From the c.r. (commutation relation),

$$[L_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma})\,,$$

which we abbreviate as

$$[L,S]=iS\,,$$

it follows by taking the adjoint and using

$$S^{\dagger}=S$$

that

$$[L^{\dagger},S]=iS$$

and consequently,

$$[L-L^{\dagger},\,S]=0\,.$$

Thus $L-L^{\dagger}$ is a tensor in the $SO(3,1)_{L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu}}$ representation which commutes with $S_{\rho\sigma}$ and, therefore,

$$L-L^{\dagger}=\alpha M\,,$$

where

$$M=M_{\mu\nu}=L_{\mu\nu}-S_{\mu\nu}\,.$$

A consequence of this is

$$M^{\dagger}=(1-\alpha)M\,.$$

Inserting this into the c.r.,

$$[M^{\dagger},M^{\dagger}]=iM^{\dagger}\,,$$

we obtain

$$(1-\alpha)^2[M,M]=i(1-\alpha)M\,,$$

which, when compared with

$$[M,M]=iM\,,$$

gives

$$1-\alpha=1 \quad \text{or} \quad \alpha=0\,,$$

so that

$$L^{\dagger}=L\,.$$

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Factorization of the Balachandran Dual-Resonance Model*

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The level structure of the Balachandran generalization of the N-point Veneziano model is considered. The model is shown to exhibit consistent factorization and level structure. Using the harmonic oscillator formalism, expressions are obtained for the vertices coupling one or two excited states to any number of ground-state particles. The form of the propagator is also obtained. Both vertices and propagators are seen to reduce to the Veneziano form when an appropriate limit is taken. Asymptotically, the degeneracy of the nth level is shown to behave like exp(const $n^{2/3}$) where the constant is given explicitly. The Gross model of N-point functions is seen to exhibit a similar asymptotic degeneracy, in contradiction to other results reported in the recent literature.

I. INTRODUCTION

RECENTLY, much work has been done on dual-resonance models,[1] in particular on those which form the simplest N-point extension of the Veneziano (beta function) model.[2] The aspects on which we concentrate are the factorization and level structure of dual-resonance models. These properties have been considered for the N-point beta function model, both directly from the integral representation[3] and also using a harmonic oscillator formulation.[4]

Balachandran[5] has recently considered a set of dual N-point functions which are generalizations of the N-point Veneziano (beta) function, nontrivial in the sense that they cannot be expressed as sums of beta functions with constant coefficients.

We demonstrate that the Balachandran model can be factorized and exhibits a consistent factorization and level structure. Introducing harmonic oscillator notation, we effectively repeat the Fubini-Gordon-Veneziano[4] procedure for this model. The asymptotic degeneracy of levels is found to behave as exp(const $n^{2/3}$) for the Balachandran model.

Before proceeding with the factorization, we will briefly define the Balachandran N-point function.[5] One first defines a homeomorphism ω of the interval $[0,1]$ onto itself, such that

(i) $\omega(0)=1$, $\omega(1)=0$,

(ii) $\omega(\omega(x))=x$,

(iii) $\omega(x)$ is holomorphic on the disks $|x|\leq 1$, $|1-x|\leq 1$. An example of such an ω is

$$\omega(x)=\frac{1-x}{1-\lambda x}\,,\quad -1<\lambda<\tfrac{1}{2}\,. \tag{1.1}$$

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The Balachandran N-point function is given by

$$F_N(\alpha)=\int_0^1 \Big(\prod_{i=2}^{n-2} dx_{1\,i}\Big)\frac{1}{J(x_{12},x_{13},\ldots,x_{1\,N-2})}\times\prod_{i,j} x_{ij}{}^{-\alpha_{ij}-1}, \quad (1.2)$$

where the x_{ij}'s are defined in terms of the u_{ij}'s of the N-point beta function by

$$u_{ij}=\Omega(x_{ij})=\tfrac{1}{2}[1+x_{ij}-\omega(x_{ij})], \quad (1.3)$$

$$u_{ij}=(1-u_{1\,i-1}\cdots u_{1j})(1-u_{1i}\cdots u_{1\,j-1})/(1-u_{1i}\cdots u_{1j})(1-u_{1\,i-1}\cdots u_{1\,j-1}). \quad (1.4)$$

The domain of holomorphy of the inverse Ω^{-1} of Ω will be assumed to be that stated for Ω itself, as is true for our example (1.1) in which

$$\Omega^{-1}(u_{ij})=x_{ij}=u_{ij}+\frac{1}{\lambda}-\frac{1}{2}-\frac{\lambda}{|\lambda|}\left[\left(\frac{1}{\lambda}-\frac{1}{2}\right)^2+u_{ij}(u_{ij}-1)\right]^{1/2} \quad \text{for } \lambda\neq 0$$

$$=u_{ij} \quad \text{for } \lambda=0. \quad (1.5)$$

J is defined by

$$\frac{1}{J(x_{12},x_{13},\ldots,x_{1\,n-2})}=\frac{1}{\tilde{J}(u_{12},u_{13},\ldots,u_{1\,n-2})}\prod_{i=2}^{n-2}[1-\omega'(x_{1\,i})], \quad (1.6)$$

with

$$\tilde{J}(u_{12},u_{13},\ldots,u_{1\,n-2})=\prod_{i=2}^{n-3}(1-u_{1\,i}u_{1\,i+1}). \quad (1.7)$$

In Sec. II we perform the factorization at the poles in the variable α_{1j} and hence in s_{1j}, showing this to be consistent. Section III is devoted to the introduction of harmonic oscillator formalism, in which we isolate the propagator and vertices with one excited external state. Double factorization is investigated in Sec. IV and the expressions for vertex operators with two external excited states are obtained. In Sec. V the asymptotic degeneracy of the nth level is obtained. Section VI is the conclusion.

II. FACTORIZATION

For the purpose of factorizing the residues of the poles in α_{1j} of the Balachandran N-point function,[5] it is convenient to write

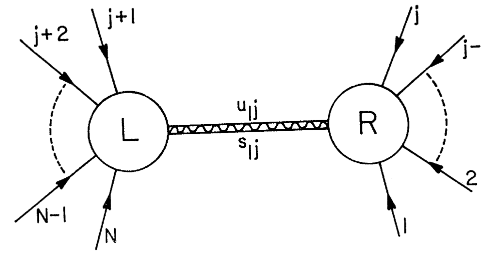

FIG. 1. Single factorization.

$$F_N(\alpha)=\int_0^1\Big(\prod_{k=j+1}^{N-2}dx_{1\,k}\Big)\frac{1}{J(x_{1\,j+1}\cdots x_{1\,N-2})}I_L\times\int_0^1\Big(\prod_{i=2}^{j-1}dx_{1\,i}\Big)\frac{1}{J(x_{12}\cdots x_{1\,j-1})}$$
$$\times I_R x_{1\,j}{}^{-\alpha_{1\,j}-1}\Big(\prod_{\substack{i,k\\ i-1\leq j\leq k;\, i\leq k}}\tilde{x}_{ik}{}^{-\alpha_{ik}-1}\Big)\times\frac{[1-\omega'(x_{1\,j})]dx_{1\,j}}{(1-u_{1\,j-1}u_{1\,j})(1-u_{1\,j}u_{1\,j+1})}, \quad (2.1)$$

where I_L and I_R are integrands corresponding to the right- and left-hand clusters of Fig. 1, respectively, evaluated at $u_{1j}=0$ and

$$\tilde{x}_{ik}=x_{ik}, \quad i\leq j\leq k-1$$
$$\tilde{x}_{j+1\,k}=\frac{x_{j+1\,k}}{x_{j+1\,k}|_{u_{1\,j}=0}}, \quad \tilde{x}_{ij}=\frac{x_{ij}}{x_{ij}|_{u_{1\,j}=0}}. \quad (2.2)$$

Since $[1-\omega'(x_{1j})]dx_{1j}=2du_{1j}$, we are interested in expanding

$$\frac{x_{1j}{}^{-\alpha_{1j}-1}}{(1-u_{1j-1}u_{1j})(1-u_{1j}u_{1j+1})}\prod_{\substack{i,k=2\\ i-1\leq j\leq k}}^{N-1}\tilde{x}_{ik}{}^{-\alpha_{ik}-1}$$

in powers of u_{1j}, knowing u_{ik} in terms of u_{1j}.

