \int_0^1 \frac{d\Omega'\delta(1-\alpha_1-\beta_1)\delta(1-\alpha_2-\beta_2)\cdots\delta(1-\alpha_N-\beta_N)}{C'^2} - \frac{1}{M|q^2|\rho^2(N!)^2}\left(\frac{g^2 K_1 \ln|q^2|\,\ln\rho}{8\pi^2}\right)^N,$$

where

$$K_1 = \int_0^1 \frac{d\beta\, \delta(1-\beta)}{\beta} = 1. \tag{3.6}$$

The first term gives no contribution to the absorptive part of $T_{\mu\nu}$ when summed over N so we drop it from consideration. Thus the contribution of the

$$p_\mu(q+k_1)_\nu P_N{}^b + p_\nu(q+k_1)_\mu P_N{}^b$$

term to the $p_\mu p_\nu$ part of $T_{N\mu\nu}$ is

$$\frac{-2p_\mu p_\nu}{M|q^2|\rho^2(N!)^2}\left(\frac{g^2 K_1 \ln|q^2|\,\ln\rho}{8\pi^2}\right)^N. \tag{3.7}$$

The next term we examine is $-g_{\mu\nu}(q+k_1)\cdot p P_N{}^b$. The calculation of the leading contribution of this term to $T_{N\mu\nu}$ is very similar to the calculation just completed. The Feynman-parameter integral for this term is given by the right-hand side of Eq. (3.1) with μ and ν summed:

$$\mathcal{G}_{N\mu\nu} = \frac{-ig_{\mu\nu}}{M}\left(\frac{g^2}{8\pi^2}\right)^N \int_0^\infty \frac{d\Omega(M^2 d_{1N} + p\cdot qC')}{C^3} e^{iD/C}, \tag{3.8}$$

where $\gamma_1 = \gamma_2 = \cdots = \gamma_N = 0$. From our previous calculation it is evident that the part of $\mathcal{G}_{N\mu\nu}$ corresponding to $p\cdot qC'$ will dominate the leading part of $\mathcal{G}_{N\mu\nu}$ by a factor of $\rho q\cdot p/M^2$ compared to the $M^2 d_{1N}$ part of the $\mathcal{G}_{N\mu\nu}$ integral. The relative factor of ρ is due to the suppression of the leading pole from $\eta = -1$ to $\eta = -2$ by the d_{1N} factor in Eq. (3.5). This factor of $q\cdot p\rho$ is the only difference between the two calculations. Therefore the leading contribution to $T_{N\mu\nu}$ coming from the term $-g_{\mu\nu}(q+k_1)\cdot pP_N{}^b$ is

$$\frac{-g_{\mu\nu}q\cdot p}{M|q^2|\rho(N!)^2}\left(\frac{g^2 K_1 \ln|q^2|\,\ln\rho}{8\pi^2}\right)^N = \frac{g_{\mu\nu}}{2M(N!)^2}\left(\frac{g^2 K_1 \ln|q^2|\,\ln\rho}{8\pi^2}\right)^N. \tag{3.9}$$

In Sec. III B we calculate the leading behavior of the remaining terms of $\mathcal{P}_N$. Although there are terms with exactly the same leading $|q^2|$ and ρ behavior as in Eqs. (3.7) and (3.9), a remarkable cancellation takes place in the numerical coefficients of those terms proportional to $p_\mu p_\nu$ and $g_{\mu\nu}$ so that the final numerical coefficients are zero. As a result, Eqs. (3.7) and (3.9) give the leading behavior of tke $p_\mu p_\nu$ and $g_{\mu\nu}$ parts of $T_{N\mu\nu}$. Section III B may be skipped without loss of continuity by the reader who is more interested in the results.

B. Completion of Spinor-Nucleon-Model Calculation

In this section we evaluate the contributions to the leading behavior of $T_{N\mu\nu}$ from the terms

$$(q+k_1)_\mu \sum_{j=1}^N k_{j\nu}\tilde{P}_N{}^j + (q+k_1)_\nu \sum_{j=1}^N k_{j\mu}\tilde{P}_N{}^j + g_{\mu\nu}\left[MP_N{}^a - (q+k_1)\cdot \sum_{j=1}^N k_j\tilde{P}_N{}^j\right].$$

The first term we consider is

$$(q+k_1)_\nu \sum_{j=1}^N k_{j\mu}\tilde{P}_N{}^j.$$

[9] Reference 4, p. 308.

According to Appendix A, $\tilde{P}_N{}^j$ has the form

$$\tilde{P}_N{}^j=2^{j-1}\prod_{r=1}^{j-1}(k_r{}^2-M^2)\sum_{s=0}^{N-j}\sum_{\{e\}}B_j{}^{e_1}B_{e_1}{}^{e_2}\cdots B_{e_s}{}^{N+1},$$

with $\{e\}=\{e_1,e_2,\ldots,e_s\}$, $j<e_1<e_2<\cdots<e_s\leqslant N$, and

$$B_i{}^j=2(4^{j-i}M^2-2^{j-i}k_i\cdot k_j)\prod_{r=i+1}^{j-1}(k_r{}^2-M^2).$$

Since we are interested in the leading behavior of $T_{N\mu\nu}$, we approximate $B_i{}^j$ by

$$B_i{}^j\sim -2^{j+1-i}k_i\cdot k_j\prod_{r=i+1}^{j-1}(k_r{}^2-M^2),\qquad(3.10)$$

and as a result we obtain an expression for $\tilde{P}_N{}^j$ which is exact to leading order:

$$\tilde{P}_N{}^j\to 2^N\prod_{i=1}^{N}(k_i{}^2-M^2)\sum_{s=0}^{N-j}(-2)^{s+1}\times\sum_{\{e\}}\frac{k_j\cdot k_{e_1}k_{e_1}\cdot k_{e_2}\cdots k_{e_{s-1}}\cdot k_{e_s}k_{e_s}\cdot p}{(k_j{}^2-M^2)(k_{e_1}{}^2-M^2)\cdots(k_{e_s}{}^2-M^2)}.$$

Following the same procedure as in the calculations of Sec. III A, we cancel $N-s-1$ factors in the denominator of the integral in Eq. (2.1) and of the N γ_i only keep the parameters $\gamma_j,\gamma_{e_1},\gamma_{e_2},\gamma_{e_3},\cdots,\gamma_{e_s}$. Therefore, with all γ_i equal to zero but these, we obtain the following contribution to $T_{N\mu\nu}$:

$$H_{N\mu\nu}=\frac{i}{M}\left(\frac{g^2}{8\pi^2 i}\right)^N\sum_{j=1}^{N}\sum_{s=0}^{N-j}i^{N-s-1}(-2)^{s+1}\sum_{\{e\}}\int_0^\infty\frac{d\Omega}{C^2}\times[(q+k_1)_\nu k_{j\mu}k_j\cdot k_{e_1}k_{e_1}\cdot k_{e_2}\cdots k_{e_s}\cdot p\times e^{iD(A)/C}]_{A_0=A_1=\cdots=0},$$

where it is understood that the loop momenta in the integral represent derivative operators. The derivative represented by $(q+k_1)_\nu$ is $(2i\alpha_0)^{-1}\,\partial/\partial A_0{}^\nu$ and use of Eq. (2.4) for the remaining $s+1$ pairs of derivatives leads to the following derivative expression:

