PHYSICAL REVIEW D VOLUME 3, NUMBER 2 15 JANUARY 1971

Bjorken Limit and the Jin-Martin Lower Bound*

KHALIL M. BITAR†
CERN, Geneva, Switzerland and Physics Department, American University of Beirut, Beirut, Lebanon

AND

N. N. KHURI
Rockefeller University, New York, New York, 10021

(Received 28 September 1970)

We investigate the restrictions that analyticity, unitarity (positivity), and current conservation place on the asymptotic behavior in q_0 at fixed $\mathbf{q}$ of the connected, covariant amplitude $\tilde{M}_{\mu\nu}{}^c(q_0,\mathbf{q})$, where $\tilde{M}_{\mu\nu}{}^c$ is the connected part of the forward amplitude for the scattering of a charged conserved current on a proton target. Our main tool is a theorem of Jin and Martin which was used to give lower bounds for forward amplitudes. Our main result is to establish a rigorous lower bound on the contribution of the class-I intermediate states to $\tilde{M}_{00}{}^c$. The bound gives $|M_{00}{}^I+p_0/mq_0|>C|q_0|^{-2}$, $C>0$, for large complex q_0. This is larger by a factor $(q_0{}^{1+\epsilon})$ than the asymptotic behavior of $\tilde{M}_{00}{}^c$ needed in order to have the Bjorken-limit definition of the space-space equal-time commutator agree with the naive quark commutator. This bound has the following consequences: (a) The existence of operator Schwinger terms is shown to be equivalent to the existence of a frame of reference for which the contribution of class-II intermediate states does not exactly cancel the $O(q_0{}^{-2})$ contribution of the class-I states. (b) The existence of the Bjorken-limit definition of the space-space commutator depends on yet another cancellation between the $O(q_0{}^{-3})$ contributions to $\tilde{M}_{00}{}^c$ of the class-I and class-II states, and the cancellation also of all terms between $O(q_0{}^{-2})$ and $O(q_0{}^{-3})$. The sum rule whose validity is needed to guarantee this cancellation is known to diverge in perturbation theory. It is also shown to be divergent when the $\mathbf{P}\to\infty$ limit is allowed. (c) The convergence of the Cottingham formula can only be guaranteed if in all frames of reference all the contributions between $O(q_0{}^{-2})$ and $O(q_0{}^{-4})$ of the class-I states are exactly canceled by the class-II contributions, and if in addition the remaining contribution at $O(q_0{}^{-4})$ of the class-I states is also exactly canceled out. The separation of the absorptive part of the amplitude into class-I and class-II contributions is carried out explicitly through a fully reduced Lehmann-Symanzik-Zimmermann formula. The class-I states are the usual s-channel states and the so-called Z-graph states. The class-II states are those that couple directly to the current, contribute only for timelike q, and are physically quite distinct from the class-I states.

I. INTRODUCTION

IN 1966 Bjorken[1] pointed out that the asymptotic behavior of the Fourier transform of the time-ordered product of two currents is related to the equal-time commutator of the currents. More specifically, the Bjorken limit gives

$$\lim_{q_0\to i\infty;\,\mathbf{q}\text{ fixed}}\int d^4x\, e^{iq\cdot x}\langle A|T(j_\mu{}^a(x)j_\nu{}^b(0))|B\rangle = \frac{i}{q_0}\int d^4x\, e^{iq\cdot x}\delta(x_0)\times\langle A|[j_\mu{}^a(x),j_\nu{}^b(0)]|B\rangle+O(1/q_0{}^2). \quad (1.1)$$

This equation makes a very specific and restrictive statement about the asymptotic behavior of the T product. There are three main questions raised by (1.1) that were discussed in Ref. 1 and that we want to examine in this paper.

First, Eq. (1.1) has been used in several instances in the case of space-space commutators to derive a class of asymptotic sum rules.[2] A study of space-space commutators can often distinguish between different models which have the same time-component algebra. Furthermore, the asymptotic sum rules derived from (1.1) have important experimental consequences in inelastic electron scattering and in neutrino reactions. Perturbation-theory calculations have raised questions about the existence of (1.1) for space-space commutators. One would like to study this question by a nonperturbative method.

The second question related to (1.1) but in a less direct way is the problem of Schwinger terms. We recall that the T product on the left-hand side in (1.1) is not a covariant quantity. The covariant T product which is related to the physical amplitude is obtained by adding Schwinger terms (or other anomalous terms) to the ordinary T product. The problem of Schwinger terms was posed by Bjorken in the following way: Given the covariant amplitude, how does one construct the T product? One needs the T product in order to derive sum rules. Bjorken's answer to this question consists of a prescription for identifying Schwinger terms from the asymptotic behavior of the covariant amplitude. He states that both the covariant and noncovari-

* Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-4204.

† Permanent address: Physics Department, American University of Beirut, Beirut, Lebanon.

[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966). See also K. Johnson and F. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966).

[2] For a review, see the lectures by J. D. Bjorken, in *Selected Topics in Particle Physics, Proceedings of the International School of Physics "Enrico Fermi" Course XLI*, edited by J. Steinberger (Academic, New York, 1968).

ant amplitudes, when considered as analytic functions of q_0 for fixed $\mathbf{q}$, have the same absorptive parts. They therefore differ at most by a polynomial in q_0. The T product he assumes must vanish as $q_0 \to \infty$. Thus one constructs the T product from the covariant amplitude by letting $q_0 \to \infty$ at fixed $\mathbf{q}$, identifying any polynomial in q_0, and subtracting it off. What is left then is supposed to satisfy (1.1).

In this prescription, c-number Schwinger terms only show up in the asymptotic behavior of the disconnected part of the amplitude. Any constant or polynomial behavior in q_0 for large q_0 in the connected part of the amplitude will be the result of the existence of operator Schwinger terms or other operator anomalies.

Finally, the third important problem raised by Ref. 1 concerns the term $O(1/q_0^2)$ in (1.1). This term is proportional to the equal-time commutator $[\dot{j},j]$. When j_μ and j_ν are electromagnetic currents, Bjorken showed that the nonvanishing of this commutator leads to a logarithmic divergence in the Cottingham formula for the electromagnetic mass of the proton.

The usefulness and simplicity of (1.1) led Adler and Tung[3] to carry out detailed perturbation-theory calculations to check its validity. Similar conclusions were also reached by Jackiw and Preparata.[4] Using the "gluon" model of strong interactions, it is shown in Ref. 3 that to second order in the gluon-fermion coupling constant, Eq. (1.1) fails to give the same space-space equal-time commutator as the "naive" commutator obtained from canonical commutation relations. Thus the asymptotic sum rules derived from the naive space-space commutator fail in perturbation theory. To fourth order in the gluon-fermion coupling, Adler and Tung calculate the leading logarithmic behavior of the SU_3-antisymmetric part of the space-space commutator and obtain a term which is $O((\ln q_0)^2/q_0)$. The existence of such a term means that the Bjorken-limit definition of the space-space commutator does not even exist in fourth order but diverges like $(\ln q_0)^2$.

The question immediately arises whether nonperturbative effects could conspire to make the Bjorken limit exist and give the canonical answer when all orders are summed up.

In order to shed some light on this question and on the other problems mentioned above, namely, existence of operator Schwinger terms, and divergence of the Cottingham formula, we look at the whole problem of asymptotic behavior in q_0 in a model-independent way.

We seek the restrictions that unitarity, analyticity, and current conservation force on the asymptotic behavior of the connected, covariant amplitude $\tilde{M}_{\mu\nu}{}^c$. Here the quantity $\tilde{M}_{\mu\nu}{}^c$ is the connected part of the covariant amplitude for the scattering of a charged conserved current on a proton target. Our input consists only of analyticity in q_0 at fixed $\mathbf{q}$, unitarity (positivity), and current conservation. The main tool we use is a theorem of Jin and Martin[5] which was used to give a lower bound for forward amplitudes that satisfy analyticity and positivity.

Because of current conservation, it suffices to study the time-time component $\tilde{M}_{00}{}^c$. If one succeeds in proving a lower bound for $\tilde{M}_{00}{}^c$ then the lower bound for $\tilde{M}_{ii}{}^c$ follows immediately and is larger by a factor q_0^2.

The absorptive part in q_0 for fixed $\mathbf{q}$ of $\tilde{M}_{00}{}^c(q_0,\mathbf{q})$ does not manifestly satisfy the positivity condition needed for the application of the Jin-Martin theorem. However, if we reduce out the target particles and use retarded products instead of T products, the disconnected part of the amplitude factors out explicitly and the connected part is related to the vacuum expectation value of the retarded product of two currents and two fields. Using the general identity for calculating the absorptive part of the retarded product of n operators, we immediately see that the absorptive part splits directly into two kinds of contributions. In the language of the Fubini sum rule these are the ones coming from class-I intermediate states and the others coming from class-II intermediate states. The class-I contributions are those of the direct s-channel states and those of the Z graphs. The class-II states which contribute only when q is timelike are those that couple directly to the photon—see Fig. 1. The separation is made directly from the fully reduced Lehmann-Symanzik-Zimmermann (LSZ) retarded formula and does not depend on perturbation theory.

The class-I contributions to $(\mathrm{Im}\tilde{M}_{00}{}^c)$ are manifestly positive for $q_0>0$. Our main result is to prove that the class-I contributions to $\tilde{M}_{00}{}^c$, which we denote by $\tilde{M}_{00}{}^{\mathrm{I}}$, have the following lower bound, as $|q_0| \to \infty$ in complex directions and $\mathbf{q}$ is held fixed:

$$\left|\tilde{M}_{00}{}^{\mathrm{I}}+\frac{p_0}{Mq_0}\right|>\frac{C}{|q_0|^2},\quad C>0. \tag{1.2}$$

The term (p_0/Mq_0) is added in (1.2) to exactly cancel

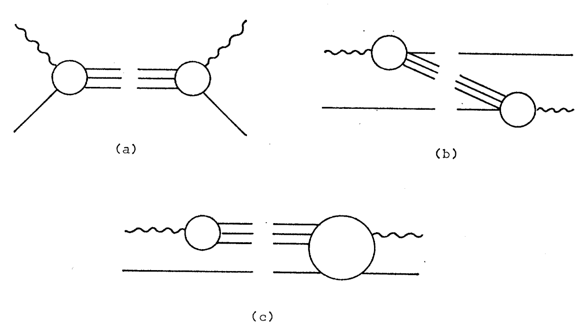

FIG. 1. Schematic representation of class-I_a, class-I_b, and class-II intermediate states which contribute to A_{00}.

[3] S. L. Adler and W.-K. Tung, Phys. Rev. Letters **22**, 978 (1969); Phys. Rev. D **1**, 2846 (1970). The latter article contains the fourth-order results.

[4] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 475 (1969).

[5] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

out the $O(1/q_0)$ term that comes from the equal-time commutator of the time components. For neutral currents this term is absent.

This lower bound is larger by a factor $(q_0)^{1+\epsilon}$ than the asymptotic behavior assumed in Eq. (4.8) in Ref. 1. It has definite consequences for all three questions related to the Bjorken limit mentioned earlier in this introduction.

For if the class-II contributions do not exactly cancel the leading behavior in (1.2), then one gets $|\tilde{M}_{00}{}^c+p_0/Mq_0|>C/|q_0|^2$, and using current conservation, $|\tilde{M}_{ii}{}^c|>\text{const}$. This could not happen if operator Schwinger terms (and other anomalies) are absent. We thus show that the existence of operator Schwinger terms is directly related to the existence of at least one frame of reference in which the contributions of the class-II states is suppressed relative to those of the class-I states or otherwise do not exactly cancel the $O(1/q_0^2)$ contribution of the class-I states.

The leading cancellation if it occurs will only guarantee the absence of operator Schwinger terms. The nonleading terms in (1.2), i.e., those between $O(1/q_0^2)$ and $O(1/q_0^3)$, will also have to be canceled if the Bjorken limit for the space-space commutator is to exist. This second cancellation (or cancellations) is more difficult to guarantee. The sum rule whose validity is necessary for the occurrence of these cancellations is divergent in perturbation theory. If one uses the infinite-momentum limit just as a guide to the convergence properties of this sum rule, one also has the indication that it is divergent in general. The only thing that could make it convergent is a cancellation between two highly divergent intergrals, one representing the contributions of the direct s-channel states and the other those of the Z graphs. This argument against the $O(1/q_0^3)$ cancellation does not provide a proof but, when coupled with the results of Adler and Tung, it makes it a reasonable conjecture.