Here it is useful to note that $\Omega^{-1}(0)=0$ and $\Omega^{-1}(1)=1$.

From our holomorphy assumptions concerning Ω^{-1}, we may expand $x_{ik}=\Omega^{-1}(u_{ik})$ as a Taylor series in u_{ik} about either $u_{ik}=0$ or $u_{ik}=1$, i.e., as

$$x_{ik}=\sum_{s=1}^{\infty}c_s u_{ik}{}^s \quad\text{or}\quad x_{ik}=\sum_{s=0}^{\infty}d_s(1-u_{ik})^s. \quad (2.3)$$

Since, for $i\leq j\leq k-1$, the condition $u_{1\,j}=0$ implies that $x_{ik}=u_{ik}=1$ and hence $\ln x_{ik}=0$, we may expand the x_{ik}'s in terms of the u_{ik}'s and hence in terms of the $u_{1\,l}$'s using (1.4).

This yields a series of the form

$$\ln x_{ik}=\ln\tilde{x}_{ik}=\sum_{\substack{p,q,r\\ p\leq r;\, q\leq r}} C_r{}^{pq}u_{1\,i-1}{}^p(u_{1i}\cdots u_{1\,k-1})^r u_{1k}{}^q$$

$$=\sum_{\substack{p,q,r\\ p\leq r;\, q\leq r}} C_r{}^{pq}u_{1\,i-1}{}^p(u_{1\,i}\cdots u_{1\,j-1})^r u_{1j}{}^r \times(u_{1\,j+1}\cdots u_{1\,k-1})^r u_{1k}{}^q, \quad (2.4)$$

where $C_r{}^{pq}=C_r{}^{qp}$.

It also follows that

$$\ln\tilde{x}_{ij}=\sum_{\substack{p,q,r\\ p\leq q;\, r\leq q}} D_r{}^{pq}u_{1\,i-1}{}^p(u_{1i}\cdots u_{1\,j-1})^q u_{1j}{}^r, \quad (2.5)$$

where $D_r{}^{pq}=C_q{}^{pr}$, $r\neq 0$, $D_0{}^{pq}=0$;

$$\ln\tilde{x}_{j+1\,k}=\sum_{\substack{p,q,r\\ q\leq p;\, r\leq p}} E_r{}^{pq}u_{1j}{}^r(u_{1\,j+1}\cdots u_{1\,k-1})^p u_{1k}{}^q, \quad (2.6)$$

where $E_r{}^{pq}=C_p{}^{rq}$, $r\neq 0$, $E_0{}^{pq}=0$. Now, from

$$x_{1j}=\sum_{s=1}^{\infty} c_s u_{1j}{}^s,$$

we see that it is convenient to define $w_{1j}=c_1u_{1j}$. Then we may write

$$\ln(x_{1j}/w_{1j})=\sum_{s=1}^{\infty} e_s u_{1j}{}^s. \quad (2.7)$$

Since the matrix C_r is a real symmetric matrix, it can be diagonalized by a real orthogonal transformation $\mathcal{O}_r$, viz.,

$$\mathcal{O}_r C_r \mathcal{O}_r{}^T=-\Delta_r G_r, \quad (2.8)$$

where both Δ_r and G_r are real $(r+1)\times(r+1)$ diagonal matrices; Δ_r has all non-negative components, while G_r, the metric tensor in the $(r+1)$-dimensional space, has all diagonal components either ± 1, depending on the sign of the eigenvalue of C_r. (We choose $+1$ to correspond to zero as well as positive eigenvalues.) Therefore,

$$-C_r=\mathcal{O}_r{}^T\Delta_r{}^{1/2}G_r\Delta_r{}^{1/2}\mathcal{O}_r$$
$$=W_r{}^TG_rW_r, \quad (2.9)$$

where $W_r=\mathcal{O}_r\Delta_r{}^{1/2}$. Thus we can write $\ln\tilde{x}_{1\,k}$ in a diagonalized form.

Therefore,

$$\ln x_{ik}=-\sum_{\substack{q,p,s\\ q\leq r;\, p\leq r;\, s\leq r}} W_r{}^{qp}u_{1\,i-1}{}^p(u_{1\,i}\cdots u_{1\,j-1})^r u_{1j}{}^r \times G_r{}^{qq}(u_{1\,j+1}\cdots u_{1\,k-1})^r W_r{}^{qs}u_{1k}{}^s, \quad (2.10)$$

and

$$-\alpha_{ik}-1=-\alpha_{ij}-\alpha_{j+1\,k}-2p_{ij}\cdot p_{j+1\,k}+b-1 \quad (\text{for } i\leq j\leq k-1),$$
$$=\tfrac{1}{2}[\alpha_{ij}-\tfrac{1}{2}(b+1)][\alpha_{j+1k}-\tfrac{1}{2}(b+1)]$$
$$-\tfrac{1}{2}[\alpha_{ij}-\tfrac{1}{2}(b-3)][\alpha_{j+1\,k}-\tfrac{1}{2}(b-3)]$$
$$-2p_{ij}\cdot p_{j+1\,k}, \quad (2.11)$$

where $p_{ij}=p_i+p_{i+1}+\cdots+p_j$; whence

$$x_{1j}{}^{-\alpha_{1j}-1}\Big(\prod_{\substack{i,k\\ i-1\leq j\leq k;\, i<k}} \tilde{x}_{ik}{}^{-\alpha_{ik}-1}\Big) \times\frac{1}{(1-u_{1j-1}u_{1j})(1-u_{1j}u_{1j+1})}$$

$$=w_{1j}{}^{-\alpha_{1j}-1}\exp\{(-\alpha_{1j}-1)\ln(x_{1j}/w_{1j})$$
$$-\ln(1-u_{1\,j-1}u_{1j})-\ln(1-u_{1j}u_{1\,j+1})$$
$$+\sum_i(-\alpha_{ij}-1)\ln(\tilde{x}_{ij})+\sum_k(-\alpha_{j+1\,k}-1)\ln\tilde{x}_{j+1\,k}$$
$$+\sum_{\substack{i,k\\ i\leq j\leq k-1}}[\tfrac{1}{2}(\alpha_{ij}-\tfrac{1}{2}(b+1))(\alpha_{j+1\,k}-\tfrac{1}{2}(b+1))$$
$$-\tfrac{1}{2}(\alpha_{ij}-\tfrac{1}{2}(b-3))(\alpha_{j+1k}-\tfrac{1}{2}(b-3))$$
$$-2p_{ij}\cdot p_{j+1\,k}]\ln\tilde{x}_{ik}\}. \quad (2.12)$$

Now let us define $(-\alpha_{1\,j}-1)e_r=\epsilon_r$:

$$A_r=\frac{1}{\sqrt{2}}\Big(\sum_{i=2}^{j-1}(-\alpha_{ij}-1)\sum_{\substack{p,q\\ p\leq q;\, q\geq r}} D_r{}^{pq}u_{1\,i-1}{}^p \times(u_{1i}\cdots u_{1j-1})^q+\frac{u_{1\,j-1}{}^r}{r}+\tfrac{1}{2}\epsilon_r+1\Big), \quad (2.13\text{a})$$

$$B_r=\frac{1}{\sqrt{2}}\Big(\sum_{k=j+2}^{N-1}(-\alpha_{j+1\,k}-1) \times\sum_{\substack{p,q\\ q\leq p;\, p\geq r}} E_r{}^{pq}(u_{1\,j+1}\cdots u_{1\,k-1})^p u_{1k}{}^q +\frac{u_{1\,j+1}{}^r}{r}+\tfrac{1}{2}\epsilon_r+1\Big), \quad (2.13\text{b})$$

$$X_r=\frac{1}{\sqrt{2}}\Big(\sum_{i=2}^{j-1}(-\alpha_{ij}-1)\sum_{\substack{p,q\\ p\leq q;\, q\geq r}} D_r{}^{pq}u_{1\,i-1}{}^p \times(u_{1i}\cdots u_{1\,i-1})^q+\frac{u_{1\,j+1}{}^r}{r}+\tfrac{1}{2}\epsilon_r-1\Big), \quad (2.13\text{c})$$