$$(q+k_1)_\nu k_{j\mu}k_j\cdot k_{e_1}\cdots k_{e_{s-1}}\cdot k_{e_s}k_{e_s}\cdot p\to\frac{(-1)^{s+1}p^\lambda}{(2i)^{s+1}Z_{e_0}Z_{e_1}\cdots Z_{e_s}}\sum_{m=0}^{s+1}\sum_{\substack{\{v\};v_1<v_2\cdots<v_m\\ v_i\in\{e_0,e_1,\ldots,e_s\}}}\frac{(-1)^m}{(2i)^{m+1}Z_{v_1}Z_{v_2}\cdots Z_{v_m}\alpha_0}\times\frac{\partial^{2m+1}}{\partial A_0{}^\nu\partial A_{v_1}{}^\mu\partial A_{v_1}{}^{\nu_1}\partial A_{v_2\nu_1}\partial A_{v_2}{}^{\nu_2}\partial A_{v_3\nu_2}\cdots\partial A_{v_{m-1}\nu_{m-1}}\partial A_{v_m}{}^{\nu_{m-1}}\partial A_{v_m}{}^\lambda},$$

where we have expanded the factors and summed over the indices of the $g_{\alpha\beta}$ which appear. We have also let $e_0=j$ and $Z_i=\beta_i+\gamma_i$. The $m=0$ term in the sum is

$$\frac{(-1)^{s+1}p_\mu}{(2i)^{s+2}\alpha_0Z_{e_0}Z_{e_1}\cdots Z_{e_s}}\frac{\partial}{\partial A_0{}^\nu}.$$

We have applied these $(2m+1)$-order derivatives to $e^{iD(A)/C}$ in Appendix C and found that the part of the resulting polynomials that gave contributions to the leading behavior of the $p_\mu p_\nu$ part of $T_{N\mu\nu}$ is [Eq. (C10)]

$$p_\mu p_\nu\sum_{h=0}^{m}\left(\frac{2i}{C}\right)^{2m+1-h}(-1)^m f(m,h)d_{1N}(d_{0N}p\cdot q)^{m-h}\times\prod_{i=1}^{m}C_{v_{i-1}}{}^0C_N{}^{v_i},\qquad(3.11)$$

where $f(m,h)$ is an integer and $C_j{}^i$ is a Feynman-parameter polynomial defined in Appendix B. After substitution of these expressions into $H_{N\mu\nu}$, we obtain

$$H_{N\mu\nu}\to\frac{ip_\mu p_\nu}{M}\left(\frac{g^2}{8\pi^2}\right)^N\sum_{j=1}^{N}\sum_{s=0}^{N-j}(-1)^{s+1}\times\sum_{\{e\}}\sum_{m=0}^{s+1}\sum_{\{v\}}\sum_{h=0}^{m}(2ip\cdot q)^{m-h}f(m,h)\mathcal{H}_{\{e\}\{v\}}{}^{Nsmh},\qquad(3.12)$$

with

$$\mathcal{H}_{\{e\}\{v\}}{}^{Nsmh}=\int_0^\infty\frac{d\Omega\, e^{iD/C}d_{1N}(d_{0N})^{m-h}}{C^{2m+3-h}Z_{e_0}Z_{e_1}\cdots Z_{e_s}}\times\prod_{r=1}^{m}Z_{v_r}C_{v_{r-1}}{}^0C_N{}^{v_r}.\qquad(3.13)$$

The calculation of the asymptotic behavior in $|q^2|$ for fixed ρ of the integral in Eq. (3.13), and then the leading ρ behavior, is similar to the calculations of Sec. III A. The Mellin transform in $|q^2|$ of $\mathcal{H}_{\{e\}\{v\}}{}^{Nsmh}$ is

$$\tilde{\mathcal{H}}_{\{e\}\{v\}}{}^{Nsmh}=i^\eta\Gamma(-\eta)\times\int_0^\infty\frac{d\Omega\, d_{1N}{}^{m-h+1}\alpha_0{}^{m-h+\eta}(\rho d_{1N}+C')^\eta e^{iJ}U}{Z_{e_0}Z_{e_1}\cdots Z_{e_s}C^{2m+3-h+\eta}},$$

where

$$U=\prod_{r=1}^{m}Z_{v_r}C_{v_{r-1}}{}^0C_N{}^{v_r}.$$

If we scale as in Eqs. (3.3), modifying the scaling to include the γ_{e_i} for $i=0,1,2,\ldots,s$,

$$\gamma_{e_i}=x_{e_i}x_{e_i+1}\cdots x_N\tilde{\gamma}_{e_i},$$

and including $\tilde{\gamma}_{e_i}$ in the appropriate δ functions, then we obtain

$$\mathcal{K}_{\{e\}\{v\}}{}^{Nsmh}=i^{\eta}\Gamma(-\eta)\int_0^1 d\tilde{\Omega}\int_0^{\infty}\frac{dx_1\cdots dx_N\tilde{d}_{1N}{}^{m-h+1}(\tilde{\alpha}_0x_1x_2\cdots x_N)^{m-h+\eta}}{\tilde{Z}_{e0}\tilde{Z}_{e1}\cdots\tilde{Z}_{e_s}\tilde{C}^{2m+3-h+\eta}}(\rho\tilde{d}_{1N}+\tilde{C}')^{\eta}e^{iJ}\tilde{U}\delta(1-\tilde{\alpha}_0-\tilde{\alpha}_1-\tilde{\beta}_1)\cdots.$$

Partially integrating with respect to $\tilde{\alpha}_0, x_1,x_2, \ldots, x_N$ gives a pole of order $N+1$ at $\eta=h-m-1$. This implies that the leading $|q^2|$ behavior (ρ fixed) is

$$\mathcal{K}_{\{e\}\{v\}}{}^{Nsmh}\to\frac{\Gamma(m+1-h)(\ln|q^2|)^N}{(i|q^2|)^{m+1-h}N!}\int_0^1\frac{d\tilde{\Omega}'\tilde{d}_{1N}{}^{m-h+1}\tilde{U}\delta(1-\tilde{\alpha}_1-\tilde{\beta}_1)\cdots\delta(1-\tilde{\alpha}_{ei}-\tilde{\beta}_{ei}-\tilde{\gamma}_{ei})\cdots}{\tilde{Z}_{e0}\cdots\tilde{Z}_{e_s}\tilde{C}'^{m+2}(\rho\tilde{d}_{1N}+\tilde{C}')^{m+1-h}} \qquad (3.14)$$

after evaluating the residue at $\eta=h-m-1$.

We now investigate the $|\rho|\gg1$ behavior of the remaining Feynman-parameter integral in Eq. (3.14). The Mellin transform in ρ is[9]

$$\frac{\Gamma(m+1-h)}{N!(i|q^2|)^{m+1-h}}(\ln|q^2|)^N(-1)^{m-h}[-\pi\csc(\pi\eta)]$$
$$\times\binom{-\eta-1}{m-h}\int_0^1\frac{d\Omega' d_{1N}{}^{m-h+1+\eta}U\delta(1-\alpha_1-\beta_1)\cdots}{Z_{e0}Z_{e1}\cdots Z_{e_s}C'^{2m+3-h+\eta}}.$$

Partially integrating $\alpha_1, \alpha_2, \cdots, \alpha_N$ exhibits a pole at $\eta=h-m-2$ of order $N+1$ [counting the pole due to $\csc(\pi\eta)$]. This is the leading pole which results in an eventual contribution to the absorptive part of $T_{N\mu\nu}$. The leading behavior is

$$\mathcal{K}_{\{e\}\{v\}}{}^{Nsmh}\to\frac{-(m+1-h)!(\ln|q^2|\ \ln\rho)^N}{(i|q^2|\rho)^{m-h}i|q^2|\rho^2(N!)^2}K_1{}^{N-s-1}K_2{}^{s+1},$$

with K_1 given by Eq. (3.6) and

$$K_2=\int_0^1\frac{d\beta d\gamma\delta(1-\beta-\gamma)}{(\beta+\gamma)^2}. \qquad (3.15)$$

The factors of K_2 come from the $s+1$ γ_{e_i}'s which occurred in this calculation. There was a pole at $\eta=h-m-1$ of first order due to $\csc(\pi\eta)$ which we ignored since that term does not lead to a contribution to the absorptive part of $T_{N\mu\nu}$.