Finally, even if cancellations occur at $O(1/q_0^2)$ and $O(1/q_0^3)$, we show that $\tilde{M}_{00}{}^{\mathrm{I}}$ will still have a contribution of $O(1/q_0^4)$. This will have to be exactly canceled by the class-II contributions if the Cottingham formula is to converge. Again we cannot rule out such a cancellation in general, but in view of the preceding discussion we find it unreasonable to assume that all these cancellations do occur exactly without pointing out a model or a mechanism which will guarantee their occurrence.

In Sec. II we review the results of Bjorken and give detailed definitions of the amplitudes we consider, and show how the problem is related to that of deriving lower bounds. In Sec. III we state the Jin-Martin theorem and make a few remarks about it that are relevant to the present discussion. We also give a proof due to Simon[6] of a weakened vision of the Jin-Martin theorem.

[6] B. Simon, Phys. Rev. D 1, 1240 (1970).

To motivate the discussion we apply the Jin-Martin theorem to the full covariant amplitude $\tilde{M}_{\mu\nu}$, where the disconnected part is still included. This is done in Sec. IV. Because of the presence of the disconnected part, the lower bound we get here could come purely from the c-number Schwinger terms and leads us in general to no new information.

In Sec. V we first show how one can explicitly factor out the disconnected part by using the retarded product and reducing out the target particles. As mentioned earlier, we use the fully reduced LSZ formula to give an explicit separation of the absorptive part of the connected amplitude into class-I and class-II contributions. We then proceed to derive our main result which is the bound (1.2).

The question of cancellations between class-I and class-II contributions is discussed in Sec. VI. It is shown that the cancellations are equivalent to the validity and existence of certain sum rules.

Finally, in Sec. VII we make a brief comparison between our work and that of Ref. 3. We point out that in perturbation theory one loses positivity even for the class-I contributions. Thus one cannot in general establish the bound (1.2) for each order in perturbation theory.

II. REVIEW OF BJORKEN LIMIT

In this section we first define the amplitudes that will be studied, and then give a brief summary of Bjorken's results. We then turn to a formulation of the questions we consider in the later sections.

We consider the time-ordered product

$$M_{\mu\nu}=+i\int d^4x\, e^{iq\cdot x}\langle p'|T(j_\mu{}^+(x)j_\nu{}^-(0))|p\rangle\,, \quad (2.1)$$

where an averaging over proton spins is understood. Here $j_\mu{}^\pm$ are the conserved $\Delta S=0$ isovector currents with $[Q^+,Q^-]=2Q_3$. (The amplitude we have defined above differs by an over-all minus sign from Bjorken's amplitude.[7]) It is well known that the quantity defined in (2.1) is not a covariant quantity. A covariant amplitude is constructed by adding an equally noncovariant object to $M_{\mu\nu}$. This term is usually called a Schwinger term or a seagull term. Thus the covariant amplitude is given by

$$\tilde{M}_{\mu\nu}\equiv M_{\mu\nu}+\Sigma_{\mu\nu}\,. \quad (2.2)$$

$\Sigma_{\mu\nu}$ is usually determined by the Schwinger terms appearing in the equal-time commutators.

Bjorken[1] proposed some time ago that $M_{\mu\nu}$ and $\tilde{M}_{\mu\nu}$, considered as analytic functions of q^0, for fixed $|\mathbf{q}|$, have the same absorptive parts and that they differ only by polynomials in q_0. The analyticity of $M_{\mu\nu}$ in q^0

[7] Our amplitude differs by an over-all sign from that of Bjorken's paper. For some reason the excellent textbook of Drell and Bjorken defines the forward amplitude in such a way that $\mathrm{Im}f\sim-\sigma_{\text{tot}}$.

is trivially evident and $M_{\mu\nu}$ satisfies a dispersion relation (known otherwise as the Low equation) of the form

$$M_{\mu\nu}(q_0,|\mathbf{q}|) = \frac{1}{\pi}\int_0^\infty dq_0'\left[\frac{\rho_{\mu\nu}(q_0',\mathbf{q})}{q_0'-q_0}+\frac{\bar{\rho}_{\mu\nu}(-q_0',+\mathbf{q})}{q_0'+q_0}\right], \quad (2.3)$$

where

$$\begin{aligned}\rho_{\mu\nu}(q_0,\mathbf{q}) &= \tfrac{1}{2}\sum_n (2\pi)^4\delta(q+p'-p_n) \\ &\quad\times\langle p'|j_\mu^+(0)|n\rangle\langle n|j_\nu^-(0)|p\rangle, \\ \bar{\rho}_{\mu\nu}(q_0,\mathbf{q}) &= \tfrac{1}{2}\sum_n (2\pi)^4\delta^4(q-p+p_n) \\ &\quad\times\langle p'|j_\nu^-(0)|n\rangle\langle n|j_\mu^+(0)|p\rangle.\end{aligned} \quad (2.4)$$

Bjorken's prescription for constructing $M_{\mu\nu}$ from $\tilde{M}_{\mu\nu}$ is to go to the limit $q_0\to i\infty$ at fixed $\mathbf{q}$, identify any polynomial in q_0, and subtract it off. The remaining $M_{\mu\nu}$, he argues, should vanish as $q_0\to\infty$ as one can see by truncating the sum over states in (2.3) and (2.4). For $q_0\to i\infty$ one then gets from (2.3)

$$M_{\mu\nu}(q_0,\mathbf{q}) \underset{q_0\to i\infty}{\sim} \frac{-1}{q_0}\int d^3x \times e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p'|[j_\mu^+(0,\mathbf{x}),j_\nu^-(0,0)]|p\rangle+O(1/q_0^2). \quad (2.5)$$

This equation provides a definition of the equal-time commutator of any two components of the currents. If one identifies the commutator with the "naive" commutator relations, one can obtain several interesting sum rules.[2] The legitimacy of such an identification has recently been questioned by Adler and Tung.[3]

To formulate our problem in a more precise way and see how it relates to the question of lower bounds, we consider $\tilde{M}_{\mu\nu}$ for the forward case $p'=p$ and average over spins.

We recall that by construction the term $\Sigma_{\mu\nu}$ in (2.2) is just chosen so as to cancel exactly the contribution of the Schwinger term when we compute $q^\mu M_{\mu\nu}$. We thus have for the covariant amplitude $\tilde{M}_{\mu\nu}$ the identity

$$\begin{aligned}q^\mu\tilde{M}_{\mu\nu} &= -2\langle p'|j_\nu^3|p\rangle \\ &= -p_\nu/M.\end{aligned} \quad (2.6)$$

One can write $\tilde{M}_{\mu\nu}$ as defined in (2.1) and (2.2) as a sum of a connected part and a disconnected part,

$$\tilde{M}_{\mu\nu} = \tilde{M}_{\mu\nu}{}^c+\langle p'|p\rangle\tilde{D}_{\mu\nu}, \quad (2.7)$$

when $\tilde{D}_{\mu\nu}$ is given by the covariantized vacuum expectation value of the T product,

$$\tilde{D}_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu}q^2)\frac{1}{2\pi}\int\frac{\rho(\sigma^2)}{\sigma^2-q^2}d\sigma^2. \quad (2.8)$$

Here ρ is the Källén-Lehmann weight function. The disconnected term $\tilde{D}_{\mu\nu}$ contains all the contributions of the c-number Schwinger terms, and $q^\mu\tilde{D}_{\mu\nu}=0$. Thus if q-number Schwinger terms are absent, then according to Bjorken's prescription,

$$\lim_{q_0\to i\infty}\tilde{M}_{\mu\nu}{}^c(q_0,|\mathbf{q}|)=0, \quad (2.9)$$

or, more precisely, $\tilde{M}_{\mu\nu}{}^c(q_0,\mathbf{q})$ will have no polynomial growth in q_0 when q-number Schwinger terms are absent. Now *if* one is able to prove that as $|q_0|\to\infty$ in complex directions

$$|\tilde{M}_{ii}{}^c(q_0,|\mathbf{q}|)|>\text{const}>0, \quad (2.10)$$

then one has the strong result that there must be a q-number Schwinger term at best. If the lower bound (2.10) is not reached but the function behaves as $C(\ln q_0)^p$, then (2.5) can have no meaning for space-space commutators.

In his paper Bjorken assumes that there are only c-number Schwinger terms and writes $\tilde{M}_{\mu\nu}{}^c$ as[8]

$$\begin{aligned}\tilde{M}_{\mu\nu}{}^c(q,p) &= [q^2p_\mu p_\nu - q\cdot p(q_\mu p_\nu+q_\nu p_\mu)+(q\cdot p)^2g_{\mu\nu}]M_1(q^2,\nu) \\ &\quad+(q_\mu q_\nu-g_{\mu\nu}q^2)M_2(q^2,\nu)+(B_{\mu\nu}+\Lambda_{\mu\nu}).\end{aligned} \quad (2.11)$$

The Born term $B_{\mu\nu}$ is given by

$$B_{\mu\nu} = -\frac{2(p_\mu+\frac{1}{2}q_\mu)(p_\nu+\frac{1}{2}q_\nu)}{M(q^2+2q\cdot p)}\left[F_{1V}{}^2(q^2)-\frac{q^2}{4M^2}F_{2V}{}^2(q^2)\right] + \frac{(q_\mu q_\nu-g_{\mu\nu}q^2)}{2M(q^2+2q\cdot p)}(F_{1V}+F_{2V})^2, \quad (2.12)$$

where F_{1V} and F_{2V} are the Dirac isovector form factors normalized to 1 and $(\kappa_p-\kappa_n)$, respectively. The term $\Lambda_{\mu\nu}$ is given by

$$\begin{aligned}\Lambda_{\mu\nu} &= [q_\mu p_\nu+q_\nu p_\mu - g_{\mu\nu}(q\cdot p)] \\ &\quad\times\left\{\frac{-1}{Mq^2}\left[1-\left(F_{1V}{}^2(q^2)-\frac{q^2}{4M^2}F^2{}_{2V}(q^2)\right)\right]\right\} \\ &\quad+g_{\mu\nu}\frac{1}{2M}\left(F_{1V}{}^2-\frac{q^2}{4M^2}F_{2V}{}^2\right).\end{aligned} \quad (2.13)$$

The first two tensors in (2.11) are conserved and $\Lambda_{\mu\nu}$ is constructed such that

$$q^\mu(B_{\mu\nu}+\Lambda_{\mu\nu}) = -p_\nu/M. \quad (2.14)$$

The connected amplitude still satisfies $q^\mu\tilde{M}_{\mu\nu}{}^c=-p_\nu/M$. This separation in Bjorken's paper is not unique since

[8] The relation between $\text{Im}M_1$ and $\text{Im}M_2$ and the now more familiar inelastic form factors W_1 and W_2 is given by

$$\text{Im}M_1 = W_2/M^2q^2$$

and

$$\text{Im}M_2 = \frac{1}{q^2}\left[W_1+\frac{(q\cdot p)^2}{q^2M^2}W_2\right].$$

Here the Born contributions are included on the left-hand side and q is taken to be spacelike.

one can always add gauge-invariant terms to $(B_{\mu\nu}+\Lambda_{\mu\nu})$ and subtract them from the rest of the amplitude. However, this nonuniqueness is not relevant to our discussion and we stick to Bjorken's definition of M_1 and M_2.

The absence of q-number Schwinger terms is equivtalent to the assumption, made explicitly in Ref. 1, that

$$\begin{aligned} &\lim_{q_0\to i\infty} q_0^2 M_1=0\,,\\ &\lim_{q_0\to i\infty} q_0^2 M_2=0\,,\\ &\lim_{q_0\to i\infty} q_0^2 F_{1,2}<\infty\,. \end{aligned} \tag{2.15}$$

The only Schwinger terms come from $\tilde{D}_{\mu\nu}$ and are c-number ones.