$$Y_r=\frac{1}{\sqrt{2}}\Big(\sum_{k=j+1}^{N-1}(-\alpha_{j+1\,k}-1) \times\sum_{\substack{p,q\\ p\leq q;\, q\geq r}} E_r{}^{pq}(u_{1\,j+1}\cdots u_{1\,k-1})^p u_{1k}{}^q +\frac{u_{1\,j+1}{}^r}{r}+\tfrac{1}{2}\epsilon_r-1\Big), \quad (2.13\text{d})$$

$$R_r{}^q=\frac{1}{\sqrt{2}}\sum_{i=2}^{j}[\alpha_{ij}-\tfrac{1}{2}(b+1)]\sum_{p\leq r} W_r{}^{qp}u_{1\,i-1}{}^p \times(u_{1i}\cdots u_{1j-1})^r, \quad (2.13\text{e})$$

$$S_r{}^q=\frac{1}{\sqrt{2}}\sum_{k=j+1}^{N-1}\left[\alpha_{j+1\,k}-\tfrac{1}{2}(b+1)\right]\times\sum_{p\leq r}W_r{}^{qp}u_{1k}{}^p(u_{1j+1}\cdots u_{1k-1})^r,\quad(2.13\text{f})$$

$$T_r{}^q=\frac{1}{\sqrt{2}}\sum_{i=2}^{j}\left[\alpha_{ij}-\tfrac{1}{2}(b-3)\right]\times\sum_{p\leq r}W_r{}^{qp}u_{1i-1}{}^p(u_{1i}\cdots u_{1j-1})^r,\quad(2.13\text{g})$$

$$U_r{}^q=\frac{1}{\sqrt{2}}\sum_{k=j+1}^{N-1}\left[\alpha_{j+1\,k}-\tfrac{1}{2}(b-3)\right]\times\sum_{p\leq r}W_r{}^{qp}u_{1k}{}^p(u_{1j+1}\cdots u_{1k-1})^r,\quad(2.13\text{h})$$

$$\mathcal{P}_r{}^q=\sqrt{2}\sum_{i=2}^{j}p_{ij}\sum_{p\leq r}W_r{}^{qp}u_{1\,i-1}{}^p(u_{1i}\cdots u_{i\,j-1})^r,\quad(2.13\text{i})$$

$$\mathcal{Q}_r{}^q=\sqrt{2}\sum_{k=j+1}^{N-1}p_{j+1\,k}\sum_{p\leq r}W_r{}^{qp}u_{1k}{}^p\times(u_{1\,j+1}\cdots u_{1\,k-1})^r.\quad(2.13\text{j})$$

We will denote $M_{rq}=G_r{}^{qq}M_r{}^q$ for any M. Hence

$$\frac{x_{1\,j}{}^{-\alpha_{1\,j}-1}}{(1-u_{1\,j-1}u_{1j})(1-u_{1\,j}u_{1\,j+1})}\prod_{\substack{i,k\\ i-1\leq j\leq k}}\tilde{x}^{-\alpha_{ik}-1}$$
$$=w_{1j}{}^{-\alpha_{1j}-1}\exp\{\sum_{r=1}^{\infty}[A_rB_r-X_rY_r+\sum_{q\leq r}(-R_r{}^qS_{rq}+T_r{}^qU_{rq}+\mathcal{P}_r{}^q\cdot\mathcal{Q}_{rq})]u_{1j}{}^r\},\quad(2.14)$$

which clearly demonstrates that the model is factorizable. The corresponding form for the N-point beta function would be[3,4]

$$u_{1j}{}^{-\alpha_{1j}-1}\exp\sum_r\left[\frac{u_{1j}{}^r}{r}(P_r\cdot Q_r-b+1)\right],\quad(2.15)$$

where

$$P_r=\sqrt{2}\sum_{i=2}^{j}(u_{1\,i}\cdots u_{1\,j-1})^r p_i,$$
$$Q_r=\sqrt{2}\sum_{k=j+1}^{N-1}(u_{1j+1}\cdots u_{1k-1})^r p_k.\quad(2.16)$$

Now for $r=1$, we can easily see that $C_1{}^{00}=-C_1{}^{10}=-C_1{}^{01}=C_1{}^{11}$ for any Ω. We now put $C_1{}^{10}\equiv\frac{1}{2}\lambda$, where λ clearly depends on the form of Ω.

Then we have

$$C_1=\tfrac{1}{2}\lambda\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix},\quad \mathcal{O}_1=(1/\sqrt{2})\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},$$
$$\Delta_1=\begin{bmatrix}0 & 0\\ 0 & |\lambda|\end{bmatrix},\quad G_1=\begin{bmatrix}1 & 0\\ 0 & \mathrm{sgn}\lambda\end{bmatrix}.\quad(2.17)$$

Hence the $r=1$ term will be $A_1B_1-X_1Y_1+R_1{}^1S_{11}-T_1{}^1U_{11}+\mathcal{P}_1{}^1\cdot\mathcal{Q}_{11}$. Although there appear to be four factored "scalar" terms, it can easily be shown that only two are independent.

Thus the leading trajectory will be nondegenerate, having a contribution from $(\mathbf{P}_1{}^1\cdot\mathbf{Q}_{11})^n$ only.

For the first daughter trajectory at level n, angular momentum $n-1$, we have contributions from each of the two $r=1$ scalar terms, as well as $\mathcal{P}_{10}{}^1\mathcal{Q}_{110}$, multiplying $(\mathbf{P}_1{}^1\cdot\mathbf{Q}_{11})^{n-1}$. There are three more contributions coming from terms $(\mathbf{P}_2{}^q\cdot\mathbf{Q}_{2q})(\mathbf{P}_1{}^1\cdot\mathbf{Q}_{11})^{n-2}$, $q=0, 1, 2$. Hence the degeneracy of the first daughter trajectory (ignoring possible Ward identities) will be six.

In general, one can show that the kth daughter trajectory will have a constant degeneracy (for fixed k), provided that $n\geq 2k$.

Thus the model exhibits a consistent factorization. The degeneracies are independent of the number of external particles and the position along the multiperipheral configuration where the factorization is performed, provided we factorize at a pole which is not too near the end of a given configuration.

III. HARMONIC OSCILLATOR MODEL

For the purpose of applying the Fubini-Gordon-Veneziano-type[4] harmonic oscillator model to our problem, we first write

$$\sum_{q\leq r}(-R_r{}^qS_{rq}+T_r{}^qU_{rq}+\mathcal{P}_r{}^q\cdot\mathcal{Q}_{rq})=\sum_{q\leq r}P_r{}^q\cdot Q_{rq},\quad(3.1)$$

where

$$P_r{}^q=(\mathcal{P}_r{}^q,R_r{}^q,T_r{}^q)$$

and

$$Q_r{}^q=(\mathcal{Q}_r{}^q,S_r{}^q,U_r{}^q)\quad(3.2)$$

are vectors in a six-dimensional metric space of indefinite metric $g^{\mu\nu}=g_{\mu\nu}$ ($g^{00}=g^{55}=+1$, $g^{11}=g^{22}=g^{33}=g^{44}=-1$). Since we wish the space components of the four-vectors to give rise to normal states for positive $G_r{}^{qq}$, then the zeroth components give rise to ghosts (imaginary coupling constants). In the region of resonance formation, the right-hand particles are incoming and the left outgoing, while our notation treats all momenta as ingoing. The change of sign associated with this convention reverses the sign of the four-vector part with respect to the other two components. Therefore, the fifth components give rise to normal states, and the fourth to ghosts. If $G_r{}^{qq}$ is negative, the normal states become ghosts, and vice versa. The A_rB_r term is normal, while the $-X_rY_r$ is ghostlike.