If we put the last expressions and Eq. (3.12) together, we obtain the leading behavior of the $p_\mu p_\nu$ part of $(q+k_1)_\nu\sum k_{j\mu}\tilde{P}_N{}^j+(q+k_1)_\mu\sum k_{j\nu}\tilde{P}_N{}^j$:

$$\frac{-2p_\mu p_\nu}{M|q^2|\rho^2(N!)^2}\left(\frac{g^2K_1\ln|q^2|\ \ln\rho}{8\pi^2}\right)^N\sum_{j=1}^{N}\sum_{s=0}^{N-j}\left(\frac{-K_2}{K_1}\right)^{s+1}$$
$$\times\binom{N-j}{s}\sum_{m=0}^{s+1}\binom{s+1}{m}\sum_{h=0}^{m}(-1)^{m-h}$$
$$\times(m-h+1)!f(m,h), \qquad (3.16)$$

where the sum over $\{e\}$, such that $j<e_1<e_2<\cdots e_s\leqslant N$, was replaced by

$$\binom{N-j}{s},$$

which counts the number of these (identical) terms, the sum over $\{v\}$ was replaced by

$$\binom{s+1}{m}$$

for similar reasons, and we used $2p\cdot q=-|q^2|\rho$. In Appendix C we prove that

$$\sum_{h=0}^{m}(-1)^h(m-h+1)!f(m,h)=1$$

for all m. Substituting this into Eq. (3.16) and noting

$$\sum_{m=0}^{s+1}\binom{s+1}{m}(-1)^m=(1-1)^{s+1}=0$$

proves that the numerical coefficient is zero and that these terms do not contribute to the leading behavior of the $p_\mu p_\nu$ part of $T_{N\mu\nu}$.

We now find the leading behavior coming from

$$-g_{\mu\nu}(q+k_1)\sum_{j=1}^{N}k_j\tilde{P}_N{}^j.$$

The calculation proceeds in the same manner as the previous calculation. However, the polynomial resulting from applying derivatives to $e^{iD(A)/C}$ has a different leading part. Instead of having the leading part given by Eq. (3.11), we now have the expression given by Eq. (C13):

$$\sum_{h=0}^{m-1}\left(\frac{2i}{C}\right)^{2m+1-h}(-1)^m p\cdot qC'(d_{0N}p\cdot q)^{m-h}$$
$$\times f_1(m,h)\prod_{i=1}^{m}C_{vi-1}{}^0C_N{}^{v_i},$$

with $f_1(m,h)$ an integer.

The calculation with this expression is virtually unchanged from the above except for the pole being shifted one unit to the right when the $|\rho|\gg1$ behavior is being found. Thus the contribution of this term is

$$\frac{1}{2M(N!)^2}\left(\frac{g^2K_1\ln|q^2|\ \ln\rho}{8\pi^2}\right)^N\sum_{j=1}^{N}\sum_{s=0}^{N-j}\left(\frac{-K_2}{K_1}\right)^{s+1}\binom{N-j}{s}$$
$$\times\sum_{m=0}^{s+1}\binom{s+1}{m}(-1)^m\sum_{h=0}^{m-1}(-1)^h(m-h)!f_1(m,h).$$

In Appendix C we prove that

$$\sum_{h=0}^{m}(-1)^h(m-h)!f_1(m,h)=1$$

and, as a result the numerical coefficient, is zero. Thus there is no contribution to leading order from these terms.

Finally, a calculation similar to the above for the leading part of the $g_{\mu\nu}P_N{}^a$ term shows that it is a factor of $\ln|q^2|$ less than that of Eq. (3.9) and thus does not contribute to leading order. There is also a part of $(q+k_1)_\nu\sum k_{j\mu}\tilde{P}_N{}^j$ proportional to $g_{\mu\nu}$ (after evaluating derivatives of $e^{iD(A)/C}$) which is a factor of $\ln|q^2|$ less than that of Eq. (3.9).

In summary, no terms investigated in this section contribute to the leading behavior of the $p_\mu p_\nu$ or $g_{\mu\nu}$ parts of $T_{N\mu\nu}$.

IV. TRUSS-GRAPH MODEL

In Sec. III A we used factors of $k_i{}^2-M^2$ occurring in the spinor trace to cancel factors in the denominator of the $T_{N\mu\nu}$ integral of Eq. (2.1). The denominator factors we canceled corresponded to the Feynman parameters $\gamma_1, \gamma_2, \ldots, \gamma_N$. If we had considered the alternative possibility that the Feynman parameters $\beta_1, \gamma_2, \beta_3, \gamma_4, \ldots$, corresponded to the canceled denominator factors and eliminated these parameters in the subsequent calculations, we would have been essentially calculating a truss bridge which Bjorken and Wu[3] have studied (Fig. 3). Except for numerical factors and a factor of d_{1N}/C, the integral of Eq. (3.2) is the truss-bridge-graph integral of Bjorken and Wu. The factor of d_{1N}/C changes the final result by a factor of ρ. Thus it is no coincidence that their result involves a Bessel function $I_1(x)$ in a similar fashion to our results [Eqs. (1.2)–(1.5)] for spin-$\frac{1}{2}$ nucleons. Since they considered a meson theory with mixed ϕ^3-ϕ^4 interactions, it would be interesting to see if we could obtain the same behavior in such a model of spinless particles as we found for spin-$\frac{1}{2}$ nucleons.

The interaction Lagrangian is given by

$$\tfrac{1}{3}(2\pi)^3g\phi^3+\tfrac{1}{6}(2\pi)^2G\phi^4-ie\phi\partial_\mu\phi A^\mu. \tag{4.1}$$

We associate ig/π with a three-point vertex and iG/π^2 with a four-point vertex following Bjorken and Wu[3] (with masses set equal to 1). The virtual Compton amplitude corresponding to Fig. 3 is

$$T_{N\mu\nu}=-g^2G^{N-1}\int_0^\infty\frac{d\Omega}{C^2}[(2k_1+q)_\mu(2k_1+q)_\nu \times e^{iD(A)/C}]_{A_0=A_0=\cdots=0}, \tag{4.2}$$

where the loop momenta in brackets represent derivative operators and we use the Feynman-parameter expressions of Appendix B with appropriate parameters