If in addition to (2.15) one also assumes that as $q_0\to i\infty$,

$$q_0^3 M_1\to 0\,,\quad q_0^3 M_2\to 0\,,\quad q_0^2 F_i<\infty\,, \tag{2.16}$$

then as $q_0\to i\infty$ the only contribution to $O(1/q_0)$ is made by the first term of $\Lambda_{\mu\nu}$, and one obtains

$$\tilde{M}_{\mu\nu}{}^c \underset{q_0\to i\infty}{\sim} -(\eta_\mu P_\nu+\eta_\nu P_\mu-g_{\mu\nu}\eta\cdot P)\frac{1}{Mq_0}, \tag{2.17}$$

where $\eta_\mu=(1;0,0,0)$. Thus, aside from a Schwinger term from $\tilde{D}$, one finds

$$\begin{aligned} \tfrac{1}{2}\sum_s \int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle p'|[j_\mu{}^+(0,\mathbf{x}),j_\nu{}^-(0,0)]|p\rangle \\ =(\eta_\mu P_\nu+\eta_\nu P_\mu-g_{\mu\nu}\eta\cdot P)/M\,. \end{aligned} \tag{2.18}$$

For $\mu=\nu=i$ this gives precisely what is expected from quark currents using the naive canonical commutation relations, i.e.,

$$[j_i{}^+(0,\mathbf{x}),j_i{}^-(0,0)]=2j_0{}^3\delta^3(\mathbf{x})\,. \tag{2.19}$$

One can now turn the problem around and remark that if one is able to obtain lower bounds on M_1 or M_2 such that (2.16) fails as $q_0\to i\infty$, then that is tantamount to proving the existence of anomalous terms in (2.19), as was shown in perturbation theory by Adler and Tung.

Finally, we remark that in the case of the electromagnetic current the behavior of $M_1{}^\gamma$ and $M_2{}^\gamma$ as $q_0\to i\infty$ and $|\mathbf{q}|$ is fixed is intimately connected with the convergence or divergence of the Cottingham formula for the electromagnetic self-mass of the proton. Thus unless

$$\lim_{q_0\to i\infty} q_0^4 M_{1,2}{}^\gamma=0\,, \tag{2.20}$$

the mass formula will be divergent, as was emphasized by Bjorken.

It is thus evident that anything one can say in a model-independent way about lower bounds for $|\tilde{M}_{\nu\nu}{}^c(q_0,\mathbf{q})|$ for large q_0 is quite significant for the whole set of questions discussed above. It is the main purpose of this paper to use methods developed by Jin and Martin to examine the question of lower bounds for $\tilde{M}_{\mu\nu}{}^c$.

Current conservation simplifies our task tremendously since it is easy to show that it suffices to prove lower bounds for the time-time component $\tilde{M}_{00}{}^c(q_0,\mathbf{q})$. This can most easily be seen by considering $\tilde{M}^c$ for neutral currents or by symmetrizing the amplitudes we have. We define

$$\begin{aligned} \tilde{M}_{\mu\nu}{}^{cs}=\tfrac{1}{2}[\tilde{M}_{\mu\nu}{}^c(j^-+p\to j^-+p) \\ +\tilde{M}_{\mu\nu}{}^c(j^++p\to j^++p)]\,. \end{aligned} \tag{2.21}$$

This function will be even in q_0. It will have a similar decomposition as in (2.11), but the first term in $\Lambda_{\mu\nu}$ in Eq. (2.13) will cancel out in the symmetrization. Instead of (2.6), $\tilde{M}_{\mu\nu}{}^{cs}$ now satisfies the conservation condition,

$$q^\mu \tilde{M}_{\mu\nu}{}^{cs}=0\,. \tag{2.22}$$

Thus it is trivial to show that if one can prove that

$$|\tilde{M}_{00}{}^{cs}(q_0,\mathbf{q})|>C/|q_0|^n\,, \tag{2.23}$$

then

$$|\tilde{M}_{03}{}^{cs}|>C'/|q_0|^{n-1} \tag{2.24}$$

and

$$|\tilde{M}_{33}{}^{cs}(q_0,\mathbf{q})|>C''/|q_0|^{n-2}\,, \tag{2.25}$$

where $q=(q_0,0,0,q_3)$ and $q_3\neq 0$.

When we work with $\tilde{M}_{\mu\nu}{}^c$ unsymmetrized, then Λ_{00} already has a term which behaves like $(-q_0)^{-1}$ for large q_0. This is the term that gives the correct equal-time commutator for the time-time component. But it can easily be subtracted out and one tries to derive lower bounds for $(\tilde{M}_{00}{}^c+p_0/Mq_0)$.

The method needed to derive lower bounds and the input needed is discussed in Sec. III.

Finally, in closing, we stress that the conservation condition (2.6) already makes the time-time and space-time commutators well defined aside from operator Schwinger terms. This is guaranteed by the kinematic tensors in front of M_1 and M_2 in (2.11). When we calculate $\tilde{M}_{00}{}^c$ and $\tilde{M}_{0k}{}^c$, these components have a factor $\mathbf{q}^2$ and q_k, respectively, multiplying both M_1 and M_2. Any violation of Bjorken's assumed asymptotic behavior for M_1 and M_2 given in (2.15) will lead to a contribution to $\tilde{M}_{0\nu}{}^c$ proportional to q_k or $q_k{}^2$ which by definition is due to an operator Schwinger term.

III. JIN–MARTIN THEOREM

It was first pointed out by Jin and Martin[5] that the forward elastic scattering amplitude cannot decrease arbitrarily rapidly for large energy. The input used to derive this lower bound was just analyticity in energy and positivity. The inequality they obtained was of the form

$$\int_0^E E'|f(E')|dE'>C(\ln E)^{1/2}\,. \tag{3.1}$$

The ingredients needed to prove the above were (i) $f(E)$ is analytic in E with two cuts $A<E<\infty$, and $-\infty<E<-B$; (ii) $f(E)$ is polynomially bounded in all directions; and (iii) $\operatorname{Im}f(E)\geqslant 0$ on the right-hand cut and $\operatorname{Im}f(E)\leqslant 0$ on the left-hand cut. The last condition, "positivity," follows from the optical theorem and crossing, since $\operatorname{Im}f(E)$ for negative E is minus the absorptive part of the u-channel process.

Martin[9] improved this result to show that on the real axis for some sequence $\{E_n\}$, $E_n\to\infty$ as $n\to\infty$, there exists a constant $C>0$ such that as $n\to\infty$,

$$|F(E_n)|>CE_n^{-2}. \tag{3.2}$$

A similar result holds for a sequence $\{E_n\}\to-\infty$. In the complex plane, the theorem is stronger and one gets the result

$$\lim_{|E|\to\infty}|E^2f(E)|>C,\quad C>0 \tag{3.3}$$

where the limit is taken along any radius from the origin not coinciding with the real axis.

We shall not give a complete proof of this theorem here. For a detailed discussion one can consult Ref. 10. At the end of this section we will give a proof of a slightly weakened version of (3.2) due to Simon.[6]

Before we do that, we would like to make a few remarks about this theorem which are helpful for the rest of this paper. (a) The constant C that appears in (3.2) and (3.3) is strictly greater than zero. A sufficient condition to guarantee that would be, for example,

$$\operatorname{Im}f(E)\geqslant 0,\ E>0;\quad \operatorname{Im}f(E)\leqslant 0,\ E<0;$$

and

$$\int_a^b \operatorname{Im}f(E')dE'>\text{const}>0 \tag{3.4}$$

for some finite interval $b>a>A$, where A is the threshold of the right-hand cut. In fact, a study of the proofs of the Jin-Martin theorem shows that C is "roughly" related to the size of $\operatorname{Im}f$. In a free-field theory, $\operatorname{Im}f(E)\equiv 0$ everywhere and in that case $C\equiv 0$, but so is f. We make this remark to point out that the case of free fields does not provide a counterexample to the theorem or to our applications of it. (b) The theorem can be generalized to the case where $\operatorname{Im}f(E)$ makes ν changes of sign as E varies from $E=-\infty$ to $E=+\infty$. In the case of the forward elastic amplitude, $\nu=1$. But in the general case of a function $f(E)$ satisfying (i) and (ii) but for which $\operatorname{Im}f$ has ν changes of sign on the real axis, one gets for large complex E

$$|f(E)|>C|E|^{-1-\nu}. \tag{3.5}$$

For $\nu=1$, this coincides with (3.3). (c) Born poles do not, in general, spoil the result. If the s-channel pole lies to the right of the u-channel pole, then since the discontinuity at the s-channel pole is positive we can consider it as part of the right-hand cut and the u-channel pole as part of the left-hand cut. The same result still holds. However, in some cases, like $\pi^0+p\to\pi^0+p$, the s-channel pole is at negative E and the u-channel pole is at positive E. For $\operatorname{Im}f(\pi^0p\to\pi^0p)$ we have three changes of sign as E varies from $-\infty$ to $+\infty$. Thus $f(\pi^0p)$ satisfies (3.5) with $\nu=3$. Clearly, if we have only one pole then regardless of the sign of its discontinuity (3.3) still holds. (d) The Jin-Martin lower bound looks very weak indeed. It is three powers of E lower than what we expect forward amplitudes to behave like. However, the input that goes into it is quite minimal. If one has any extra information about $f(E)$, one can immediately improve (3.2) and (3.3). For example, if one knows that $f(E)$ is negative at any real point E_0 which lies in the gap between the discontinuities, $-B<E_0<A$, then one gets instead of (3.2)

$$\lim_{n\to\infty}|f(E_n)|>C, \tag{3.6}$$

on the real axis and $\lim_{|E|\to\infty}|f(E)|>C$, $C>0$, in all complex directions. Finally, even if $f(E_A)$ is positive it suffices to find an E_1 such that

$$0>\left\{f(E_A)-\int_{E_A}^{E_1}\left[\frac{\operatorname{Im}f(E')}{E'-E_A}-\frac{\operatorname{Im}f(-E')}{E'+E_A}\right]dE'\right\} \tag{3.7}$$

in order to obtain the bound (3.6). We recall here that in (3.7), $\operatorname{Im}f(-E')$ is negative in the region of integration.

Finally, we close this section by repeating Simon's short and elegant proof of a slightly weakened version of (3.2). Simon's statement is that there exists a sequence $\{E_n\}$, $E_n\to\infty$ as $n\to\infty$, and a constant $C>0$, such that as $n\to\infty$, *either*

$$|f(E_n)|>C|E_n|^{-2} \tag{3.8}$$

or

$$|f(-E_n)|>C|E_n|^{-2}.$$

The proof is immediate. One assumes that (3.8) is false and produces a contradiction. In fact, if (3.8) is *false*, then along the real axis

$$\lim_{E\to\pm\infty}E^2|f(E)|=0. \tag{3.9}$$

However, $f(E)$ is polynomially bounded and the Phragmén-Lindelöf principle then tells us that

$$\lim_{|E|\to\infty}E^2|f(E)|=0$$

$$\text{(uniformly in all complex directions)}. \tag{3.10}$$

But if (3.10) is true, one has a superconvergence relation

$$\int_{-\infty}^{+\infty}(E'-E_0)\operatorname{Im}f(E')dE'=0, \tag{3.11}$$

[9] A. Martin (unpublished); see footnote 11 of T. Kinoshita, Phys. Rev. **154**, 1438 (1967).

[10] For a detailed discussion of the Jin-Martin theorem, consult the lectures by N. N. Khuri, BNL Report No. BNL 50212 (C-58), 1969 (unpublished).

where E_0 is real and chosen such that $E_B < E_0 < E_A$. But $\mathrm{Im} f$ is positive for $E' > E_A$, zero in the gap, and negative for $E' > E_B$; and we immediately have a contradiction unless $f \equiv 0$. Therefore (3.9) cannot hold and hence (3.8) must be valid.

IV. APPLICATION OF JIN–MARTIN THEOREM TO $\tilde{M}_{00}$

In order to motivate the main discussion of this paper, we first apply the theorem of Jin and Martin to the function $\tilde{M}_{00}(q_0,|\mathbf{q}|)$. Here we recall from (2.7) that the covariant $\tilde{M}_{\mu\nu}$ has a disconnected part in it which is proportional to the propagator. We delay our study of the connected part, $\tilde{M}_{00}{}^c$, by itself to Sec. V.

The discussion in this section is more or less heuristic and we only go through it to clarify the problem. The main results we have are proved in Sec. V.