We then define boson annihilation operators $a_{r\mu}{}^q$, $\mu=0, 1, 2, 3, 4, 5$, obeying

$$[a_{r\mu}{}^q,a_{s\nu}{}^{p\dagger}]=\delta_{rs}G_r{}^{qp}g_{\mu\nu},$$
$$[a_{r\mu}{}^q,a_{s\nu}{}^{p\dagger}]=0,\quad(3.3)$$

and boson annihilation operators b_{1r} and b_{2r} obeying

$$
\begin{aligned}
&[b_{1r},b_{1s}{}^{\dagger}]=\delta_{rs}\,, \quad [b_{2r},b_{2s}{}^{\dagger}]=-\delta_{rs}\,,\\
&[b_{1r},b_{1s}]=[b_{2r},b_{2s}]=[b_{1r},b_{2s}]=[b_{1r},b_{2s}{}^{\dagger}]=0\,. \qquad (3.4)
\end{aligned}
$$

All commutators between the a's ($a^{\dagger}$'s) and the b's ($b^{\dagger}$'s) vanish. Now we may write

$$
\begin{aligned}
&\exp\{\sum_{r=1}^{\infty}(A_rB_r-X_rY_r+\sum_{q\leq r}P_r{}^q\cdot Q_{rq})u_{1j}{}^r\}\\
&=\langle 0|\exp[\sum_r(B_rb_{1r}-Y_rb_{2r}+\sum_{q\leq r}Q_r{}^q\cdot a_{rq})]\\
&\quad\times u_{1j}{}^{-\Sigma_r r(b_{1r}{}^{\dagger}b_{1r}-b_{2r}{}^{\dagger}b_{2r}+\Sigma_{q\leq r}a_{rq}{}^{\dagger}\cdot a_{rq})}\exp[\sum_r(A_rb_{1r}{}^{\dagger}\\
&\quad-X_rb_{2r}{}^{\dagger}+\sum_{q\leq r}P_r{}^q\cdot a_{rq}{}^{\dagger})]|0\rangle\,, \qquad (3.5)
\end{aligned}
$$

where we have used the identities[4]

$$
\begin{aligned}
&e^{fa}z^{a^{\dagger}a}=z^{a^{\dagger}a}e^{faz}\,,\\
&e^{fa}e^{ga^{\dagger}}=e^{ga^{\dagger}}e^{fa}e^{fg}\,. \qquad (3.6)
\end{aligned}
$$

We may now write

$$
F_N(\alpha)=\langle 0|\mathcal{G}(p_L,a,b)D(R,s_{1j})\mathcal{G}(p_R,a^{\dagger},b^{\dagger})|0\rangle\,, \qquad (3.7)
$$

where

$$
\begin{aligned}
&\mathcal{G}(p_L,a,b)\\
&=\int_0^1(\prod_{k=j+1}^{N-2}dx_{1k})I_L\frac{1}{J(x_{1\,j+1}\cdots x_{1\,N-2})}\\
&\quad\times\exp[\sum_r(B_rb_{1r}-Y_rb_{2r}+\sum_{q\leq r}Q_r{}^q\cdot a_{rq})]\,,\\
&\mathcal{G}(p_R,a^{\dagger},b^{\dagger})\\
&=\int_0^1(\prod_{i=2}^{j-1}dx_{1i})I_R\frac{1}{J(x_{12}\cdots x_{1\,j-1})}\\
&\quad\times\exp[\sum_r(A_rb_{1r}{}^{\dagger}-X_rb_{2r}{}^{\dagger}+\sum_{q\leq r}P_r{}^q\cdot a_{rq}{}^{\dagger})]\,, \qquad (3.8)
\end{aligned}
$$

where we have adopted the notation

$$
p_L\equiv(p_1\cdots p_j)\,,\quad p_R\equiv(p_N\cdots p_{j+1})\,, \qquad (3.9)
$$

$$
\begin{aligned}
D(R,s_{1j})&=2\int_0^1du_{1j}w_{1j}{}^{-\alpha_{1j}-1}u_{1j}{}^R\\
&=2c_1{}^{-\alpha_{1j}-1}\int_0^1u_{1j}{}^{-\alpha_{1j}-1+R}\\
&=\frac{2c_1{}^{-\alpha_{1j}-1}}{R-\alpha_{1j}}\,, \qquad (3.10)
\end{aligned}
$$

where

$$
R=\sum_r(b_{1r}{}^{\dagger}b_{1r}-b_{2r}{}^{\dagger}b_{2r}+\sum_{q\leq r}a_{r}{}^{q\dagger}a_{rq})\,. \qquad (3.11)
$$

The above are to be compared with the harmonic oscillator formulation of the beta-function model, in which we define $a_{r\mu}$ and c_r obeying

$$
\begin{aligned}
&[a_{r\mu},a_{s\nu}{}^{\dagger}]=\delta_{rs}g_{\mu\nu}\quad(\mu,\nu=0,1,2,3)\,,\\
&[c_r,c_s{}^{\dagger}]=\delta_{rs}\,, \qquad (3.12)
\end{aligned}
$$

whereby we can write the N-point beta function as[4]

$$
B_N(\alpha)=\langle 0|\mathcal{G}(p_L,a,c)D(R,s_{1j})\mathcal{G}(p_R,a^{\dagger},c^{\dagger})|0\rangle\,, \qquad (3.13)
$$

where

$$
\begin{aligned}
\mathcal{G}(p_L,a,c)&=\int_0^1(\prod_{k=j+1}^{N-2}du_{1k})\frac{I_L(u)}{\mathcal{J}(u_{1\,j+1}\cdots u_{1\,N-2})}\\
&\quad\times\sum_r\frac{Q_r\cdot a_r+(1-b)^{1/2}c_r}{\sqrt{r}}\,, \qquad (3.14)
\end{aligned}
$$

$$
D(R,s_{1j})=\frac{1}{R-\alpha_{1j}}\,, \qquad (3.15)
$$

$$
R=\sum_r r(a_r{}^{\dagger}\cdot a_r+c_r{}^{\dagger}c_r)\,. \qquad (3.16)
$$

If in the expression for $F_N(\alpha)$ we wish to bring all the α_{1j} dependence into the propagator, we may do this by redefining A_r, B_r, X_r, and Y_r without the ϵ_r term and moving this into our definition of the propagator, whence D would become

$$
D(R,s_{1j})=2\int_0^1du_{1j}x_{1j}{}^{-\alpha_{1j}-1}u_{1j}{}^R\,,
$$

which is a sum of pole terms.

One now introduces the occupation number space basis

$$
|\lambda\kappa\mu\rangle=\prod_r\prod_{q\leq r}\frac{(a_r{}^{q\dagger})^{\lambda_{rq}}}{(\lambda_r{}^q!)^{1/2}}\prod_s\frac{(b_{1s}{}^{\dagger})^{\kappa_{1s}}}{(\kappa_{1s}!)^{1/2}}\frac{(b_{2s}{}^{\dagger})^{\kappa_{2s}}}{(\kappa_{2s}!)^{1/2}}|0\rangle\,, \qquad (3.17)
$$

which is a "short-hand" form, since each $a_r{}^q$ is a six-component entity.

Then, introducing these states between the factors of the above expression for $F_N(\alpha)$, we get

$$
F_N(\alpha)=\sum_{\lambda\kappa}\mathcal{G}(p_L,\lambda,\kappa)D(\lambda,\kappa,s_{1j})\mathcal{G}(p_R,\lambda,\kappa)\,, \qquad (3.18)
$$

where

$$
\mathcal{G}(p_L,\lambda,\kappa)=\langle 0|\mathcal{G}(p_L,a,b)|\lambda\kappa\rangle\,, \qquad (3.19)
$$

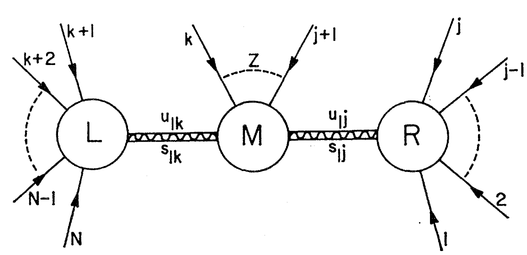

FIG. 2. Double factorization.

$$D(\lambda,\kappa,s_{1j})=\langle\lambda\kappa|D(R,s_{1j})|\lambda\kappa\rangle \tag{3.20a}$$

$$=D(\sum_n n(\lambda_n+\kappa_n),s_{1j}), \tag{3.20b}$$

$$\mathcal{G}(p_R,\lambda,\kappa)=\langle\lambda\kappa|\mathcal{G}(p_R,a^\dagger,b^\dagger)|0\rangle, \tag{3.21}$$

where $\mathcal{G}(p_R,\lambda,\kappa)$ can be identified with the vertex coupling the excited state (λ,κ) to the ground-state particles $(i,\ldots,j)$; similarly for $\mathcal{G}(p_L,\lambda,\kappa)$. The quantity $D(\lambda,\kappa,s_{1j})$ can be identified with the propagator for the excited state (λ,κ).