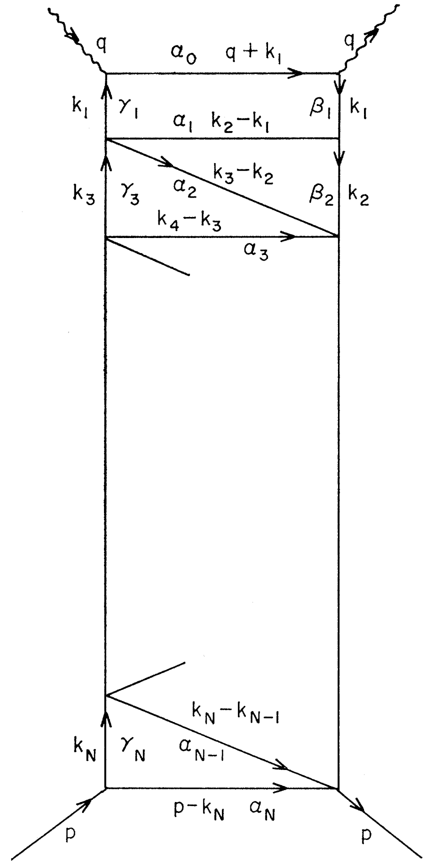

FIG. 3. Typical truss-bridge graph.

set equal to zero. After applying the derivatives, we find that the leading part of T_2 is contained in

$$T_{2N}\sim -4g^2G^{N-1}\int_0^\infty\frac{d\Omega}{C^4}d_{1N}{}^2e^{iD/C}, \tag{4.3}$$

which has the asymptotic behavior $|q^2|^{-1}f(\rho)$ as $|q^2|\to\infty$ for fixed ρ. If we follow the same procedure as we did in Sec. II, we find that the leading asymptotic behavior as $|q^2|\to\infty$ with ρ fixed and $|\rho|\gg1$ is

$$T_{2N}\to\frac{4g^2(GK_1)^{N-1}(\ln\rho)^{2N-1}}{|q^2|\rho^3(2N-1)!}, \tag{4.4}$$

which results in Eq. (1.7) upon taking the absorptive part and summing over N. The reason we did not obtain factors of $\ln|q^2|$ in Eq. (4.4) is the presence of both β_1 and γ_1 in the above integral without any compensating factors in the integral's denominator to enhance its asymptotic $|q^2|$ dependence through scalings such as Eq. (3.3).

The situation for the leading part of T_1 is quite different—a factor of $(\beta_1+\gamma_1)^{-1}$ in the integral's denominator allows us to scale as in Eq. (3.3). We begin with

$$T_{1N}\sim 2ig^2G^{N-1}\int_0^\infty \frac{d\Omega}{C^2}\frac{1}{\beta_1+\gamma_1}e^{iD/C}. \tag{4.5}$$

The leading behavior as $|q^2|\to\infty$ with ρ fixed has the form $|q^2|^{-1}(\ln|q^2|)^N f_N(\rho)$ and if $\omega\gg 1$, we find the asymptotic behavior

$$T_{1N}\to\frac{g^2}{|q^2|\rho GK_1(N!)^2}(GK_1\ln|q^2|\ln\rho)^N. \tag{4.6}$$

Upon taking the absorptive part and summing, we obtain Eq. (1.6).

V. CONCLUSION

As a result of the calculations of Secs. II, III A, and III B, we have found that the leading behavior of T_{2N} in the neutral-vector-meson model is given by Eq. (3.7). After taking the absorptive part and summing, we obtain Eq. (1.3). Similarly the leading behavior of T_{1N} comes from Eq. (3.9), which implies Eq. (1.2).

We have also calculated the γ_5 neutral-pseudoscalar-meson ladder model and noted that the only difference between this model and the vector-meson model arises from constant factors. Therefore, we can immediately obtain the asymptotic behavior of structure functions in this model which is given in Eqs. (1.4) and (1.5). The proof that the spinor polynomials are the same in both models except for over-all constants is in Appendix A [compare Eqs. (A27)–(A30) to Eqs. (A18)–(A20)].

Our results for the $N=1$ box graph agree with those of Adler and Tung and Jackiw and Preparata[10] in the neutral-vector-meson model.

The truss-bridge-graph model appears to be intermediate between the ϕ^3 ladder-model calculation of Altarelli and Rubinstein and our spin-$\frac{1}{2}$ nucleon models in the case of W_1. However, W_2 has the same $|q^2|$ dependence in the ϕ^3-ladder model and the truss-bridge-graph model. The scaling behavior of the inelastic structure functions in the various ladder models is summarized in Table I.

TABLE I. Scaling properties of W_1 and νW_2 in various ladder models. Only the $|q^2|$ behavior of the $(N+1)$-rung diagram is shown in the $|q^2|\to\infty$ (for fixed ρ) limit.

Ladder model	Nonscaling $\lvert q^2\rvert$ behavior of the $(N+1)$-rung ladder contribution to W_1	νW_2
1. All particles spinless.	$1/\lvert q^2\rvert$	
2. Spin-1 photons. All other particles are mesons in a ϕ^3 theory.	$\ln\lvert q^2\rvert/\lvert q^2\rvert$	1
3. Spin-1 photons. All other particles are mesons in a mixed ϕ^3-ϕ^4 theory.	$(\ln\lvert q^2\rvert)^N/\lvert q^2\rvert$	1
4. Spin-1 photons, spin-$\frac{1}{2}$ nucleons, neutral vector mesons *or* pseudoscalar mesons on the ladder's rungs.	$(\ln\lvert q^2\rvert)^N$	$(\ln\lvert q^2\rvert)^N$
5. Model 4 with a transverse momentum cutoff at pion-nucleon vertices.	1	1

Drell, Levy, and Yan[11] have studied a model similar to our pseudoscalar-meson model. The major difference is their introduction of a cutoff in transverse momenta at pion-nucleon vertices. The structure functions scale in their model. This is apparently a consequence of the use of a momentum cutoff.

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APPENDIX A: SPINOR POLYNOMIAL

The spinor polynomial

$$\begin{aligned}P_N=&(\boldsymbol{k}_1+M)\gamma^{\alpha_1}(\boldsymbol{k}_2+M)\gamma^{\alpha_2}\cdots(\boldsymbol{k}_N+M)\gamma^{\alpha_N}(\boldsymbol{p}+M)\\&\times\gamma_{\alpha_N}(\boldsymbol{k}_N+M)\gamma_{\alpha_{N-1}}\cdots(\boldsymbol{k}_2+M)\gamma_{\alpha_1}(\boldsymbol{k}_1+M)\end{aligned} \tag{A1}$$

can be calculated from a set of recurrence relations. If we let $L_{N+1-i}=k_i$, then

$$P_N=(L_N+M)\gamma^\alpha P_{N-1}\gamma_\alpha(L_N+M), \tag{A2}$$

with $P_0=\boldsymbol{p}+M$. A simple induction in which the form of P_0 and the recurrence relations to be derived play the key roles proves that we may write

$$P_N=P_N{}^a+\boldsymbol{p}P_N{}^b+\sum_{i=1}^N P_N{}^iL_i, \tag{A3}$$

with $P_N{}^j$ being a scalar function of inner products of the momenta.