Let us consider the function $\tilde{M}_{00}(q^0,|\mathbf{q}|)$. This function is an analytic function of q_0 with two cuts. Its discontinuity across the cuts is the same as that of $\tilde{M}_{00}(q_0,\mathbf{q})$, which can be calculated from (2.3) and (2.4); one gets

$$\mathrm{Im}\tilde{M}_{00}(q_0,|\mathbf{q}|)=\rho_{00}(q_0,|\mathbf{q}|), \qquad q_0>0 \qquad (4.1)$$

and

$$\mathrm{Im}\tilde{M}_{00}(q_0,|\mathbf{q}|)=-\bar{\rho}_{00}(q_0,|\mathbf{q}|), \quad q_0<0.$$

It is clear from (2.4) that the discontinuity is positive on the right-hand cut and negative on the left-hand cut.

The function $\tilde{M}_{00}(q_0,|\mathbf{q}|)$ also has a Born pole in q_0 and a Z-graph pole. The Born pole is at

$$q_0=+[(M^2+\mathbf{p}^2+\mathbf{q}^2)^{1/2}-(M^2+\mathbf{p}^2)^{1/2}], \qquad (4.2)$$

where we always take $\mathbf{P}\cdot\mathbf{q}=0$. The Z-graph pole is at

$$q_0=-[(M^2+\mathbf{p}+\mathbf{q}^2)^{1/2}+(M^2+\mathbf{p}^2)^{1/2}]. \qquad (4.3)$$

The Born pole has the same sign of discontinuity as the right-hand cut. The Z-graph pole has the same sign of discontinuity as the left-hand cut.

There is one problem left before we apply the Jin-Martin theorem to $\tilde{M}_{00}$, namely, one can see from (2.7) that the disconnected part of $\tilde{M}_{00}$ is proportional to $\langle\mathbf{p}'|\mathbf{p}\rangle$. In our normalization we have $\langle p'|p\rangle =(2\pi)^3(E/M)\delta^3(\mathbf{p}'-\mathbf{p})$. In the forward direction this is infinite. However, we shall imagine for the sake of the discussion that we have averaged the expression in (2.1) over sharp wave packets in $\mathbf{p}$ and $\mathbf{p}'$ near the forward direction. Since we are only considering analyticity in q_0 at fixed $\mathbf{q}$, this smoothing operation will not affect the analyticity nor the positivity. Thus we can replace $\langle\mathbf{p}'|\mathbf{p}\rangle$ by a constant and proceed with our heuristic discussion. In Sec. V we shall completely factor out the disconnected part from the reduction formula and study only the connected part where this problem does not arise.

Finally, before we apply the theorem, we note that Λ_{00} in Eq. (2.3) already has a term that behaves asymptotically as $-(p_0/M)q_0^{-1}$. This is essentially the term that gives the commutator. So instead of considering $\tilde{M}_{00}$ directly, we define a new function G_{00}, where

$$G_{00}(q_0,\mathbf{q})\equiv[\tilde{M}_{00}(q_0,|\mathbf{q}|)+p_0/Mq_0]. \qquad (4.4)$$

This new function has exactly the same analyticity properties as $\tilde{M}_{00}(q_0,|\mathbf{q}|)$ except for the added pole at $q_0=0$. The discontinuity at this pole is negative, i.e.,

$$\frac{p_0}{Mq_0}=\frac{1}{\pi}\int_{-\infty}^{+\infty}dq_0'\frac{(-p_0/M)\delta(q_0')}{q_0'-q_0}. \qquad (4.5)$$

Hence we can consider the contribution at this pole as part of the left-hand discontinuity. In all, $G(q_0,|\mathbf{q}|)$ has a positive right-hand discontinuity starting at $q_0=(M^2+\mathbf{p}^2+\mathbf{q}^2)^{1/2}-(M^2+\mathbf{q}^2)^{1/2}$, and a negative left-hand discontinuity extending from $-\infty$ to 0. Thus, applying the theorem of Jin and Martin, we get that in all complex-q_0 directions,

$$|G_{00}(q_0,|\mathbf{q}|)|>C/|q_0|^2, \quad C>0 \qquad (4.6)$$

where $|q_0|$ is large and $|\mathbf{q}|$ is fixed; $|q_0|\gg|\mathbf{q}|$. This means that

$$\left|\tilde{M}_{00}(q_0,|\mathbf{q}|)+\frac{p_0}{Mq_0}\right|>\frac{C}{|q_0|^2}. \qquad (4.7)$$

Using the fact that $q^\mu\tilde{M}_{\mu\nu}=-(P_\nu/M)$, we get

$$|\tilde{M}_{30}(q_0,|\mathbf{q}|)|>\frac{C/|q_3|}{q_0} \qquad (4.8)$$

and

$$|\tilde{M}_{33}(q_0,|\mathbf{q}|)|>C/q_3^2=C'>0. \qquad (4.9)$$

In (4.8) and (4.9), we have set

$$q=(q_0,0,0,q_3), \qquad (4.10)$$

and

$$\mathbf{p}\cdot\mathbf{q}=0.$$

(In fact, throughout this paper we take $\mathbf{p}\cdot\mathbf{q}=0$ unless otherwise noted.)

At first sight the result implied in (4.9) seems essentially trivial. We have only succeeded in rediscovering c-number Schwinger terms. For, recalling Eq. (2.7), we have

$$\tilde{M}_{33}(q_0,|\mathbf{q}|)=\tilde{M}_{33}{}^c(q_0,\mathbf{q})+\langle p'|p\rangle\tilde{D}_{33}(q^2). \quad (4.11)$$

Certainly the second term in (4.11) gives at least a constant as $q_0\to i\infty$. This constant is just the well-known Schwinger term. In fact $\tilde{M}_{33}{}^c$ could decrease as fast as we please without violating (4.9). The leading powers could in principle all come from $\tilde{D}_{33}(q^2)$.

However, a further look at the problem shows that the situation is not so simple. To see this, let us symmetrize the function $\tilde{M}_{\mu\nu}$ in the charge indices and consider

$$\tilde{M}_{\mu\nu}{}^s=\tfrac{1}{2}[\tilde{M}_{\mu\nu}((+)+p\to(+)+p) +\tilde{M}_{\mu\nu}((-)+p\to(-)+p)]. \quad (4.12)$$

In this case $\tilde{M}_{00}{}^s(q_0,\mathbf{q})$ would be an even function of q_0, and $\mathrm{Im}\tilde{M}_{00}{}^s$ will be odd in q_0. Now since we are seeking lower bounds we can safely assume that $\tilde{M}_{00}{}^s$ satisfies an unsubtracted dispersion relation in q_0. For if that is not true, then one can easily show that there is a real sequence $\{q_n{}^0\}$, $q_n{}^0\to\infty$ as $n\to\infty$, such that

$$\lim_{n\to\infty}|\tilde{M}_{00}{}^s(q_n{}^0,|\mathbf{q}|)|>\mathrm{const}. \qquad (4.13)$$

Using the fact that $q^\mu\tilde{M}_{\mu\nu}{}^s=0$, this gives

$$\lim_{n\to\infty}|(q_n{}^0)^{-2}\tilde{M}_{33}{}^s(q_n{}^0,|\mathbf{q}|)|>\mathrm{const}. \qquad (4.14)$$

This is already a strong lower bound. The behavior (4.14) cannot come from the propagator if the c-number Schwinger term is finite. Below, we consider only the case where $\tilde{M}_{00}{}^s$ satisfies an unsubtracted dispersion relation. The argument can be easily modified if subtractions are needed. In any case the rest of this section is just to provide a motivation for the theorem of Sec. V.

We thus write

$$\tilde{M}_{00}{}^s(q_0,|\mathbf{q}|)=\frac{1}{\pi}\int_0^\infty q_0{}'dq_0{}'\frac{\rho_{00}{}^s(q_0{}',|\mathbf{q}|)}{(q_0{}'^2-q_0{}^2)}. \qquad (4.15)$$

From (2.4) we have

$$\rho_{00}{}^s(q_0,|\mathbf{q}|)=\tfrac{1}{2}\sum_n(2\pi)^4\delta^4(q+p'-p_n)|\langle p'|j_0{}^+(0)|n\rangle|^2. \qquad (4.16)$$

Suppose now we divide the sum over states n into two classes, A and B. In class A we keep only states which have no proton in them. For example, class A could consist of 17 states with strange hadrons in them but *no* protons. Class B would consist of all remaining intermediate states.

We can write

$$\rho_{00}{}^s(q_0,|\mathbf{q}|)=\rho_{00}{}^{(A)}(q_0,|\mathbf{q}|)+\rho_{00}{}^{(B)}(q_0,|\mathbf{q}|), \qquad (4.17)$$

where

$$\rho_{00}{}^{(A)}=\tfrac{1}{2}\sum_{n\ A}(2\pi)^4\delta^4(q+p'-p_n)|\langle p'|j_0{}^+(0)|n\rangle|^2,$$
$$\rho_{00}{}^{(B)}=\tfrac{1}{2}\sum_{n\ B}(2\pi)^4\delta^4(q+p'-p_n)|\langle p'|j_0{}^+(0)|n\rangle|^2. \qquad (4.18)$$

Then obviously both $\rho_{00}{}^{(A)}\geqslant 0$ and $\rho_{00}{}^{(B)}\geqslant 0$. We can then rewrite (4.15) as

$$\tilde{M}_{00}{}^s(q_0,|\mathbf{q}|)=\frac{1}{\pi}\int_0^\infty q_0{}'dq_0{}'\frac{\rho_{00}{}^{(A)}(q_0{}',|\mathbf{q}|)}{q_0{}'^2-q_0{}^2}+\frac{1}{\pi}\int_0^\infty q_0{}'dq_0{}'\frac{\rho_{00}{}^{(B)}(q_0{}',|\mathbf{q}|)}{q_0{}'^2-q_0{}^2}. \qquad (4.19)$$

Since the states $\{A\}$ contain no protons, they cannot contribute to the disconnected part. Thus all the contribution of the disconnected part is by definition in the $\{B\}$ part. Now we have

$$\tilde{M}_{00}{}^s=\tilde{M}_{00}{}^{(A)}+\tilde{M}_{00}{}^{(B)}. \qquad (4.20)$$

Both functions satisfy the Jin-Martin bounds

$$|\tilde{M}_{00}{}^{(A)}|>C/|q_0|^2,\quad |\tilde{M}_{00}{}^{(B)}|>C'/|q_0|^2. \qquad (4.21)$$

In fact, as $q_0\to i\infty$, both C and C' above are positive. Thus for all separations $\{A\}$ and $\{B\}$ and in all frames of reference there should always be a cancellation between the two leading terms in (4.20) such that we are always left with just the contribution of the propagator. Otherwise we have Q-number Schwinger terms or even worse singularities. The cancellation has to occur in all orders above $O(q_0{}^{-2})$ and at $O(q_0{}^{-2})$. It has also to occur at $O(q_0{}^{-3})$ and $O(q_0{}^{-4})$ if we want a convergent Cottingham formula for the electromagnetic self-mass of the proton.

To make the problem clearer, we have to deal explicitly with the connected part of $\tilde{M}_{\mu\nu}$. This we can do directly by reducing out all the target particles in (2.1) and explicitly factoring out the disconnected part. This is done in Sec. V.

V. LOWER BOUND ON CLASS-I CONTRIBUTION TO $\tilde{M}_{00}{}^c$

In this section we write a fully reduced LSZ formula for $\tilde{M}_{\mu\nu}$ and thereby explicitly factor out the disconnected part. We choose to use the R-product form of the reduction formula since the vacuum expectation value of an R product is always connected. We then take the absorptive part of the four-field retarded product and show how it explicitly separates into class-I and class-II contributions. We obtain a lower bound on the contribution of the class-I states to $\tilde{M}_{00}{}^c$.

We have defined $\tilde{M}_{\mu\nu}$ by the following:

$$\tilde{M}_{\mu\nu}=i\int d^4x\, e^{iq\cdot x}\langle\pi^+(p')|T(j_\mu{}^+(x)j_\nu{}^-(0))|\pi^+(p)\rangle+\Sigma_{\mu\nu}. \qquad (5.1)$$

We use pions as targets in this section to simplify the algebra.