IV. DOUBLE FACTORIZATION

We are now interested in the factorization of the residues at simultaneous poles in α_{1j}, (s_{1j}) and α_{1k}, (s_{1k}) of the amplitude $F_N(\alpha)$, where $j+2\leq k$.

In order to facilitate the above-mentioned decomposition, we write

$$F_N(\alpha)=\int_0^1(\prod_{i=2}^{N-2}dx_{1i})\frac{1}{J(x_{12},x_{13},\ldots,x_{1\,N-2})}(\prod_{i,l}x_{il}^{-\alpha_{il}-1})$$

$$=\int_0^1(\prod_{l=k+1}^{N-2}dx_{1l})\frac{1}{J(x_{1\,k+1},\ldots,x_{1\,N-2})}I_L\int_0^1(\prod_{m=j+1}^{k-1}dx_{1m})\frac{1}{J(x_{1\,j+1},\ldots,x_{1\,k-1})}I_M$$

$$\times\int_0^1(\prod_{i=2}^{j-1}dx_{1i})\frac{1}{J(x_{12},\ldots,x_{1\,j-1})}I_R x_{1j}^{-\alpha_{1j}-1}\prod_{\substack{i,m\\ i-1\leq j\leq m\leq k;\, i<m}}\tilde{x}_{im}^{-\alpha_{im}-1}\frac{[1-\omega'(x_{1j})]dx_{1j}}{(1-u_{1\,j-1}u_{1j})(1-u_{1j}u_{1\,j+1})}$$

$$\times x_{1k}^{-\alpha_{1k}-1}\prod_{\substack{n,l\\ j\leq n-1\leq k\leq l;\, n<l}}\bar{x}_{nl}^{-\alpha_{nl}-1}\frac{[1-\omega'(x_{1k})]dx_{1k}}{(1-u_{1\,k-1}u_{1k})(1-u_{1k}u_{1\,k+1})}\prod_{\substack{i,l\\ i-1\leq j;\, k\leq l;\, i<l}}\hat{x}_{il}^{-\alpha_{il}-1}, \tag{4.1}$$

where I_L refers to the left-hand cluster evaluated at $u_{1k}=0$, I_R refers to the right-hand cluster evaluated at $u_{1j}=0$, and I_M refers to the middle cluster evaluated at $u_{1j}=u_{1k}=0$. See Fig. 2.

Our new variables $\tilde{x}$, $\bar{x}$, and $\hat{x}$ are defined by

$$\begin{gathered}
\tilde{x}_{im}=x_{im},\quad i\leq j\leq m-1\leq k-2;\quad \tilde{x}_{ik}=x_{ik}|_{u_{1k}=0},\quad i\leq j;\\
\tilde{x}_{j+1\,m}=\frac{x_{j+1\,m}}{x_{j+1\,m}|_{u_{1j}=0}},\quad j+1\leq m-1\leq k-1;\quad \tilde{x}_{ij}=\frac{x_{ij}}{x_{ij}|_{u_{1j}=0}},\quad i\leq j-1;\\
\tilde{x}_{j+1\,k}=\frac{x_{j+1\,k}|_{u_{1k}=0}}{x_{j+1\,k}|_{u_{1j}=u_{1k}=0}};\quad \bar{x}_{nl}=x_{nl},\quad j+2\leq n\leq k\leq l-1;\quad \bar{x}_{nk}=\frac{x_{nk}}{x_{nk}|_{u_{1k}=0}},\quad j+2\leq n\leq k-1;\\
\bar{x}_{j+1\,l}=x_{j+1\,l}|_{u_{1j}=0};\quad k\leq l-1,\quad \bar{x}_{k+1\,l}=\frac{x_{k+1\,l}}{x_{k+1\,l}|_{u_{1k}=0}},\quad k+2\leq l;\\
\bar{x}_{j+1\,k}=\frac{x_{j+1\,k}|_{u_{1j}=0}}{x_{j+1\,k}|_{u_{1j}=u_{1k}=0}};\quad \hat{x}_{il}=x_{il},\quad i\leq j,\quad k\leq l-1;\quad \hat{x}_{ik}=\frac{x_{ik}}{x_{ik}|_{u_{1k}=0}},\quad i\leq j;\\
\hat{x}_{j+1\,l}=\frac{x_{j+1\,l}}{x_{j+1\,l}|_{u_{1j}=0}},\quad k\leq l-1;\quad \hat{x}_{j+1\,k}=\frac{x_{j+1\,k}x_{j+1\,k}|_{u_{1j}=u_{1k}=0}}{x_{j+1\,k}|_{u_{1j}=0}x_{j+1\,k}|_{u_{1k}=0}}.
\end{gathered} \tag{4.2}$$

Thus the factor containing the $\tilde{x}$'s depends on u_{1j} but not on u_{1k}, and is identical in form to an expression obtained for the factorization of the graph without the left-hand cluster, at the poles in α_{1j}. Similarly, the factor containing the $\bar{x}$'s is independent of u_{1j} and represents the factorization at poles in α_{1k} of the graph without the right-hand cluster. The factor containing the $\hat{x}$'s depends on both u_{1k} and u_{1j} and gives the explicit factorization at the simultaneous poles in α_{1j} and α_{1k}.

It can now be shown that

$$x_{1j}^{-\alpha_{1j}-1} \prod_{\substack{i,\, m=2 \\ i-1\leq j\leq m\leq k;\, i<m}}^{N-1} \tilde{x}_{im}^{-\alpha_{im}-1} \frac{1}{(1-u_{1\,j-1}u_{1j})(1-u_{1j}u_{1\,j+1})} x_{1k}^{-\alpha_{1k}-1}$$

$$\times \prod_{\substack{n,\, l=2 \\ j\leq n-1\leq k\leq l;\, n<l}}^{N-1} \bar{x}_{nl}^{-\alpha_{nl}-1} \frac{1}{(1-u_{1\,k-1}u_{1k})(1-u_{1k}u_{1\,k+1})} \prod_{\substack{i,\, l=2 \\ i-1\leq j;\, k\leq l}}^{N-1} \hat{x}_{il}^{-\alpha_{il}-1}$$

$$=w_{1j}^{-\alpha_{1j}-1}w_{1k}^{-\alpha_{1k}-1} \exp[\sum_r [(A_rL_r-X_rM_r+\sum_{q\leq r} P_r^q\cdot K_{rq})u_{1j}^r+(\bar{L}_rB_r-\bar{M}_rY_r+\sum_{q\leq r} \bar{K}_r^q\cdot Q_{rq})u_{1k}^r$$

$$+\sum_{q\leq r} P_r^q\cdot Q_{rq}u_{1j}^rZ^ru_{1k}^r]+\sum_r \sum_{q\leq r} \{(T_r^q-R_r^q)(U_{rq}-S_{rq})u_{1j}^r[\tfrac{1}{2}(\alpha_{j+1k}-b)Z^r]u_{1k}^r$$

$$+(T_r^q-R_r^q)\mathcal{Q}_{rq}u_{1j}^r\cdot(p_{j+1k}Z^r)u_{1k}^r+\mathcal{P}_r^q(U_{rq}-S_{rq})u_{1j}^r\cdot(p_{j+1k}Z^r)u_{1k}^r\}$$