Using Eq. (A3) on both sides of Eq. (A2) and the Dirac algebra of the γ matrices, we obtain the following recurrence relations:

$$P_N{}^a=4(L_N{}^2+M^2)P_{N-1}{}^a-4Mp\cdot L_NP_{N-1}{}^b-4M\sum_{j=1}^{N-1}P_N{}^jL_j\cdot L_N, \quad \text{(A4}$$

$$P_N{}^b=2(L_N{}^2-M^2)P_{N-1}{}^b, \tag{A5}$$

$$P_N{}^j=2(L_N{}^2-M^2)P_{N-1}{}^j,\quad j=1,2,\cdots,N-1 \tag{A6}$$

$$P_N{}^N=8MP_{N-1}{}^a-4p\cdot L_NP_{N-1}{}^b-4\sum_{j=1}^{N-1}P_{N-1}{}^jL_j\cdot L_N. \tag{A7}$$

We now solve this set of recurrence relations. Equation

[10] S. Adler and Wu-ki Tung, Phys. Rev. Letters 22, 978 (1969); R. Jackiw and G. Preparata, *ibid.* 22, 975 (1969).

[11] S. D. Drell, D. J. Levy, and T. Yan, Phys. Rev. Letters 22, 744 (1969).

(A5) implies

$$P_N{}^b=2^N\prod_{i=1}^{N}(L_i{}^2-M^2)\,, \tag{A8}$$

and similarly Eq. (A6) implies

$$P_N{}^j=2^{N-j}P_j{}^j\prod_{i=j+1}^{N}(L_i{}^2-M^2) \tag{A9}$$

for $j=1,2,\ldots,N-1$. In addition, Eqs. (A4) and (A7) lead to

$$P_N{}^a-MP_N{}^N=4(L_N{}^2-M^2)P_{N-1}{}^a\,. \tag{A10}$$

We now iterate Eq. (A10), which results in

$$P_N{}^a=M\sum_{j=0}^{N}4^{N-j}P_j{}^j\prod_{i=j+1}^{N}(L_i{}^2-M^2)\,, \tag{A11}$$

with $P_0{}^0=1$. The use of Eqs. (A11), (A8), and (A9) in Eq. (A7) enables us to put $P_N{}^N$ in the following form:

$$P_N{}^N=\sum_{j=0}^{N-1}A_j{}^NP_j{}^j\,, \tag{A12}$$

where

$$A_i{}^j=2(M^24^{j-i}-L_j\cdot L_i2^{j-i})\prod_{r=i+1}^{j-1}(L_r{}^2-M^2)\,. \tag{A13}$$

Equation (A12) leads directly to the solution for $P_N{}^N$:

$$P_N{}^N=\sum_{s=0}^{N-1}\sum_{1\leqslant i_1<i_2<\cdots<i_s<N}A_0{}^{i_1}A_{i_1}{}^{i_2}A_{i_2}{}^{i_3}\cdots A_{i_{s-1}}{}^{i_s}A_{i_s}{}^N\,, \tag{A14}$$

with the $s=0$ term being $A_0{}^N$ and with $L_0=p$. We have now solved the recurrence relations since Eq. (A14) can be used in Eqs. (A9) and (A11), respectively, in order to give

$$P_N{}^j=2^{N-j}\prod_{i=j+1}^{N}(L_i{}^2-M^2)\times\sum_{s=0}^{j-1}\sum_{1\leqslant i_1<\cdots<i_s<j}A_0{}^{i_1}A_{i_1}{}^{i_2}\cdots A_{i_s}{}^j \tag{A15}$$

and

$$P_N{}^a=M\sum_{j=0}^{N}4^{N-j}\prod_{i=j+1}^{N}(L_i{}^2-M^2)\times\sum_{s=0}^{j-1}\sum_{1\leqslant i_1<\cdots<i_s<j}A_0{}^{i_1}A_{i_1}{}^{i_2}\cdots A_{i_s}{}^j \tag{A16}$$

$$=M\sum_{j=0}^{N}2^{N-j}P_N{}^j\,. \tag{A17}$$

We now express (A8) and (A15) in terms of the loop momenta $k_i=L_{N+1-i}$:

$$P_N{}^b=2^N\prod_{i=1}^{N}(k_i{}^2-M^2) \tag{A18}$$

and

$$\tilde{P}_N{}^i=P_N{}^{N+1-i}=2^{i-1}\prod_{r=1}^{i-1}(k_r{}^2-M^2)\times\sum_{s=0}^{N-i}\sum_{i<a_1<a_2<\cdots<a_s\leqslant N}B_i{}^{a_1}\times B_{a_1}{}^{a_2}\cdots B_{a_{s-1}}{}^{a_s}B_{a_s}{}^{N+1}\,, \tag{A19}$$

with

$$B_b{}^a=2(4^{a-b}M^2-2^{a-b}k_a\cdot k_b)\prod_{r=b+1}^{a-1}(k_r{}^2-M^2) \tag{A20}$$

and the conventions $k_{N+1}=p$, $\tilde{P}_N{}^{N+1}=1$, and

$$\prod_{r=a+1}^{a}(k_r{}^2-M^2)=1$$

whenever it occurs. The reason for having the superscript $N+1-i$ on $P_N{}^j$ in Eq. (A19) becomes clear when we rewrite Eq. (A3):

$$P_N=P_N{}^a+\mathbf{p}P_N{}^b+\sum_{i=1}^{N}P_N{}^{N+1-i}\mathbf{k}_i$$

$$=P_N{}^a+\mathbf{p}P_N{}^b+\sum_{i=1}^{N}\tilde{P}_N{}^i\mathbf{k}_i\,. \tag{A21}$$

The spinor polynomial for the γ_5 meson-nucleon case can be calculated in a similar fashion. Let $P_N{}^a$, $P_N{}^b$, and $P_N{}^i$ be defined as in Eq. (A3). Then using

$$P_N=(\mathbf{L}_N+M)\gamma_5P_{N-1}\gamma_5(\mathbf{L}_N+M)\,, \tag{A22}$$

we obtain the following recurrence relations:

$$P_N{}^a=(L_N{}^2+M^2)P_{N-1}{}^a-2Mp\cdot L_NP_{N-1}{}^b-2M\sum_{j=1}^{N-1}P_{N-1}{}^jL_j\cdot L_N\,, \tag{A23}$$

$$P_N{}^b=(L_N{}^2-M^2)P_{N-1}{}^b\,, \tag{A24}$$

$$P_N{}^i=(L_N{}^2-M^2)P_{N-1}{}^i\,,\quad i=1,2,\ldots,N-1 \tag{A25}$$

$$P_N{}^N=2MP_{N-1}{}^a-2p\cdot L_NP_{N-1}{}^b-2\sum_{j=1}^{N-1}P_{N-1}{}^jL_j\cdot L_N\,. \tag{A26}$$

If we proceed in exactly the same fashion as the previous case, we obtain

$$P_N{}^b=\prod_{i=1}^{N}(k_i{}^2-M^2)\,, \tag{A27}$$

$$P_N{}^a=M\sum_{j=0}^{N}\tilde{P}_N{}^{N+1-j}\,, \tag{A28}$$

$$\tilde{P}_N{}^j=\prod_{r=1}^{j-1}(k_r{}^2-M^2)$$

$$\times\sum_{s=0}^{N-j}\sum_{j<a_1<\cdots<a_s\leqslant N}E_j{}^{a_1}E_{a_1}{}^{a_2}\cdots E_{a_s}{}^{N+1},\quad \text{(A29)}$$

with

$$E_i{}^j=2(M^2-k_i\cdot k_j)\prod_{r=i+1}^{j-1}(k_r{}^2-M^2),\quad \text{(A30)}$$

and the convention $\tilde{P}_N{}^{N+1}=1$, $k_{N+1}=p$, and

$$\prod_{r=k+1}^{k}(k_r{}^2-M^2)$$

is taken to be 1 wherever it might appear.