Now if we define

$$\tilde{R}_{\mu\nu}=-\int d^4x\, e^{iq\cdot x}\langle\pi^+(p')|R(j_\mu{}^+(x);\,j_\nu{}^-(0))|\pi^-(p)\rangle+\Sigma_{\mu\nu}, \qquad (5.2)$$

then, since $R(j_\mu{}^+(x);\,j_\nu{}^-(0))=-i\theta(x_0)[j_\mu{}^+(x),j_\nu{}^-(0)]$, the only difference between (2.2) and (2.3) is a term that is proportional to the matrix element of $ij_\nu{}^-(0)j_\mu{}^+(x)$. The contribution of this term is zero for

$q_0>0$, $\mathbf{q}$ fixed. Thus we have

$$\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)\equiv\tilde{R}_{\mu\nu}(q_0,|\mathbf{q}|),\quad q_0>0. \tag{5.3}$$

The analytic function defined by continuing either $\tilde{M}_{\mu\nu}$ or $\tilde{R}_{\mu\nu}$ in q_0 is the same. We have one analytic function $G_{\mu\nu}(z,|\mathbf{q}|)$ such that on the right-hand cut, $G_{\mu\nu}(z=q_0+i\epsilon)=\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)=\tilde{R}_{\mu\nu}(q_0;\,|\mathbf{q}|)$, $q_0>0$. On the left-hand cut, $q_0<0$, we have $G_{\mu\nu}(z=q_0+i\epsilon)=\tilde{R}_{\mu\nu}(q_0)$ and $G_{\mu\nu}(z=q_0-i\epsilon)=\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)$. For $q_0<0$ the functions $\tilde{R}$ and $\tilde{M}$ are just complex conjugates of each other. We go through this trivial exercise to show that no difficulties are introduced by going from the T product to the R product. In the end we are mainly interested in limits as $q_0\to i\infty$ in the purely imaginary direction.

From here on we shall identify the two functions with the analytic function and use the label $\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)$ to denote the analytic continuation from (5.3) into the complex plane. We have

$$\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)=-\int d^4x \times e^{iq\cdot x}\langle p'|R(j_\mu^+(x);\,j_\nu^-(0))|p\rangle+P(q_0), \tag{5.4}$$

where $P(q_0)$ denotes a polynomial in q_0 coming from the Schwinger terms.

We proceed to reduce out the two pions in (5.4):

$$\begin{aligned}\tilde{M}_{\mu\nu}(q_0,|\mathbf{q}|)&=-\int d^4x\, e^{iq\cdot x}\langle 0|a_{\rm in}{}^{(+)}(\mathbf{p}')R(j_\mu^+(x);\,j_\nu^-(0))|p\rangle+P(q_0)\\&=-\int d^4x\, e^{iq\cdot x}\langle 0|[a_{\rm in}{}^{(+)}(\mathbf{p}'),R(j_\mu^+(x);\,j_\nu^-(0))]|p\rangle\\&\qquad-\int d^4x\, e^{iq\cdot x}\langle 0|R(j_\mu^+(x);\,j_\nu^-(0))a_{\rm in}{}^{(+)}(\mathbf{p}')|p\rangle+P(q_0).\end{aligned} \tag{5.5}$$

As in the usual derivation of the retarded reduction formula (see, for example, Ref. 11), the first term above gives the three-field retarded product. The second term is the disconnected part, since

$$a_{\rm in}{}^{(+)}(\mathbf{p}')|p\rangle=a_{\rm in}{}^{(+)}(p')a_{\rm in}{}^{(+)\dagger}(\mathbf{p})|0\rangle=(2\pi)^3 2E\delta^3(\mathbf{p}'-\mathbf{p})|0\rangle.$$

One obtains

$$\begin{aligned}\tilde{M}_{\mu\nu}=+\int e^{iq\cdot x}d^4x\,[K_y\langle 0|R(j_\mu^+(x);\,j_\nu^-(0)\phi^+(y))|p\rangle]e^{+ip'\cdot y}d^4y\\-\langle p'|p\rangle\int d^4x\, e^{iq\cdot x}\langle 0|R(j_\mu^+(x);\,j_\nu^-(0))|0\rangle+P(q_0),\end{aligned} \tag{5.6}$$

where $K_y=\Box_y+\mu^2$. The second term above is proportional to the retarded propagator. For $q_0>0$ this is identical with $\tilde{D}_{\mu\nu}$ of Eq. (2.7), after we add to it the contribution of the c-number Schwinger terms in $P(q_0)$ to make it covariant.

We thus obtain, for $q_0>0$,

$$\tilde{M}_{\mu\nu}{}^c(q_0,|\mathbf{q}|)=+\int e^{iq\cdot x}d^4x \times[K_y\langle 0|R(j_\mu^+(x);\,j_\nu^-(0)\phi^+(y))|p\rangle]e^{+ip'\cdot y}d^4y +P'(q_0), \tag{5.7}$$

where now all the c-number Schwinger terms have been taken out. The polynomial $P'(q_0)$ comes from the q-number Schwinger terms if they exist. The connected amplitude is defined by analytic continuation of the expression (5.7) from the region $q_0>0$ to the whole cut q_0 plane. As we stressed at the beginning of this section this gives the same analytic function as the continuation of the connected part of the T product.

We reduce out the second target pion in (5.7) and get

$$\tilde{M}_{\mu\nu}{}^c(q_0,|\mathbf{q}|)=-\int d^4xd^4yd^4z\, e^{iq\cdot x}e^{ip'\cdot y}e^{-ip\cdot z} \times K_yK_z\langle 0|R(j_\mu^+(x);\,j_\nu^-(0)\phi^+(y)\phi^-(z)) \times|0\rangle+P'(q_0). \tag{5.8}$$

For the definition of the multiple retarded product, see Ref. 11.

We want an explicit expression for the imaginary part of $\tilde{M}_{00}{}^c$,

$$A_{00}\equiv{\rm Im}M_{00}{}^c. \tag{5.9}$$

[11] H. Lehmann, Nuovo Cimento Suppl. **14**, 153 (1959).

To obtain that, we need to calculate the difference:

$$R(j_0^+(x);\, j_0^-(0)\phi^+(y)\phi^-(z)) \\ -R(j_0^-(0);\, j_0^+(x)\phi^+(y)\phi^-(z)).$$

(We are dealing with the forward case only where $p'=p$.) The result is given by

$$\begin{aligned} &R(j_0^+(x);\, j_0^-(0)\phi^+(y)\phi^-(z)) \\ &\quad -R(j_0^-(0);\, j_0^+(x)\phi^+(y)\phi^-(z)) \\ &= -i[j_0^+(x),R(j_0^-(0);\,\phi^+(y)\phi^-(z))] \\ &\quad -i[R(j_0^+(x);\,\phi^+(y)),R(j_0^-(0);\,\phi^-(z))] \\ &\quad -i[R(j_0^+(x);\,\phi^-(z)),R(j_0^-(0);\,\phi^+(y))] \\ &\quad -i[R(j_0^+(x);\,\phi^+(y)\phi^-(z)),j_0^-(0)], \end{aligned} \tag{5.10}$$

where to get (5.10) we have used the identity[12]

$$\begin{aligned} &R(x;\,y,x_1,\ldots,x_n)-R(y;\,x,x_1,\ldots,x_n) \\ &= -i\sum_{i_1\cdots i_n}\sum_{k=0}^{n}\frac{1}{k!(n-k)!} \\ &\quad\times[R(x;\,x_{i_1},\ldots,x_{i_k}),R(y;\,x_{i_{k+1}},\ldots,x_{i_n})]. \end{aligned} \tag{5.11}$$

Substituting (5.10) into (5.8), we get an expression for the absorptive part A_{00} which contains the four different commutators in (5.10). Inserting sums over intermediate states and using translation invariance, we see that A_{00} can be written as the sum of three different types of terms. We write

$$A_{00}=A_{00}{}^{\mathrm{I}a}+A_{00}{}^{\mathrm{I}b}+A_{00}{}^{\mathrm{II}}, \tag{5.12}$$

where

$$\begin{aligned} A_{00}{}^{\mathrm{I}a}=&\tfrac{1}{2}\sum_n(2\pi)^4\delta(q_0+p_0-E_n)\left|\int e^{ip'\cdot y}d^4y\,K_y\langle 0|R(j_0^+(0);\,\phi^+(y))|n,\,M_n,\,\mathbf{p}+\mathbf{q}\rangle\right|^2 \\ &-\tfrac{1}{2}\sum_n(2\pi)^4\delta(q_0-p_0+E_n)\left|\int e^{+ip'\cdot y}d^4y\,K_y\langle 0|R(j_0^-(0);\,\phi^+(y))|n;\,M_n;\,+\mathbf{p}-\mathbf{q}\rangle\right|^2 \end{aligned} \tag{5.13}$$

and

$$E_n=(M_n{}^2+\mathbf{p}^2+\mathbf{q}^2)^{1/2},\quad p_0=(\mu^2+\mathbf{p}^2)^{1/2}.$$

These are the usual direct s-channel contributions of dispersion theory. But since we vary q_0 at fixed $\mathbf{q}$, we get the additional contributions

$$\begin{aligned} A_{00}{}^{\mathrm{I}b}=&\tfrac{1}{2}\sum_n(2\pi)^4\delta(q_0-p_0-E_n)\left|\int e^{-ip\cdot z}d^4z\,K_z\langle 0|R(j_0^+(0);\,\phi^-(z))|n,\,M_n{}^2,\,\mathbf{q}-\mathbf{p}\rangle\right|^2 \\ &-\tfrac{1}{2}\sum_n(2\pi)^4\delta(q_0+p_0+E_n)\left|\int e^{-ip\cdot z}d^4z\,K_z\langle 0|R(j_0^-(0);\,\phi^-(z))|n,\,M_n{}^2,\,-\mathbf{p}-\mathbf{q}\rangle\right|^2. \end{aligned} \tag{5.14}$$

These are nothing but the so-called Z-graph contributions; see Fig. 1(b). Finally, we have the class-II contributions

$$\begin{aligned} A_{00}{}^{\mathrm{II}}=&\tfrac{1}{2}\sum_n\delta^4(q-p_n)\int e^{ip'\cdot y}e^{-ip\cdot z}d^4y d^4z\,K_yK_z\langle 0|j_0^+(0)|n\rangle\langle n|R(j_0^-(0);\,\phi^+(y)\phi^-(z))|0\rangle(2\pi)^4 \\ &-\tfrac{1}{2}\sum_n\delta^4(q+p_n)\int e^{ip'\cdot y}e^{-ip\cdot z}d^4y d^4z\,K_yK_z\langle 0|R(j_0^-(0);\,\phi^+(y)\phi^-(z))|n\rangle\langle n|j_0^+(0)|0\rangle(2\pi)^4 \\ &+[\text{complex conjugate}]. \end{aligned} \tag{5.15}$$

A schematic representation of the class-II contributions is shown in Fig. 1(c).

Both the direct-channel and Z-graph contributions satisfy our positivity requirement. That is, we have

$$A_{00}{}^{\mathrm{I}}=A_{00}{}^{\mathrm{I}a}+A_{00}{}^{\mathrm{I}b} \tag{5.16}$$

and

$$\begin{aligned} &A_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)\geqslant 0,\quad q_0>0; \\ &A_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)\leqslant 0,\quad q_0<0. \end{aligned} \tag{5.17}$$

This is evident from (5.10), (5.13), and (5.14). On the other hand, $A_{00}{}^{\mathrm{II}}$ has no clear positivity properties. Thus after factoring out the disconnected part, the full absorptive part of the connected amplitude, $\mathrm{Im}M_{00}{}^c$, no longer has any obvious positivity property.

However, since we are only seeking to derive lower bounds, we can separate the contributions of the class-I and class-II states to $M_{00}{}^c$, obtain a lower bound for the class-I contribution, and then study the possibility of cancellations between the two contributions.

A similar situation will hold for the spin-averaged proton amplitude of Secs. II and IV. The same splitting can be achieved and the class-I contributions will be positive. In order not to repeat the definitions of Sec. II for spin-zero targets, we shall from here on use

[12] K. Nishijima, Phys. Rev. 119, 485 (1960).

the kinematics of the spin-averaged proton amplitude defined previously.