$$+\sum_{p\leqslant r}\sum_{q\leq r} \{(T_r^q-R_r^q)(B_p-Y_p)u_{1j}^r[\tfrac{1}{2}W_{rq}^p(\alpha_{j+1k}-\tfrac{1}{2}(b-1))Z^r]u_{1k}^p+\mathcal{P}_r^q(B_p-Y_p)u_{1j}^r\cdot(W_{rq}^pp_{j+1k}Z^r)u_{1k}^p$$

$$+(R_r^q+T_r^q)(B_p-Y_p)u_{1j}^r(\tfrac{1}{2}W_{rq}^pZ^r)u_{1k}^p\}+\sum_{p\leq r}\sum_{q\leq r} \{(A_p-X_p)(U_{rq}-S_{rq})u_{1j}^p[\tfrac{1}{2}W_r^{qp}(\alpha_{j+1k}-\tfrac{1}{2}(b-1))]u_{1k}^r$$

$$+(A_p-X_p)\mathcal{Q}_r^qu_{1j}^p\cdot(W_r^{pq}p_{j+1k}Z^r)u_{1k}^r+(A_p-X_p)(U_{rq}+S_{rq})u_{1j}^p(\tfrac{1}{2}W_r^{qp}Z^r)u_{1k}^r\}$$

$$+\sum_{1\leq p\leq r}\sum_{1\leq q\leq r} \tfrac{1}{2}(A_p-X_p)(B_q-Y_q)u_{1j}^p[C_r^{pq}(-\alpha_{j+1k}-1)Z^r]u_{1k}^q]. \quad (4.3)$$

Each term in the exponent shows explicitly a factorized form, indicating that the residues at the simultaneous poles can be written in a factored form, being a sum of terms which are products of a factor depending only on the left-hand cluster, with one depending only on the middle cluster, and one depending only on the right-hand cluster.

Using the harmonic oscillator notation, the above exponential in (4.3) may be written

$$\langle 0|\exp[\sum_r (B_rb_{1r}-Y_rb_{2r}+\sum_{q\leq r} Q_r^q\cdot a_{rq})]u_{1k}^{\Sigma_r r(b_{1r}^\dagger b_{1r}-b_{2r}^\dagger b_{2r}+\Sigma_{q\leq r}a_r^{q\dagger}\cdot a_{rq})} \exp[\sum_r (\bar{L}_rb_{1r}^\dagger-\bar{M}_rb_{2r}^\dagger+\sum_{q\leq r} \bar{K}_r^qa_{rq})]$$

$$\times\exp[\sum_r\sum_{q\leq r} (a_{r5}^{q\dagger}-a_{r4}^{q\dagger})a_{rqv}\cdot p_{j+1k}] \exp[\sum_r\sum_{q\leq r} \tfrac{1}{4}(a_{r5}^{q\dagger}-a_{r4}^{q\dagger})(a_{rq5}-a_{rq4})(b-\alpha_{j+1k})]$$

$$\times\exp\{\sum_r\sum_{q\leq r}\sum_{p\leq r;\, p\geq 1} (b_{1p}^\dagger-b_{2p}^\dagger)[p_{j+1k}\cdot a_{rv}^q+\tfrac{1}{2}(a_{r4}^q+a_{r5}^q)-\tfrac{1}{4}(a_{r5}^q-a_{r4}^q)(\alpha_{j+1k}-2b-1)]W_{rq}^p\}Z^{\Sigma_{r,q\leq r}a_r^{q\dagger}\cdot a_{rq}}|0\rangle_b$$

$$\times{}_b\langle 0|\exp\{\sum_r\sum_{q\leq r}\sum_{p\leq r;\, p\geq 1} W_r^{qp}[p_{j+1k}\cdot a_{rqv}^\dagger+\tfrac{1}{2}(a_{rq4}^\dagger+a_{rq5}^\dagger)-\tfrac{1}{4}(a_{rq5}^\dagger-a_{rq4}^\dagger)(\alpha_{j+1k}-2b-1)](b_{1p}-b_{2p})\}$$

$$\times\exp[\sum_r\sum_{q\leq r} \tfrac{1}{4}(a_{r5}^{q\dagger}-a_{r4}^{q\dagger})(a_{rq5}-a_{rq4})(b-\alpha_{j+1k})] \exp[\sum_r\sum_{q\leq r} p_{j+1k}\cdot a_{rv}^{q\dagger}(a_{rq5}-a_{rq4})]$$

$$\times\exp[\sum_r (L_rb_{1r}-M_rb_{2r}+\sum_{q\leq r} K_r^q\cdot a_{rq})]u_{1j}^{r\,\Sigma_r(b_{1r}^\dagger-b_{2r}^\dagger b_{2r}+\Sigma_{q\leq r}a_r^{q\dagger}\cdot a_{rq})}$$

$$\times\exp[\sum_r (A_rb_{1r}^\dagger-X_rb_{2r}^\dagger+\sum_{q\leq r} P_r^q\cdot a_{rq})]|0\rangle, \quad (4.4)$$

where A_r, X_r, and P_r^q refer to the right-hand cluster as before, while L_r, M_r, and K_r^q are the equivalents of B_r, Y_r, and Q_r^q when the middle cluster is treated as the left-hand cluster. B_r, Y_r, and Q_r^q refer to the new left-hand cluster, while $\bar{L}_r$, $\bar{M}_r$, and $\bar{K}_r^q$ are the equivalents of A_r, X_r, and P_r^q when the middle cluster is treated as the right-hand cluster. We notice that the projection operator on to the subspace of occupation number space in which the number of particles of type b is zero, $|0\rangle_b\,{}_b\langle 0|$, may be written

$$|0\rangle_b\,{}_b\langle 0| = 1/\Gamma(1-\sum_r (b_{1r}^\dagger b_{1r}-b_{2r}^\dagger b_{2r})). \quad (4.5)$$

Now we may write

$$F_N(\alpha)=\langle 0|\mathcal{G}(p_L,a,b)D(R,s_{1k})\Gamma(p_M,(s_{1j}),a,b;\, a^\dagger,b^\dagger)D(R,s_{1j})\mathcal{G}(p_R,a,b)|0\rangle, \quad (4.6)$$

where

$$\Gamma(p_M,(s_{1j}),a,b;\,a^\dagger,b^\dagger)$$

$$=\int_0^1\prod_{m=j+1}^{k-1}dx_{1m}\frac{1}{J(x_{1j+1}\cdots x_{1k-1})}I_M\exp[\sum_r(\bar{L}_rb_{1r}^\dagger-\bar{M}_rb_{2r}^\dagger+\sum_{q\le r}\bar{K}_r{}^q\cdot a_{rq}{}^\dagger)]$$

$$\times\exp[\sum_r\sum_{q\le r}(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})a_{rqv}\cdot p_{j+1\,k}]\exp\{\sum_r\sum_{q\le r}[\tfrac14(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})(a_{rq5}-a_{rq4})(b-\alpha_{j+1\,k})]\}$$

$$\times\exp\{\sum_r\sum_{p\le r}\sum_{q\le r;\,q\ge1}(b_{1p}{}^\dagger-b_{2p}{}^\dagger)W_{rq}{}^p[p_{j+1\,k}\cdot a_{rv}{}^q+\tfrac12(a_{r4}{}^q+a_{r5}{}^q)-\tfrac14(a_{r5}{}^q-a_{r4}{}^q)(\alpha_{j+1\,k}-2b-1)]\}$$

$$\times Z^{\Sigma_{r,q\le r}a_{rq}{}^\dagger\cdot a_{rq}{}^\dagger}|0\rangle_b\,{}_b\langle0|\exp\{\sum_r\sum_{p\le r}\sum_{q\le r;\,q\ge1}W_r{}^{qp}[p_{j+1\,k}\cdot a_{rqv}{}^\dagger+\tfrac12(a_{rq4}{}^\dagger+a_{rq5}{}^\dagger)-\tfrac14(a_{rq5}{}^\dagger-a_{rq4}{}^\dagger)(\alpha_{j+1\,k}-2b-1)]$$

$$\times(b_{1p}-b_{2p})\}\exp[\sum_r\sum_{q\le r}\tfrac14(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})(a_{rq5}-a_{rq4})(b-\alpha_{j+1\,k})]\exp[\sum_r\sum_{q\le r}p_{j+1\,k}\cdot a_{rv}{}^{q\dagger}(a_{rq5}-a_{rq4})]$$

$$\times\exp\{\sum_r[L_rb_{1r}-M_rb_{2r}+\sum_{q\le r}K_r{}^q\cdot a_{rq}]\},\quad(4.7)$$

where the s_{1j} dependence is removed, if we include the ϵ_r factor in the propagator.