APPENDIX B: POLYNOMIALS C AND D

We define[12] $C_j{}^i=C_j{}^i(\alpha_i,\alpha_{i+1},\ldots,\alpha_j,\beta_{i+1},\beta_{i+2},\ldots,\beta_j,\gamma_{i+1},\gamma_{i+2},\ldots,\gamma_j)$ to be the homogeneous polynomial (of order $j-i$, in the Feynman parameters) which is associated with the graph obtained from Fig. 2 by deleting all lines above the rung labeled with α_i and below the rung labeled with α_j. Thus the polynomial C associated with the entire graph in Fig. 2 would be denoted $C_N{}^0$. If we let

$$d_{ij}=\alpha_i\alpha_{i+1}\cdots\alpha_j \quad \text{(B1)}$$

for $i\leqslant j$, then the recurrence relations for the $C_j{}^i$ are given by

$$C_j{}^i=d_{i+1,j}+(\alpha_i+\beta_{i+1}+\gamma_{i+1})C_j{}^{i+1}+\sum_{k=i+1}^{j-1}d_{i+1,k}(\beta_{k+1}+\gamma_{k+1})C_j{}^{k+1}\quad \text{(B2)}$$

$$=d_{i,j-1}+(\alpha_j+\beta_j+\gamma_j)C_{j-1}{}^i+\sum_{k=i+1}^{j-1}d_{k,j-1}(\beta_k+\gamma_k)C_{k-1}{}^i,\quad \text{(B3)}$$

with $C_k{}^k=1$. In particular,

$$C=C_N{}^0=d_{1N}+(\alpha_0+\beta_1+\gamma_1)C_N{}^1+\alpha_1(\beta_2+\gamma_2)C_N{}^2+\cdots+d_{1,N-1}(\beta_N+\gamma_N)\quad \text{(B4)}$$

$$=d_{0,N-1}+(\alpha_N+\beta_N+\gamma_N)C_{N-1}{}^0+\alpha_{N-1}(\beta_{N-1}+\gamma_{N-1})\times C_{N-2}{}^0+\cdots+d_{1,N-1}(\beta_1+\gamma_1).\quad \text{(B5)}$$

The homogeneous polynomial D of order $N+1$ in the Feynman parameters and containing scalar products of the external momenta in Fig. 2, has the form

$$D=G(q+p+A_0)^2+\sum_{i=1}^{N}[(q+A_0-A_i)^2G_i+(p+A_i)^2H_i]+\sum_{i,j=1,(i<j)}^{N}(A_i-A_j)^2I_{ij}-\alpha_0M^2C-\sum_{i=1}^{N}[M^2(\beta_i+\gamma_i)+m^2\alpha_i]C,\quad \text{(B6)}$$

with G, G_i, H_i, and I_{ij} functions of Feynman parameters only. We now give the explicit form of these functions which follow immediately from rules given in Ref. 8:

$$G=d_{0N},\quad \text{(B7)}$$

$$G_i=d_{0,i-1}(\beta_i+\gamma_i)C_N{}^i,\quad \text{(B8)}$$

$$H_i=d_{iN}(\beta_i+\gamma_i)C_{i-1}{}^0,\quad \text{(B9)}$$

$$I_{ij}=d_{i,j-1}(\beta_i+\gamma_i)(\beta_j+\gamma_j)C_{i-1}{}^0C_N{}^j.\quad \text{(B10)}$$

If we note that Eqs. (B4) and (B5) imply

$$G+\sum_{i=1}^{N}G_i=\alpha_0(C-\alpha_0C_N{}^1),\quad \text{(B11)}$$

$$G+\sum_{i=1}^{N}H_i=\alpha_N(C-\alpha_NC_{N-1}{}^0),\quad \text{(B12)}$$

then

$$D(0)=D|_{A_0=A_1=\cdots=A_N=0}=\alpha_0q^2(\rho d_{1N}+C')+\alpha_NM^2(C-\alpha_NC_{N-1}{}^0)-\alpha_0M^2C-C\sum_{i=1}^{N}[M^2(\beta_i+\gamma_i)+m^2\alpha_i]$$
$$=\alpha_0q^2(\rho d_{1N}+C')+JC,\quad \text{(B13)}$$

with $\rho=2q\cdot p/q^2$ and $C'=C-\alpha_0C_N{}^1=C|_{\alpha_0=0}$.

We now calculate

$$I_j=\sum_{k<j}I_{kj}+\sum_{k>j}I_{jk}$$

from Eqs. (B2), (B3), and (B10):

$$I_j=(\beta_j+\gamma_j)[C_N{}^j\sum_{k=1}^{j-1}d_{k,j-1}(\beta_k+\gamma_k)C_{k-1}{}^0+C_{j-1}{}^0\sum_{k=j+1}^{N}d_{j,k-1}(\beta_k+\gamma_k)C_N{}^k]$$

$$=(\beta_j+\gamma_j)[\alpha_{j-1}C_N{}^j(C_{j-1}{}^0-d_{0,j-2}-\alpha_{j-1}C_N{}^{j+1})+\alpha_jC_{j-1}{}^0(C_N{}^j-d_{j+1,N}-C_N{}^{j+1}\alpha_j)]\quad \text{(B14)}$$

for $j=2,3,\ldots,(N-1)$. For $j=1$ and $j=N$, I_j has the following form:

$$I_1=(\beta_1+\gamma_1)\alpha_1(C_N{}^1-d_{2N}-\alpha_1C_N{}^2),\quad \text{(B15)}$$

$$I_N=(\beta_N+\gamma_N)\alpha_{N-1}(C_{N-1}{}^0-d_{0,N-2}-\alpha_{N-1}C_{N-2}{}^0).\quad \text{(B16)}$$

The functions I_j occur in the polynomials resulting from applying derivatives to $e^{iD(A)/C}$ in Appendix C.

[12] Reference 7, pp. 34–36.

APPENDIX C: DERIVATIVES OF $\exp[iD(A)/C]$

We begin by defining the following quantities:

$$(i)_\nu = p_\nu d_{iN} C_{i-1}{}^0 - q_\nu d_{0i-1} C_N{}^i, \quad i=1,2,\ldots,N \tag{C1}$$

$$(0)_\nu = p_\nu d_{1N} + q_\nu C', \tag{C2}$$

$$(ij) = -d_{ij-1} C_{i-1}{}^0 C_N{}^j, \quad i<j;\ i,j=1,2,\ldots,N \tag{C3}$$

$$(0i) = -d_{1i-1} C_N{}^i, \quad i=2,3,\ldots,N \tag{C4}$$

$$(01) = -C_N{}^1, \tag{C5}$$

$$(ii) = \frac{\alpha_{i-1} C_N{}^i (C_{i-1}{}^0 - \alpha_{i-1} C_{i-2}{}^0) + \alpha_i C_{i-1}{}^0 (C_N{}^i - \alpha_i C_N{}^{i+1})}{\beta_i + \gamma_i}, \quad i=2,3,\ldots,(N-1) \tag{C6}$$

$$(11) = \frac{(\alpha_0+\alpha_1) C_N{}^1 - \alpha_1{}^2 C_N{}^2}{\beta_1+\gamma_1}, \tag{C7}$$

$$(NN) = \frac{(\alpha_N + \alpha_{N-1}) C_{N-1}{}^0 - \alpha_{N-1}{}^2 C_{N-2}{}^0}{\beta_N + \gamma_N}. \tag{C8}$$