For the analytic functions $\tilde{M}_{00}{}^c$ we shall distinguish the following three possible alternatives which exhaust all possibilities:

(i) $\tilde{M}_{00}{}^c(q_0,|\mathbf{q}|)$ does *not* satisfy an unsubtracted dispersion relation. This means that on the real axis on some sequence of points we have the lower bound

$$\lim_{n\to\infty} |\tilde{M}_{00}{}^c(q_n{}^0,|\mathbf{q}|)|>C>0\,, \tag{5.18}$$

where $\{q_n\}$ is a real sequence tending either to $+\infty$ or $-\infty$. (We always take $|\mathbf{q}|\neq 0$; otherwise $\tilde{M}_{00}{}^c$ will trivially vanish except for the contact terms as $|\mathbf{q}|\to 0$.) This is the strongest statement we can make about a function that requires subtractions. Of course the function might grow faster than shown above. It might also be greater than a constant along complex directions, but this does not follow just from the fact that subtractions are needed. (One can easily construct a function that goes to zero in all complex directions but oscillates along the cuts on the real axis.) If (5.18) is not true for any sequence $\{q_n\}$, then $\tilde{M}_{00}{}^c(q_0,|\mathbf{q}|)\to 0$ along the real axis as $q_0\to\infty$, and by the Phragmén-Lindelöf theorem it will also approach zero in all complex directions, enabling us to write an unsubtracted dispersion relation. Thus the negation of (5.18) is equivalent to the requirement of no subtractions.

Translating (5.18) to $\tilde{M}_{33}{}^c$, we get

$$\lim_{n\to\infty} |\tilde{M}_{33}{}^c(q^0,|\mathbf{q}|)|\,|q_n{}^0|^{-2}>C'>0\,. \tag{5.19}$$

This lower bound is clearly too large and leads to very singular behavior. If it is true, then the Cottingham formula will badly diverge anyway. Thus we either get strong oscillations as $q_0\to\infty$ or we have highly singular q-number Schwinger terms.

(ii) The second possibility is that $\tilde{M}_{00}{}^c$ satisfies an unsubtracted dispersion relation. In that case we write

$$\tilde{M}_{00}{}^c(q_0,|\mathbf{q}|)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|)}{q_0'-q_0}dq_0' +\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{A_{00}{}^{\mathrm{II}}(q_0',|\mathbf{q}|)}{q_0'-q_0}dq_0'\,. \tag{5.20}$$

Here we have taken the case where the above two integrals are separately convergent. The case where we need no subtractions but where each of the two integrals above diverges separately while their sum remains finite will be dealt with under case (iii).

From (5.20) we can write

$$\tilde{M}_{00}{}^c\equiv M_{00}{}^{\mathrm{I}}+M_{00}{}^{\mathrm{II}}\,. \tag{5.21}$$

The separation in (5.21) is of course frame dependent. (It is also not necessary that $M_{\mu\nu}{}^{\mathrm{I}}$ and $M_{\mu\nu}{}^{\mathrm{II}}$ satisfy gauge conditions separately.)

We now consider the function

$$G_{00}=M_{00}{}^{\mathrm{I}}+P_0/Mq_0\,. \tag{5.22}$$

This function satisfies all the positivity and analyticity properties needed for the Jin-Martin theorem to hold. The added pole at $q_0=0$ has a negative discontinuity and can be considered as part of the left-hand discontinuity. The Born pole and the Z-graph pole are both included in the first integral in (5.20). We thus obtain the lower bound:

$$\left|M_{00}{}^{\mathrm{I}}+\frac{P_0}{Mq_0}\right|>\frac{C}{|q_0|^2},\quad C>0 \tag{5.23}$$

for large $|q_0|$ in any complex direction including $q_0\to i\infty$. The term (P_0/Mq_0) just exactly cancels out the $O(1/q_0)$ term coming from Λ_{00}; see (2.11) and (2.13).

For $|\mathbf{q}|\neq 0$ the constant C is strictly greater than zero.[13] In addition to being proportional to $\mathbf{q}^2$ for small $\mathbf{q}^2$, C also depends on the frame of reference, i.e., it is also a function of p_0. Just in order to give an intuitive picture of what C might depend on, we shall in the Appendix give a derivation of a lower bound on C for the symmetrized case or the neutral-current case. It turns out that this lower bound in the large-$|\mathbf{p}|$ frame is proportional to the integral $\int_{\nu_t}^{a\nu_t}\nu d\nu\, W_2(\nu,q^2=-\mathbf{q}^2)$, where W_2 is the deep inelastic scattering form factor defined in Ref. 8, a is a finite real number, $a\gg 1$, and $\nu_t=\frac{1}{2}(2M\mu+\mu^2+\mathbf{q}^2)$, $\nu=q\cdot p$.

Thus if the leading asymptotic behavior of $M_{00}{}^{\mathrm{I}}$ implied in (5.23) is not canceled by the contribution from $M_{00}{}^{\mathrm{II}}$, we then get

$$\left|\tilde{M}_{00}{}^c+\frac{P_0}{Mq_0}\right|>\frac{C}{|q_0|^2},\quad C>0\,. \tag{5.24}$$

If this bound holds, we can use the fact that $q^\mu\tilde{M}_{\mu\nu}{}^c=-p_\nu/M$ to get the following lower bounds for the space-time and space-space components. We go to a frame where $q=(q_0;0,0,q_3)$, $(\mathbf{p}\cdot\mathbf{q})=0$, and $|q_3|\neq 0$, and get the lower bounds

$$|\tilde{M}_{30}{}^c|>\frac{C}{|q_3|\,|q_0|}$$

and

$$|\tilde{M}_{33}{}^c|>C|q_3|^{-2}\,, \tag{5.24'}$$

where both bounds hold for large $|q_0|$ in any complex direction. As a consequence the following will hold:

(a) We will have either a q-number Schwinger term, or worse, the Bjorken definition of the space-space commutator will not exist. We recall here that we have

[13] In the limit where the strong interactions are turned off, $A_{00}{}^{\mathrm{I}}$ will only have the Born pole contribution and the Z-graph pole. However, in that limit if we calculate $\tilde{M}_{\mu\nu}{}^c$ from an unsubtracted dispersion relation like (5.20), the pole term one gets does not satisfy the conservation condition. One needs an additional contact term to get (2.6). With that contact term, one still gets the standard free-field results. There is no contradiction between our bound and the free-field limit.

chosen $\mathbf{p}\cdot\mathbf{q}=0$ and no trivial $(1/q_0^2)$ term comes from Λ_{00}.

(b) If the trouble is due simply to the existence of a q-number Schwinger term, then one can subtract it out of $\tilde{M}_{\mu\nu}{}^c$ in defining the noncovariant connected T product $M_{\mu\nu}{}^c$. In this way the $O(q_0^{-2})$ term in (5.24) will not negate the existence of a Bjorken limit. However, the next-to-leading terms in (5.23) and (5.24) could generally be $O((\ln q_0)^n/q_0^{-3})$ or $O(q_0^\alpha/q_0^{-3})$, $0<\alpha<1$. The existence of such terms will lead to the nonexistence of the Bjorken limit for the space-space commutators. Even if the next-to-leading terms are exactly $O(1/q_0^3)$, this, in general, will lead to anomalous terms in the space-space equal-time commutators. Thus, to avoid these anomalous terms we also have to guarantee another cancellation between the class-I and class-II contributions at the $O(1/q_0^3)$ level in $\tilde{M}_{00}{}^c$. We shall discuss these cancellations in more detail in Sec. VI.

(c) Finally, even if cancellations occur at the $O(1/q_0^2)$ and $O(1/q_0^3)$ levels, for the electromagnetic case we also need a cancellation at the $O(1/q_0^4)$ level in $\tilde{M}_{00}{}^{c\gamma}$ if the Cottingham formula for the electromagnetic self-mass is to converge.[1] However, in this case we shall see in Sec. VI that even when Bjorken's argument fails, it fails in such a way as to lead to a worse divergence.

(iii) The third case we have to consider is when $\tilde{M}_{00}{}^c$ satisfies an unsubtracted dispersion relation in q_0 but each of the integrals in (5.20) diverges when taken separately. This would happen if the sum $A_{00}{}^{\mathrm{I}}+A_{00}{}^{\mathrm{II}}$ vanishes as $q_0\to\infty$ while each term separately does not. This means that, excluding oscillations, $A_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)>(C/\ln q_0)$ as $q_0\to\infty$. Such a bound will lead to a divergence in the contribution of the direct-channel states to the Fubini sum rule for the time-time commutator and we can easily exclude it from our discussion. However, even if the above bound holds for the part of $A_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)$ which is odd in q_0 so that it drops out of the Fubini sum rule, we can still modify our arguments by introducing dummy subtractions and write

$$\tilde{M}_{00}{}^c(q_0,\mathbf{q})=\tilde{M}_{00}{}^c(0,|\mathbf{q}|)+\frac{q_0}{\pi}\int_{-\infty}^{+\infty}\frac{A_{00}{}^{\mathrm{I}}(q_0',\mathbf{q})}{q_0'(q_0'-q_0)}dq_0' +\frac{q_0}{\pi}\int_{-\infty}^{+\infty}\frac{A_{00}{}^{\mathrm{II}}(q_0',\mathbf{q})}{q_0'(q_0'-q_0)}dq_0'. \quad (5.25)$$

We now define $\tilde{M}_{00}{}^{\mathrm{I}\prime}$ as

$$\tilde{M}_{00}{}^{\mathrm{I}\prime}=\frac{q_0}{\pi}\int\frac{A_{00}{}^{\mathrm{I}}(q_0',\mathbf{q})}{q_0'(q_0'-q_0)}dq_0'. \quad (5.26)$$

A result analogous to (5.23) still follows:

$$\left|\tilde{M}_{00}{}^{\mathrm{I}\prime}+\frac{P_0}{Mq_0}\right|>\frac{C}{|q_0|^2},\quad C>0 \quad (5.27)$$

for large $|q_0|$ in any complex direction. The leading cancellation, $O((q_0)^0)$, in (5.25) is now trivial, since by definition

$$\tilde{M}_{00}{}^c(0,\mathbf{q})=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{A^{\mathrm{I}}(q_0',\mathbf{q})+A^{\mathrm{II}}(q_0',\mathbf{q})}{q_0'}dq_0'. \quad (5.28)$$

The subtraction introduced in (5.25) was only needed to allow us to separate the class-I and class-II integrals. The right-hand side of (5.28) converges because by assumption $\tilde{M}_{00}{}^c$ satisfies an unsubtracted dispersion relation.

To order $(1/q_0)$, Eq. (5.25) gives us just the Fubini sum rule. For $O(1/q_0^2)$ and lower orders, we are essentially back to a situation identical to case (ii).

VI. DISCUSSION OF CANCELLATIONS BETWEEN CLASS-I AND CLASS-II CONTRIBUTIONS

In this section we shall discuss the possible cancellations between the contributions of class-I and class-II intermediate states. We stress here at the onset that with the minimal input that we have used so far we will *not* be able to rule out these cancellations, especially for the leading order, $O(1/q^2{}_0)$. However, the existence of the lower bound (5.23) does in our opinion change the picture of what is, in the language of Ref. 1, the "reasonable" behavior to expect for $\tilde{M}_{00}{}^c$. For example, we no longer see why it is "reasonable" to assume that the invariant amplitudes M_1 and M_2 are such that $q_0^2M_i\to 0$ as $q_0\to i\infty$. The absence of Q-number Schwinger terms apparently is the exception to the rule instead of the other way around.

Before we discuss the cancellations in detail, we wish to make the following general remark. The contributions of the two integrals in (5.20) are Lorentz-frame-dependent. However, the cancellations we need must occur in *all* frames of reference if we want to avoid the consequences of the inequality (5.24). Experience with saturating sum rules with a few intermediate states shows that while in one frame of reference most of the contribution to the sum rule comes from one state of class I, in another frame of reference most of the contribution to the same sum rule comes from a single state of class II. This situation is most striking if one looks at the Fubini sum rule for two axial-vector currents between proton states.[14] Here in the $\mathbf{P}\to\infty$ frame most of the contribution to the sum rule comes from the one-neutron state; while in the $\mathbf{P}=0$ frame most of the contribution comes from the one-pion state, a class-II state. Thus, even though we cannot rule out cancellations, it is unlikely that the class-I and class-II contributions are so matched that we can have recurring cancellations at different orders of $1/q_0{}^n$ and $(\ln q_0)^\alpha/q_0{}^n$.