The above expressions (4.6) and (4.7) are to be compared with those for the beta function,[4] viz.,

$$B_N(\alpha)=\langle0|\mathcal{G}(p_L,a,c)D(R,s_{1j})\Gamma(p_M,a,c;\,a^\dagger,c^\dagger)\times D(R,s_{1k})\mathcal{G}(p_R,a,{}^\dagger c^\dagger)|0\rangle,\quad(4.8)$$

where

$$\Gamma(p_M,a,c;\,a^\dagger,c^\dagger)=\int_0^1\prod_{m=j+1}^{k-1}dx_{1m}\frac{1}{\tilde{J}(u_{1j+1}\cdots u_{1N-2})}I_M(u)$$
$$\times\exp\{\sum_r[\bar{K}_r\cdot a_r{}^\dagger+(1-b)^{1/2}c_r{}^\dagger]/\sqrt{r}\}Z^{\Sigma_r a_r{}^\dagger\cdot a_r}|0\rangle_c$$
$$\times{}_c\langle0|\exp\{\sum_r[K_ra_r+(1-b)^{1/2}c_r]/\sqrt{r}\}.\quad(4.9)$$

Introducing our occupation number states between the factors of the above expression for $F_N(\alpha)$ yields

$$F_N(\alpha)=\sum_{\lambda,\kappa}\mathcal{G}(p_L,\lambda,\kappa)D(\lambda,\kappa,s_{1k})\Gamma(p_M,(s_{1j}),\lambda,\kappa;\,\lambda',\kappa')\times D(\lambda',\kappa;\,s_{1j})\mathcal{G}(p_R,\lambda',\kappa'),\quad(4.10)$$

where

$$\Gamma(p_M,(s_{1j}),\lambda,\kappa;\,\lambda',\kappa')=\langle\lambda\kappa|\Gamma(p_M,(s_{1j}),a,b;\,a^\dagger,b^\dagger)|\lambda',\kappa'\rangle\quad(4.11)$$

can be identified as a many-particle amplitude, with two of its external lines corresponding to the excited states (λ,κ) and (λ',κ'), while the rest will be ground-state particles $j+1,\ldots,k$. It is easy to see that by the insertion of such vertices, together with propagators, we may build up amplitudes with a larger number of external lines.

For the special case where $k=j+1$, a similar analysis can be performed. Such an analysis reveals that the expression for the exponential has the same form as for the more general factorization, in which

$$L_r\equiv\frac{1}{\sqrt2}\left(\frac{\epsilon_r}{2}+1\right),\quad M_r=\frac{1}{\sqrt2}\left(\frac{\epsilon_r}{2}+1\right),$$

$$K_r{}^q\equiv\left[\sqrt2W_r{}^{q0}p_{j+1},\frac{-(b+1)}{2\sqrt2}W_r{}^{q0},\frac{-(b-3)}{2\sqrt2}W_r{}^{q0}\right],$$

$$\bar{L}_r\equiv\frac{1}{\sqrt2}\left(\frac{\bar\epsilon_r}{2}+1\right),\quad\bar{M}_r\equiv\frac{1}{\sqrt2}\left(\frac{\bar\epsilon_r}{2}-1\right),\quad(4.12)$$

$$\bar\epsilon_r\equiv(-\alpha_{1j+1}-1)e_r,$$

$$\bar{K}_r{}^q\equiv\left[\sqrt2W_r{}^{q0}p_{j+1},\,\frac{-(b+1)}{2\sqrt2}W_r{}^{q0},\,\frac{-(b-3)}{2\sqrt2}W_r{}^{q0}\right],$$

$$Z\equiv1,\quad p_{j+1j+1}\equiv p_{j+1},\quad\alpha_{j+1j+1}\equiv0.$$

In order to demonstrate this equivalence, we make use of the identity

$$\sum_{r=1}^\infty\frac{u_{1j}{}^ru_{1k}{}^r}{r}=\sum_{r=1}^\infty\sum_{1\le p\le r}\sum_{1\le q\le r}u_{1j}{}^p(-C_r{}^{pq})u_{1j+1}{}^q$$
$$=\sum_{r=1}^\infty\sum_{1\le p\le r}\sum_{1\le q\le r}\tfrac12(A_p-X_p)(B_q-Y_q)\times u_{1j}{}^p(-C_r{}^{pq})u_{1\,j+1}{}^q.\quad(4.13)$$

Thus we may write our three vertex (the equivalent of

Γ, with only one external ground-state particle, $j+1$) as

$$\mathcal{V}(p_{j+1},(s_{1j}),a,b;\,a^\dagger,b^\dagger)$$
$$=\exp\Bigg(\sum_r\bigg\{\frac{1}{\sqrt{2}}(\tfrac{1}{2}\bar{\epsilon}_r+1)b_{1r}{}^\dagger-\frac{1}{\sqrt{2}}(\tfrac{1}{2}\bar{\epsilon}_r-1)b_{2r}{}^\dagger+\sum_{q\leq r}\bigg[-\bigg(\frac{-(b+1)}{2\sqrt{2}}\bigg)W_r{}^{q0}a_{rq4}{}^\dagger$$
$$+\bigg(\frac{-(b-3)}{2\sqrt{2}}\bigg)W_r{}^{q0}a_{rq5}{}^\dagger+\sqrt{2}W_r{}^{q0}p_{j+1}\cdot a_{rqV}{}^\dagger\bigg]\bigg\}\Bigg)\exp[\sum_r\sum_{q\leq r}(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})a_{qrV}\cdot p_{j+1}]$$
$$\times\exp[\sum_r\sum_{q\leq r}\tfrac{1}{4}b(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})(a_{rq5}-a_{rq4})]\exp\{\sum_{r=1}^{\infty}\sum_{p\leq r;\,p\geq 1}\sum_{q\leq r}(b_{1p}{}^\dagger-b_{2p}{}^\dagger)W_{rq}{}^p[p_{j+1}\cdot a_{rV}{}^q$$
$$+\tfrac{1}{2}(a_{r4}{}^q+a_{r5}{}^q)+\tfrac{1}{4}(a_{r5}{}^q-a_{r4}{}^q)(2b+1)]\}\,|0\rangle_b\,{}_b\langle 0|\exp\{\sum_{r=1}^{\infty}\sum_{p\leq r;\,p\geq 1}\sum_{q\leq r}W_r{}^{qp}[p_{j+1}\cdot a_{rqV}{}^\dagger+\tfrac{1}{2}(a_{rq4}{}^\dagger+a_{rq5}{}^\dagger)$$
$$+\tfrac{1}{4}(a_{rq5}{}^\dagger-a_{rq4}{}^\dagger)(2b+1)](b_{1p}-b_{2p})\}\exp[\sum_r\sum_{q\leq r}\tfrac{1}{4}(a_{r5}{}^{q\dagger}-a_{r4}{}^{q\dagger})(a_{rq5}-a_{rq4})b]\exp[\sum_r\sum_{q\leq r}p_{j+1}a_{rV}{}^{q\dagger}(a_{rq5}-a_{rq4})]$$
$$\times\exp\Bigg(\sum_r\bigg\{\frac{1}{\sqrt{2}}(\tfrac{1}{2}\epsilon_r+1)b_{1r}-\frac{1}{\sqrt{2}}(\tfrac{1}{2}\epsilon_r-1)b_{2r}+\sum_{q\leq r}\bigg[-\bigg(\frac{-(b+1)}{2\sqrt{2}}\bigg)W_r{}^{q0}a_{rq4}$$
$$+\bigg(\frac{-(b-3)}{2\sqrt{2}}\bigg)W_r{}^{q0}a_{rq5}+\sqrt{2}W_r{}^{q0}p_{j+1}\cdot a_{rqV}\bigg]\bigg\}\Bigg)\,. \quad (4.14)$$

Using this vertex, we may build up the general vertex Γ, viz.,

$$\Gamma(p_M,(s_{1j}),a,b;\,a^\dagger,b^\dagger)$$
$$=\mathcal{V}(p_{1j+1},(s_{1j}),a,b;\,a^\dagger,b^\dagger)D(R,s_{1j+1})$$
$$\times\mathcal{V}(p_{1j+2},(s_{1j+1}),a,b;\,a^\dagger,b^\dagger)D(R,s_{1j+2})\cdots$$
$$\times D(R,s_{1k-1})\mathcal{V}(p_{1k},(s_{1k-1}),a,b;\,a^\dagger,b^\dagger)\,, \quad (4.15)$$

which is of the form vertex, propagator, vertex, etc., the form one would expect for the Born term of a field theory.