These definitions will enable us to calculate the derivatives of $e^{iD(A)/C}$ for the general case and write the result in a compact form. The following examples of second- and fourth-order derivatives illustrate the use of quantities defined above (k_1, k_2, k_3, and k_4 are integers from the set $\{0,1,\ldots,N\}$):

$$\left(\frac{1}{Z_{k_1}Z_{k_2}} \frac{\partial^2}{\partial A_{k_2}{}^{\nu_2} \partial A_{k_1}{}^{\nu_1}} e^{iD(A)/C}\right)_{A=0} = \left[\frac{2i}{C} g_{\nu_1\nu_2}(k_1k_2) + \left(\frac{2i}{C}\right)^2 (k_1)_{\nu_1}(k_2)_{\nu_2}\right] e^{iD/C},$$

$$\begin{aligned}\left[\left(\prod_{j=1}^4 \frac{1}{Z_{k_j}} \frac{\partial}{\partial A_{k_j}{}^{\nu_j}}\right) e^{iD(A)/C}\right]_{A=0} = \{&(2i/C)^2[g_{\nu_1\nu_2}(k_1k_2) g_{\nu_3\nu_4}(k_3k_4) + g_{\nu_1\nu_3}(k_1k_3) g_{\nu_2\nu_4}(k_2k_4) + g_{\nu_1\nu_4} g_{\nu_2\nu_3}(k_1k_4)(k_2k_3)] \\ &+ (2i/C)^3 [g_{\nu_1\nu_2}(k_1k_2)(k_3)_{\nu_3}(k_4)_{\nu_4} + g_{\nu_1\nu_3}(k_1k_3)(k_2)_{\nu_2}(k_4)_{\nu_4} + g_{\nu_1\nu_4}(k_1k_4)(k_2)_{\nu_2}(k_3)_{\nu_3} + g_{\nu_2\nu_3}(k_2k_3)(k_1)_{\nu_1}(k_4)_{\nu_4} \\ &+ g_{\nu_2\nu_4}(k_2k_4)(k_1)_{\nu_1}(k_3)_{\nu_3} + g_{\nu_3\nu_4}(k_3k_4)(k_1)_{\nu_1}(k_2)_{\nu_2}] + (2i/C)^4 (k_1)_{\nu_1}(k_2)_{\nu_2}(k_3)_{\nu_3}(k_4)_{\nu_4}\} e^{iD/C},\end{aligned}$$

with $Z_i = \beta_i + \gamma_i$ (and $Z_0 = \alpha_0$). It is clear from these examples that the terms in the coefficient of each $(2i/C)^n$ are isomorphic to the permutations in a certain class of the symmetric group. Thus $g_{\nu_1\nu_2}(k_1k_2) g_{\nu_3\nu_4}(k_3k_4)$ corresponds to the permutation (12)(34), and $g_{\nu_1\nu_2}(k_1k_2)(k_3)_{\nu_3}(k_4)_{\nu_4}$ to (12). This isomorphism gives us an algorithm for determining the coefficient of $(2i/C)^n$ from the appropriate corresponding class in the symmetric groups. As a result, the fourth-order derivative expression may be rewritten

$$(2i/C)^2\{2^2\} + (2i/C)^3\{21^2\} + (2i/C)^4\{1^4\},$$

with the terms corresponding to each bracket easily determined from the class of S_4 (the symmetric group on four objects) having the cycle structure given in the bracket.[13]

The general case of the pth-order derivative operating on $e^{iD(A)/C}$ is given by

$$\begin{aligned}\left(\prod_{j=1}^p \frac{1}{Z_{k_j}} \frac{\partial}{\partial A_{k_j}{}^{\nu_j}} e^{iD(A)/C}\right)_{A=0} &= \sum_{h_1+2h_2=p} \left(\frac{2i}{C}\right)^{h_1+h_2} \{2^{h_2}1^{h_1}\} \\ &= \sum_{h=0}^{[p/2]} \left(\frac{2i}{C}\right)^{p-h} \{2^h 1^{p-2h}\},\end{aligned} \tag{C9}$$

with $[p/2]$ being the greatest integer less than $p/2$, and the terms in $\{2^h1^{p-2h}\}$ isomorphic to the permutations in the class of S_p with cycle structure 2^h1^{p-2h}. With this expression for the pth derivative, we can proceed to evaluate the parts of these Feynman-parameter polynomials which contribute to the leading behavior of the $(N+1)$-rung diagram integral.

We now investigate the polynomial resulting from

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[13] M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962), p. 25.

applying the derivative $(0<L_1<L_2<\cdots<L_m)$,

$$(p^\lambda/\alpha_0 \prod_{i=1}^m Z_{L_i}{}^2)\times\frac{\partial^{2m+1}}{\partial A_0{}^\nu \partial A_{L_1}{}^\mu \partial A_{L_1}{}^{\nu_1}\partial A_{L_2\nu_1}\partial A_{L_2}{}^{\nu_2}\cdots\partial A_{L_m\nu_{m-1}}\partial A_{L_m}{}^\lambda},$$

to $e^{iD(A)/C}$. This polynomial, denoted

$$\sum_{h=0}^m \left(\frac{2i}{C}\right)^{2m+1-h}\{2^h 1^{2m+1-2h}\},$$

will be shown to give the following contribution to the leading behavior of the $p_\mu p_\nu$ part of $T_{N\mu\nu}$:

$$p_\mu p_\nu \sum_{h=0}^m \left(\frac{2i}{C}\right)^{2m+1-h}(-1)^m f(m,h)\times d_{1N}(d_{0N}p\cdot q)^{m-h}\prod_{i=1}^m C_{L_i-1}{}^0 C_N{}^{L_i}, \quad \text{(C10)}$$

with $f(m,h)$ an integer.

Since factors of α_i in the numerator of the Feynman-parameter integral suppress the leading behavior of the integral, we wish to find the part of $\{2^h 1^{2m+1-2h}\}$ with the minimum number of such factors. More precisely, given two terms in the polynomial, count the number of powers of $q\cdot p$ and q^2 in each term and divide by d_{0N} raised to the respective power in each term. (Owing to the variables in which we have chosen to find the asymptotic behavior of the Feynman integral, the integral's asymptotic behavior is unaffected by factors of $q^2 d_{0N}$ or $q\cdot pd_{0N}$.) The term now containing the least number of powers of α_0 is leading. Should they be the same, then the term with the least number of different α_i is leading, etc. These considerations imply the following rules: (i) If a factor of $(L_i)_\alpha(L_j)^\alpha$ occurs in a term, $i<j$, then $-q\cdot pd_{0L_i-1}d_{L_jN}C_N{}^{L_i}C_{L_j-1}{}^0$ is the leading part of this factor; (ii) if $(L_i)_\lambda$ occurs in a term, then $-q_\lambda d_{0L_i-1}C_N{}^{L_i}$ is the leading part of this factor; (iii) any term with (L_iL_i) as a factor is not a leading term since there are other terms in $\{2^h 1^{2m+1-2h}\}$ in which the two L_i are in combinations contributing a $q\cdot pd_{0N}$ factor (for example), while in (L_iL_i) they contribute a factor of α_{L_i} or α_{L_i-1}; (iv) for reasons similar to those given in (iii), terms with factors having the form $(L_iL_j)(L_iL_k)$ or $(L_jL_i)(L_kL_i)$ are also not leading. As a result of these rules, we can calculate the leading parts given in the following examples:

$$\{1^{2m+1}\}\to(-1)^m d_{1N}(p\cdot qd_{0N})^m\prod_{i=1}^m C_{L_i-1}{}^0 C_N{}^{L_i}p_\mu p_\nu,$$

$$\{21^{2m-1}\}\to(-1)^m\binom{m+1}{2}d_{1N}(p\cdot qd_{0N})^{m-1}\times\prod_{i=1}^m C_{L_i-1}{}^0 C_N{}^{L_i}p_\mu p_\nu,$$

where

$$f(m,1)=\binom{m+1}{2}$$

counts the number of leading terms in the second example, which is the number of distinct choices (ab), where $a<b$ and $a,\ b\in\{0,L_1,L_2,\ldots,L_m\}$.