We shall discuss the cancellations at three different

[14] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 322 (1968). This paper also contains a nonperturbative separation of class-I and class-II contributions to the absorptive part similar to the one of Sec. V, which, however, does not use retarded product.

levels, $O(1/q_0^2)$, $O(1/q_0^3)$, and $O(1/q_0^4)$. We are of course aware that this is the simplest possible picture and the one most favorable to the validity of the Bjorken limits. There is nothing in general to prevent the existence of $\ln q_0$ powers between the different orders. Furthermore there is nothing to guarantee in general that the next-to-leading order in $M_{00}{}^{\mathrm{I}}$ is $O(q_0^{-\alpha}/q_0^2)$, or even much more complicated behavior.

To discuss the first cancellation, we first point out that (5.23) is only a lower bound. If that lower bound is not reached, we could have for example

$$M_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)+\frac{P_0}{Mq_0}\underset{q_0\to i\infty}{\sim}\frac{Cq_0{}^{\alpha}}{q_0{}^2},\quad \alpha>0. \qquad (6.1)$$

In such a case the integral $\int_{-\infty}^{+\infty} q_0 A_{00}{}^{\mathrm{I}}(q_0,\mathbf{q})dq_0$ will diverge like a power and, to achieve a cancellation, we have to have two divergent integrals canceling each other.

For the sake of the discussion, we take the simplest case, namely, the case when $M_{00}{}^{\mathrm{I}}$ decreases as fast as is allowed by unitarity and the bound (5.23) is exactly reached. We then have

$$\tilde{M}_{00}{}^{\mathrm{I}}(q_0,\mathbf{q})+\frac{p_0}{Mq_0}\underset{q_0\to i\infty}{\sim}\frac{-C}{q_0{}^2}, \qquad (6.2)$$

where C is real. It then follows trivially from (5.20) that the constant C is positive and is given by

$$C=\frac{+1}{\pi}\int_{-\infty}^{+\infty} q_0' A_{00}{}^{\mathrm{I}}(q_0',\mathbf{q})dq_0'. \qquad (6.3)$$

If the integral above exists, it defines the class-I contributions to the space-time commutator.

To obtain the first cancellation is simple. We need $M_{00}{}^{\mathrm{II}}\to +C/q_0{}^2$ and hence

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} q_0'[A_{00}{}^{\mathrm{I}}(q_0',\mathbf{q})+A_{00}{}^{\mathrm{II}}(q_0',\mathbf{q})]dq_0'=0. \qquad (6.4)$$

This gives

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} q_0' \,\mathrm{Im}\tilde{M}_{00}{}^{c}(q_0',|\mathbf{q}|)dq_0'=0$$

and

$$\frac{q^k}{\pi}\int_{-\infty}^{+\infty} \mathrm{Im}\tilde{M}_{k0}{}^{c}(q_0',|\mathbf{q}|)dq_0'=0. \qquad (6.5)$$

This last integral defines the $[j_0{}^+,j_k{}^-]$ equal-time commutator with the c-number Schwinger terms taken out. Indeed we have

$$\frac{q^k}{\pi}\int_{-\infty}^{+\infty} \mathrm{Im}\tilde{M}_{k0}{}^{c}(q_0',|\mathbf{q}|)dq_0'=q^k\int e^{-i\mathbf{q}\cdot\mathbf{x}}d^3x$$
$$\times\{\langle p'|[j_0{}^+(0,\mathbf{x}),j_k{}^-(0,0)]|p\rangle -\langle p'|p\rangle\langle 0|[j_0{}^+(0,\mathbf{x}),j_k{}^-(0,0)]|0\rangle\}. \qquad (6.6)$$

The right-hand side is in fact zero if the space-time commutator has no operator Schwinger terms. We always take $\mathbf{p}\cdot\mathbf{q}=0$. [If one works in the case where $\mathbf{p}\cdot\mathbf{q}\neq 0$, then $\Lambda_{\mu\nu}$ defined in (2.13) will contribute a term of $O(1/q_0{}^2)$ to $\tilde{M}_{00}{}^{c}$ which will appear on the right-hand side in (6.4).] Thus we see that the cancellation in (6.4) is equivalent to the absence of operator Schwinger terms (S.T.) in the space-time commutator. If, however, the right-hand side of (6.4) is not zero, then the commutation relation

$$[j_0{}^+(0,\mathbf{x}),j_k{}^-(0,0)] =2j_k{}^3\delta^3(\mathbf{x})+(c\text{-number S.T.}) \qquad (6.7)$$

cannot be true and needs an operator Schwinger term on the right-hand side.

Thus, even though we cannot rule out cancellations at this level, we can still make the following statement, from (6.3) and (6.4):

In any physical theory for which there exists at least *one* frame of reference in which the integral $\int_{-\infty}^{+\infty} q_0' A_{00}{}^{\mathrm{II}}dq_0'$ is suppressed relative to $\int_{-\infty}^{+\infty} q_0' A_{00}{}^{\mathrm{I}}dq_0'$, the commutator (6.7) cannot be correct, and operator Schwinger terms are needed. Conversely, for (6.7) to be true there must exist no frame of reference in which the contributions of the class-II states to (6.4) are suppressed.

One might ask at this stage if the contribution of the class-II states to the left-hand side of (6.4) is suppressed in the $P_0\to\infty$ frame as it is in the case of the Fubini sum rule. The answer unfortunately is no. The Fubini sum rule is related to the integral $\int_{-\infty}^{+\infty} A_{00}(q_0',\mathbf{q})dq_0'$ and has one less power of q_0'. The t-channel $I=0$ and $I=2$ contributions to A_{00} lead to an absorptive part which is odd in q_0 and hence drop out of the Fubini sum rule. One is left only with the contribution of the $I=1$, t-channel projection. This one can show by standard arguments is suppressed because of a superconvergence relation; see, for example, Ref. 15. However in our case we are dealing with $\int_{-\infty}^{+\infty} q_0' A_{00}dq_0'$ and only the $I=0$ and $I=2$ t-channel contributions survive in this case. For these the existence of the Pomeranchuk trajectory prevents us from getting a superconvergence relation and therefore a suppression of the class-II contributions.

Before going to the second cancellation, we wish to make a remark about the contribution of the class-I_a states, i.e., class I without the Z graphs, to the sum rule we are looking at. We are aware that the $P_0\to\infty$ limit is dangerous but we carry it out anyway just to give us a rough guide. Note that the main trouble with the $P_0\to\infty$ limit comes from the I_b states, the Z graphs, for the space-space and space-time cases, since in these cases one does not have sufficient convergence factors. The case we are looking at corresponds to a space-time sum rule. Looking only at class I_a contribu-

[15] S. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968), pp. 345–347.

tions, we get after interchanging limits

$$\lim_{P_0\to\infty}\int_0^\infty q_0' A_{00}^{\mathrm{I}a}(q_0',\mathbf{q})dq_0' = \int_0^\infty \nu d\nu \frac{W_2(\nu, q^2=-\mathbf{q}^2)}{M^2}. \quad (6.8)$$

The function W_2 behaves asymptotically as $\nu^{\alpha-2}$, where $\alpha=1$ in the case of Pomeranchuk dominance. Thus the right-hand side of (6.8) is linearly divergent. If the $P\to\infty$ limit is allowed, then (6.4) involves a cancellation of two divergent integrals. We point this out here because this divergence gets worse for the other cancellations we are going to consider.

Let us now assume that the leading cancellation occurs and proceed further. The first question is to determine what happens if we subtract the $(-C/q_0^2)$ term from the left-hand side of (6.2). We consider

$$G^{\mathrm{I}}(q_0,\mathbf{q})=M_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)+\frac{P_0}{Mq_0}+\frac{C}{q_0^2}. \quad (6.9)$$

We have in general no guide about the asymptotic behavior of this new function $G^{\mathrm{I}}(q_0,|\mathbf{q}|)$. All we know is that it decreases faster than $1/q_0^2$; how much faster is an open question, in general.[16] If after the $1/q_0^2$ cancellation

$$\lim_{q_0\to i\infty}\left[M_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)+M_{00}{}^{\mathrm{II}}(q_0,|\mathbf{q}|)+\frac{P_0}{Mq_0}\right]q_0^3\to\infty\,, \quad (6.10)$$

then the Bjorken definition of the space-space equal-time commutator diverges and is certainly not equal to the naive quark commutator. Thus to get a well-defined Bjorken limit for the space-space commutator, $M_{00}{}^{\mathrm{II}}$ must also cancel out all the terms in the asymptotic expansion of $M_{00}{}^{\mathrm{I}}$ that lie between $O(1/q_0^2)$ and $O(1/q_0^3)$. If there are logarithmic factors around, this will involve many cancellations.

Again the simplest case is the one in which both integrals satisfy

$$\int_{-\infty}^{+\infty} q_0'^2 A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|)dq_0'<\infty$$

and $\quad (6.10')$

$$\int_{-\infty}^{+\infty} q_0'^2 A_{00}{}^{\mathrm{II}}(q_0',|\mathbf{q}|)dq_0'<\infty\,.$$

In this case the Bjorken limit is well defined. But it does not necessarily lead to the naive quark space-space commutator unless

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} q_0^2(A_{00}{}^{\mathrm{I}}+A_{00}{}^{\mathrm{II}})dq_0=-\frac{p_0\mathbf{q}^2}{M}. \quad (6.11)$$

The term on the right-hand side comes from the $O(1/q_0^3)$ term in Λ_{00}; see Eq. (2.13).

Again on general grounds we cannot exclude the possibility that (6.11) might be exactly correct. However, we can make the following relevant remarks:

(a) The sum rule (6.11) is known to fail in lowest order perturbation theory, as was shown in Refs. 3 and 4. One gets an additional term on the right-hand side of (6.11).

(b) To order g^4, the left-hand side in (6.11) diverges logarithmically. This also follows from Ref. 3. For if $\tilde{M}_{kk}{}^c\sim(\ln^2 q_0/q_0)$ to fourth order, then again in that order $\tilde{M}_{00}{}^c\sim(\ln^2 q_0/q_0^3)$. Consequently, the fourth-order contribution to the left-hand side of (6.11) will have to diverge.

(c) In the $P_0\to\infty$ limit, the integral $\int_{-\infty}^{+\infty}q_0^2dq_0A_{00}{}^{\mathrm{I}a}$ is highly divergent. While in this case one can show that $\int_{-\infty}^{+\infty}q_0^2dq_0A_{00}{}^{\mathrm{II}}$ is suppressed by a superconvergence condition. If the contribution from the Z graphs, I_b, does not cancel out the divergent behavior from the direct-channel states, then the left-hand side of (6.11) will diverge.

Let us expand on remark (c) a bit further. In the sum rule (6.11), only terms in A_{00} which are even in q_0 will contribute. Therefore we need only consider that part of A_{00} which is pure $I=1$ in the t channel, just as in the case of the Fubini sum rule. Using arguments identical to those of Ref. 15, one can show that

$$\lim_{P_0\to\infty}\frac{1}{P_0}\int_{-\infty}^{+\infty} q_0^2A_{00}{}^{\mathrm{II}\rho}dq_0=0\,, \quad (6.12)$$

where $A_{00}{}^{\mathrm{II}\rho}$ is the contribution of the ρ-meson intermediate state as a class-II state. The result follows from the fact that only the invariant amplitude for $(\rho+N\to N+\text{current})$ which behaves like $\nu^{\alpha-2}$ survives the $P_0\to\infty$ limit. The final answer is proportional $\lim_{\nu\to\infty}\nu^{\alpha-1}$ and vanishes since $\alpha\cong\frac{1}{2}$ for an $I=1$ amplitude. The argument of course depends on the absence of fixed $J=1$ poles. For details the reader should consult Adler and Dashen.[15]

For the I_a states, we also take the $P_0\to\infty$ limit and write

$$\frac{1}{P_0}\int_0^\infty q_0^2A_{00}{}^{\mathrm{I}a}dq_0 \underset{P_0\to\infty}{\sim} \frac{1}{P_0^2}\int_0^\infty \nu^2 d\nu \times\frac{W_2(\nu, q^2=-\mathbf{q}^2)}{M^2}. \quad (6.13)$$

The right-hand side of (6.13) diverges like $\nu^{1+\alpha}/P_0^2$ as $\nu\to\infty$. We are aware that the application of the

[16] However, by arguments similar to those for Eqs. (6.14) and (6.15), one can prove that G^{I} cannot decrease faster than q_0^{-4}.