The leading part of the general term $\{2^h 1^{hm+1-2h}\}$ is given by the hth term in the sum of Eq. (C10). The factor $(p\cdot qd_{0N})^{m-h}$ arises from $2m+1-2h(L_i)_\alpha(L_j)_\beta$ factors occurring in each term of this set. Because of rules (iii) and (iv) given above, the number of leading terms $f(m,h)$ is the number of distinct objects

$$(a_1b_1)(a_1b_2)\cdots(a_hb_h),$$

which can be formed by all possible choices of a_i from the set $\{0,\ 1,\ 2,\ \ldots,\ m-1\}$ and b_i from the set $\{1,\ 2,\ \ldots,\ m-1,\ m\}$ such that $a_i<b_i$, no integer occurs twice in the set $\{a_1,a_2,\ldots,a_h\}$, and no integer occurs twice in the set $\{b_1,b_2,\ldots,b_h\}$. Two terms which are the same except for the ordering of the factors (a_jb_j) are not considered distinct.

A recurrence relation for $f(m,h)$ can be easily derived. The number of objects formed above in which a factor (a_jm) occurs for some j is given by

$$f(m,h)-f(m-1,\ h).$$

We now consider how this set of objects containing a factor (a_jm) can be formed. One method would be to take the set of objects enumerated by $f(m-1,\ h-1)$ whose general term is

$$(a_1b_1)(a_2b_2)\cdots(a_{h-1}b_{h-1})$$

and to each term in that set add an additional factor (a_jm). The possible choices for a_j are limited by the fact that $h-1$ out of the m possibilities $0,\ 1,\ 2,\ \ldots,\ m-1$ have already been chosen. Thus there are $m-(h-1)$ different choices for the (a_jm) factor which can be added to each term. Since this exhausts all possible distinct terms, we have

$$f(m,h)-f(m-1,\ h)=(m+1-h)f(m-1,\ h-1). \quad \text{(C11)}$$

We now show that

$$d_m=\sum_{h=0}^m(-1)^h(m+1-h)!f(m,h)=1. \quad \text{(C12)}$$

Since $d_0=d_1=1$, we need only to prove that d_m is independent of m. Consider

$$
\begin{aligned}
I&=d_m-d_{m-1}\\
&=\sum_{h=0}^{m}(-1)^h(m+1-h)![f(m-1,h)\\
&\qquad\qquad+(m+1-h)f(m-1,h-1)]\\
&\qquad\qquad\qquad-\sum_{h=0}^{m-1}(-1)^h(m-h)!f(m-1,h)\\
&=\sum_{h=0}^{m-1}(-1)^h(m-h)(m-h)!f(m-1,h)\\
&\qquad+\sum_{h=1}^{m}(-1)^h(m+1-h)(m+1-h)!f(m-1,h-1)\\
&=\sum_{h=0}^{m-1}(-1)^h(m-h)(m-h)!f(m-1,h)\\
&\qquad+\sum_{h'=0}^{m-1}(-1)^{h'+1}(m-h')(m-h')!f(m-1,h')=0\,,
\end{aligned}
$$

where we used the recurrence relation in the second equation, recombined terms in the third equation, and let $h'=h-1$ in the second sum of the fourth equation. [Note that $f(m,-1)=f(m,m+1)=0$.]

Finally we determine the terms with leading behavior in the polynomial resulting from the derivative (which is the previous derivative with μ and ν summed)

$$\frac{p^\lambda}{\alpha_0\prod Z_{Li}{}^2}\,\frac{\partial^{2m+1}}{\partial A_{0\mu}\partial A_{L1}{}^{\mu}\partial A_{L1}{}^{\nu_1}\partial A_{L2\nu1}\cdots\partial A_{Lm}{}^{\lambda}}$$

applied to $e^{iD(A)/C}$. Since we are summing over μ and ν, we are interested in a slightly different part of the polynomials than the previous case. In particular, we wish to find terms containing one more power of $p\cdot q$ than of α_0 so that W_1 will eventually scale (neglecting $\ln|q^2|$ factors). The factor that gives this extra power of $p\cdot q$ is the $q_\mu C'$ part of $(0)_\mu$. Therefore the only terms contributing to leading order are those with a factor of $(0)_\mu$. The leading part of the polynomial is given by

$$\sum_{h=0}^{m-1}\left(\frac{2i}{C}\right)^{2m+1-h}(-1)^m p\cdot qC'(d_{0N}p\cdot q)^{m-h}\times f_1(m,h)\prod_{i=1}^{m}C_{Li-1}{}^{0}C_N{}^{Li}\,,\quad\text{(C13)}$$

where the only differences from Eq. (C10) are to change $p_\mu d_{1N}$ to $q_\mu C'$ and sum over μ and ν, and to delete the $h=m$ term since it contains no contribution to leading order. The number of terms in $\{2^h1^{2m+1-2h}\}$ contributing to leading order is different from the previous case and as a result we must investigate $f_1(m,h)$. The integer $f_1(m,h)$ counts the number of distinct objects

$$(a_1b_1)(a_2b_2)\cdots(a_hb_h)$$

which can be formed with all possible choices of a_i from $\{1,2,\ldots,m-1\}$ and b_i from $\{1,2,\ldots,m\}$ under the same conditions as those given for $f(m,h)$. The only difference between the two enumerations is that a_i cannot equal zero here [corresponding to the requirement that we must have $(0)_\mu$] and thus no $(0L_i)$ factors occur in leading terms. The result is a small change in the derivation of the recurrence relation for $f_1(m,h)$, which is otherwise identical to the derivation for $f(m,h)$. Owing to the loss of zero from the set a_j could be chosen from, there are now only $m-1-(h-1)$ possible ways of constructing a distinct term of the form

$$(a_1b_1)\cdots(a_jm)\cdots(a_hb_h)$$

from each term in the set enumerated by $f_1(m-1,h-1)$. Thus we have the recurrence relation

$$f_1(m,h)=f_1(m-1,h)+(m-h)f_1(m-1,h-1)\,.\quad\text{(C14)}$$

Because of the boundary conditions,

$$f_1(2,1)=f(1,1)=f(1,0)\,,$$

and Eq. (C14), it is clear that

$$f_1(m,h)=f(m-1,h)\,.$$

Therefore,

$$
\begin{aligned}
d_m'&=\sum_{h=0}^{m-1}(-1)^h(m-h)!f_1(m,h)\\
&=d_{m-1}=1
\end{aligned}
$$

by our previous calculation for d_m.