$P_0 \to \infty$ limit is highly dangerous in this case because of the q_0^2 factor in the integral. We are effectively dealing with the space-space sum rule. The contributions of the I_b states in this case lead to a nonuniformity in the $P_0 \to \infty$ limit since there are not enough convergence factors to cancel the Z-graph contributions. However, the $P_0 \to \infty$ limit could still be a rough guide to whether the integral over the I_a states converges or diverges. Thus, unless the Z-graph contribution cancels exactly with the divergent one in (6.13), it is hard to avoid the conclusion that the integral in (6.11) does not converge—a conjecture which is strengthened by the fact that in perturbation theory it diverges in fourth order.

The optimist could of course still hold to the view that the $P_0 \to \infty$ limit is not allowed here and that the divergence in perturbation theory will somehow disappear when all orders are summed up.

In that case we move on to the third cancellation at $O(1/q_0^4)$. This is mainly of interest in the electromagnetic case where it is directly related to the question of the divergence of the Cottingham formula for the self-mass. For details see Bjorken's paper, Ref. 1. We discuss only the electromagnetic case, $\tilde{M}_{00}{}^{\gamma}$. Even if the two cancellations discussed above occur, one can still use the Jin-Martin theorem to prove the following inequality:

$$\left|q_0\tilde{M}_{00}{}^{\gamma\mathrm{I}}+\frac{C}{q_0}\right| > \frac{C'}{|q_0|^3}, \quad C'>0 \tag{6.14}$$

where we have assumed that

$$\tilde{M}_{00}{}^{\gamma\mathrm{I}} \underset{q_0\to i\infty}{\sim} -C/q_0^2.$$

The function $(q_0\tilde{M}_{00}{}^{\gamma\mathrm{I}}+C/q_0)$ has a positive right-hand discontinuity and a positive left-hand discontinuity. The right-hand Born pole has a positive discontinuity and the left-hand Born pole also has a positive discontinuity because of the extra q_0 factor. We have also introduced a pole at q_0 which has a negative discontinuity, $C>0$. Therefore the discontinuity of the above function makes two changes of sign as q_0 varies from $-\infty$ to $+\infty$. Hence we use (3.5) with $\nu=2$ and get (6.14). We can rewrite (6.14) as

$$\left|\tilde{M}_{00}{}^{\gamma\mathrm{I}}+\frac{C}{q_0^2}\right| > \frac{C'}{|q_0|^4}, \quad C'>0 \tag{6.15}$$

for large complex q_0. Since neutral currents commute, the term P_0/Mq_0 is absent from (6.14) and (6.15).

Hence we see from (6.15) that positivity forces us to have an $O(1/q_0^4)$ left in $\tilde{M}_{00}{}^{\mathrm{I}}$ even after we subtract its leading behavior of $O(1/q_0^2)$. If the behavior in (6.15) is not canceled by contributions from the class-II states, then the mass shift will be at least logarithmically divergent. On the other hand, if the integrals are such that

$$\int_0^\infty q_0^3 A_{00}{}^{\mathrm{I}\gamma}(q_0,|\mathbf{q}|)dq_0<\infty\,,$$
$$\int_0^\infty q_0^3 A_{00}{}^{\mathrm{II}\gamma}(q_0,|\mathbf{q}|)dq_0<\infty\,, \tag{6.16}$$

and thus both exist, then their sum

$$\int_0^\infty q_0^3(A_{00}{}^{\mathrm{I}\gamma}+A_{00}{}^{\mathrm{II}\gamma})dq_0=I_{00} \tag{6.17}$$

will give the Bjorken-limit definition of the commutator $[\dot{j}_k,j_k]$. To get a convergent electromagnetic mass I_{00} must be identically zero.

Following our previous discussion, the most likely possibility is that (6.16) is not true and the integrals are divergent. But even when (6.16) is true and the commutator is well defined, it is difficult to make reasonable models, barring the Brandt-Sucher[17] type models, for which $[\dot{j}_k,j_k]=0$. It is therefore hard to escape Bjorken's conclusion about the divergence of the Cottingham formula.

VII. REMARKS ON PERTURBATION THEORY RESULTS

We make a few remarks which compare our results here to the results of Ref. 3.

The main question that arises here is why no constant term appears in the lowest-order calculation of $\tilde{M}_{kk}{}^{c}$ in the gluon model.

The first point is that in the gluon model used in the above references it is assumed there are no q-number Schwinger terms. The calculation was done directly without splitting it up into class-I and class-II contributions.

However, unfortunately even if one were to split the perturbation theory results into class-I and class-II contributions, one might still miss the constant term we are talking about. The trouble is that in perturbation theory one *loses* the positivity for the class-I absorptive parts. This is obvious by looking at some of the diagrams considered by Adler and Tung. In Fig. 2(a) we see that the class-I cut of the box diagram has positivity, but in Fig. 2(b) we have a class-I cut that

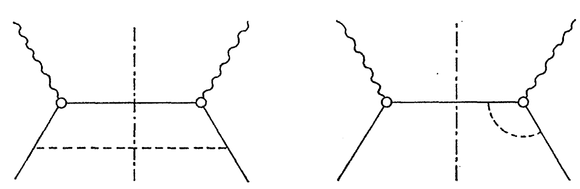

FIG. 2. Class-I cuts to lowest-order perturbation-theory diagrams.

[17] R. Brandt and J. Sucher, Phys. Rev. Letters 20, 1131 (1968).

has no positivity. Unfortunately the box diagrams by themselves do not satisfy the current-conservation condition. So even if we prove that $|M_{00}{}^{\mathrm{I}}(\text{box})| > C/q_0{}^2$, we cannot use the current-conservation condition to convert that into $|M_{kk}{}^{\mathrm{I}}(\text{box})| > C'$. When we include all the diagrams to order g^2, we lose the positivity of the class-I contributions.

This is a standard deficiency of perturbation theory. It is the main reason why even after many years of work no one was able to prove analyticity in the Martin domain to each order in perturbation theory. However, looking at the full amplitude and using positivity Martin was able to give a nonperturbative proof.

In the present case positivity seems to be working against the optimist. After the work of Adler and Tung it was still hoped[18] that the full amplitude will be less singular than perturbation theory indicates. The main lesson of this paper is to indicate that this hope is not well founded.

ACKNOWLEDGMENTS

Most of this work was done while the authors were summer visitors at CERN. The authors wish to thank the members of the CERN theory group for this hospitality. We wish also to thank R. Brandt, H. Epstein, and S. Adler for helpful discussions.

APPENDIX

In this appendix we shall calculate a lower bound for the constant C that appears in Eq. (5.23). We consider only the symmetrized case or the neutral case. In that case we have, as $|q_0| \to \infty$,

$$|M_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|)| > \frac{C}{|q_0|^2}; \quad C>0. \tag{A1}$$

The function $M_{00}{}^{\mathrm{I}}$ is even in q_0 and satisfies the dispersion relation

$$M_{00}{}^{\mathrm{I}}(q_0,|\mathbf{q}|) = \frac{1}{\pi}\int_0^\infty q_0' dq_0' \frac{A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|)}{q_0'^2 - q_0{}^2}. \tag{A2}$$

As we are mainly interested in the limit $q_0 \to i\infty$, let us set $q_0 = i\lambda$ and write

$$M_{00}{}^{\mathrm{I}}(i\lambda,|\mathbf{q}|) = \frac{1}{\pi}\int_0^\infty q_0' dq_0' \frac{A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|)}{q_0'^2 + \lambda^2}. \tag{A3}$$

Since $A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|) \geqslant 0$, the above function is real and positive and has the obvious lower bound

$$M_{00}{}^{\mathrm{I}}(i\lambda,|\mathbf{q}|) > \frac{1}{\pi}\int_{q_t{}^0}^{aq_t{}^0} q_0' dq_0' \frac{A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|)}{q_0'^2 + \lambda^2}, \tag{A4}$$

where

$$q_t{}^0 = [(M+\mu)^2 + \mathbf{p}^2 + \mathbf{q}^2]^{1/2} - (M^2 + \mathbf{p}^2)^{1/2}, \tag{A5}$$

and we choose a to be any real number such that $a>1$ but with $|\mathbf{q}|$ and $|\mathbf{p}|$ taken such that $(aq_t{}^0) < |\mathbf{q}|$. Thus in the integral above, we vary q_0' over a region for which q remains spacelike.

In terms of the invariant amplitudes M_1 and M_2 of Eq. (2.11), $A_{00}{}^{\mathrm{I}}$, in the region of integration in (A4), is given by

$$A_{00}{}^{\mathrm{I}}(q_0',|\mathbf{q}|) = -\mathbf{q}^2 p_0{}^2 \,\mathrm{Im} M_1 + \mathbf{q}^2 \,\mathrm{Im} M_2. \tag{A6}$$

The Born term is not included here since the lower limit of the integral in (A4) was conveniently chosen to be at the continuum threshold. Also q is always spacelike in (A4).

Substituting (A6) in (A4), we get for any $\lambda > 0$

$$M_{00}{}^{\mathrm{I}}(i\lambda,|\mathbf{q}|) > \frac{\mathbf{q}^2}{\pi[\lambda^2 + a^2(q_t{}^0)^2]}\int_{q_t{}^0 p_0}^{aq_t{}^0 p_0} \nu d\nu \times\left[-\mathrm{Im} M_1(\nu,q^2) + \frac{\mathrm{Im} M_2(\nu,q^2)}{p_0{}^2}\right], \quad |\mathbf{q}| \text{ fixed} \tag{A7}$$

where $\nu = q\cdot p = q_0 p_0$ since $\mathbf{q}\cdot\mathbf{p} = 0$; and $q^2 = (\nu^2/p_0{}^2 - \mathbf{q}^2)$. This last inequality can give us a lower bound on C but it will be in terms of an integral at fixed $\mathbf{q}^2$, not fixed q^2. To get a fixed-q^2 bound, we take the limit $p_0 \to \infty$. The limit is legitimate in (A7) since we are integrating over a compact region. As $p_0 \to \infty$, we get from (A5)

$$q_t{}^0 \underset{p_0\to\infty}{\sim} \frac{1}{2p_0}(2M\mu + \mathbf{q}^2 + \mu^2) + O(1/p_0{}^2). \tag{A8}$$

Thus the limits of the integral in (A7) remain finite for large p_0. The contribution from the $\mathrm{Im} M_2$ term in the integral vanishes for large p_0. Using the relation between $\mathrm{Im} M_1$ and the inelastic form factor W_2, we get

$$M_{00}{}^{\mathrm{I}}(i\lambda,|\mathbf{q}|) > \frac{1}{\pi\lambda^2}\int_{\nu_t}^{a\nu_t} \nu d\nu \frac{W_2(\nu,\, q^2 = -\mathbf{q}^2)}{M^2}, \tag{A9}$$

where $\nu_t = (M\mu + \frac{1}{2}\mu^2 + \frac{1}{2}\mathbf{q}^2)$. It is clear that starting with large $\mathbf{q}^2$ and large enough p_0, we can choose a such that $a \gg 1$. Thus the constant C in (A1) has the lower bound

$$C > \frac{1}{\pi}\int_{\nu_t}^{a\nu_t} \nu d\nu \frac{W_2(\nu,\, q^2 = -\mathbf{q}^2)}{M^2}. \tag{A10}$$

The explicit proportionality to $\mathbf{q}^2$ for small $\mathbf{q}^2$ is not manifest in (A10). However, we recall the fact that as $\mathbf{q}^2 \to 0$,

$$W_2(\nu,\, q^2 = -\mathbf{q}^2) = O(\mathbf{q}^2), \quad \mathbf{q}^2 \to 0. \tag{A11}$$

[18] See, for example, footnote 13 in P. V. Landshoff, CERN Report No. Th. 1180-CERN, 1970 (unpublished).