V. ASYMPTOTIC COUNTING OF STATES

First, we note that our approach to the beta-function model would lead to an overestimation of the number of states in the harmonic oscillator formulation because all c_r's have constant coefficients. To obtain the correct counting in this formulation, we must count only one state for each eigenvalue of $\sum_r rc_r{}^\dagger c_r$, whereas the number of states for an eigenvalue n of this operator is the number of partitions of n into integers. Since $A_r-X_r=\sqrt{2}$, $B_r-Y_r=\sqrt{2}$, there is a similar excess of states in the Balachandran model.[6] However, these do not affect the asymptotic counting for this model.

What follows is a rather detailed account of the counting procedure in the asymptotic limit. We include this detail, since our counting disagrees with that used in the literature, for the Gross model.

In order to count the total degeneracy of states in the limit as the energy of the level, and hence n becomes large, we first note that the contribution from A_rB_r and X_rY_r will be relatively unimportant. Hence we approximate R by

$$R\sim\sum_r\sum_{q\leq r}ra_r{}^{q\dagger}\cdot a_{rq}\,.$$

The number of states contributing for a level where $\alpha(s)=n$ will be the number of states (degeneracy) of the eigenvalue n of R.

Since x^R is diagonal in the occupation number basis, and becomes x^n when taken between identical states whose eigenvalue of R is n, thus

$$\mathrm{Tr}x^R=\sum_n d(n)x^n\,, \quad (5.1)$$

where $d(n)$ is the degeneracy of the level $\alpha(s)=n$.

Now

$$\mathrm{Tr}x^R=\prod_r \mathrm{Tr}x^{R_r}\,,$$

where $R_r=r\sum_{q=0}{}^r a_r{}^{q\dagger}\cdot a_{rq}$ (since the R_r's are also diagonal).

$$\mathrm{Tr}x^{R_r}=(\sum_{n=0}^{\infty}x^{rn})^{6(r+1)}$$
$$=\bigg(\frac{1}{1-x^r}\bigg)^{6(r+1)}\,.$$

[6] It is possible that our modes connected with A_rB_r and $-X_rY_r$ in the harmonic oscillator formalism, might have been able to be handled in a manner similar to that by which Frye, Susskind, and Amati *et al.* (Ref. 4) have handled the $1-b$ factor of the beta-function model, hence avoiding the introduction of excess modes.

Therefore,

$$\mathrm{Tr}x^{R}=G^{6}(x)\,, \tag{5.2}$$

where

$$G(x)=F(x)H(x) \tag{5.3}$$

and

$$F(x)=\prod_{r}(1-x^{r})^{-1},\quad H(x)=\prod_{r}(1-x^{r})^{-r}. \tag{5.4}$$

Following the techniques used by Ayoub[7] for the counting series of the normal partition function, we proceed as follows:

$$H(x)=\prod_{k}(1-x^{k})^{-k}.$$

Hence

$$\begin{aligned}\ln H(x)&=\sum_{k}k\ln\left(\frac{1}{1-x^{k}}\right)\\&=\sum_{k,l}\frac{k}{l}x^{kl}\\&=\sum_{l,k}\frac{k}{l}(x^{l})^{k}\\&=\sum_{l}\frac{1}{l}\frac{x^{l}}{(1-x^{l})^{2}}.\end{aligned}$$

Now, $1-x^{l}\leq(1-x)l$ for $x\in[0,1]$; therefore,

$$\frac{1}{(1-x)^{2}}\sum\frac{x^{l}}{l^{3}}\leq\ln H(x)\quad\text{for } x\in[0,1].$$

However,

$$\ln H(x)=\sum_{l}\frac{1}{l}\frac{1}{(x^{-l/2}-x^{l/2})^{2}}.$$

Put $x=e^{-2\phi}$, $0\leq x\leq1\Rightarrow2\phi\geq0$, and $2\phi=0\Rightarrow x=1$. Therefore,

$$\begin{aligned}\ln H(x)&=\frac{x}{(1-x)^{2}}\sum_{l}\frac{1}{l}\frac{(x^{-1/2}-x^{1/2})^{2}}{(x^{-l/2}-x^{l/2})^{2}}\\&=\frac{x}{(1-x)^{2}}\sum_{l}\frac{1}{l}\frac{\sinh^{2}\phi}{\sinh^{2}l\phi}.\end{aligned}$$

Consider

$$y(\phi)=\sinh l\phi-l\sinh\phi\,,$$

$$\frac{dy}{d\phi}=l(\cosh l\phi-\cosh\phi)\geq0\quad\text{for}\quad\phi\geq0,\quad l\geq1\,,$$

and

$$y(0)=0.$$

Hence $y(\phi)\geq0$ for $\phi\geq0$, $l\geq1$, i.e., $\sinh l\phi\geq l\sinh\phi$. Therefore,

$$\ln H(x)\leq\frac{x}{(1-x)^{2}}\sum_{l}\frac{1}{l^{3}}$$

and

$$\begin{aligned}\frac{1}{(1-x)^{2}}\sum_{l}\frac{x^{l}}{l^{3}}&\leq\ln H(x)\\&\leq\frac{x}{(1-x)^{2}}\sum_{l}\frac{1}{l^{3}}\quad\text{for } x\in[0,1].\end{aligned} \tag{5.5}$$

Hence as $x\uparrow1$, i.e., $x\rightarrow1$ from below,

$$\ln H(x)\underset{x\uparrow1}{\sim}\frac{1}{(1-x)^{2}}\sum_{l}\frac{1}{l^{3}}=\zeta(3)\frac{1}{(1-x)^{2}}.$$

Hence

$$H(x)\underset{x\uparrow1}{\sim}\exp\left[\zeta(3)\frac{1}{(1-x)^{2}}\right]. \tag{5.6}$$

To determine the asymptotic value of $d(n)$, we observe that

$$G^{6}(x)\underset{x\uparrow1}{\sim}H^{6}(x)\underset{x\uparrow1}{\sim}\exp\left[6\zeta(3)\frac{1}{(1-x)^{2}}\right]$$

and use the method of steepest descent[8] (saddle-point and integration) to calculate $d(n)$. Therefore,

$$d(n)=\frac{1}{2\pi i}\oint\frac{G^{6}(z)}{z^{n+1}}dz\,,$$

where the contour of integration surrounds the origin and has radius from the origin <1. Thus

$$d(n)=\frac{1}{2\pi i}\oint\exp[\ln G^{6}(z)-(n+1)\ln z]dz.$$

We wish to find the saddle point of the exponent, which is given by

$$\frac{d}{dz}[\ln G^{6}(z)-(n+1)\ln z]=6\left(\frac{d}{dz}\ln G(z)\right)-\frac{(n+1)}{z}=0.$$

Thus

$$6\frac{d}{dz_{n}}\ln G(z_{n})=\frac{(n+1)}{z_{n}}\,,$$

and since it can be seen (at least in the limit of large n) that the second derivative is positive, then we have a saddle point (on the real axis) at $z=z_{n}$.

As $n\rightarrow\infty$, $z_{n}\rightarrow1$ and we may use our asymptotic form, since the equation has a real solution. Therefore,

$$6\frac{d}{dz_{n}}\zeta(3)\frac{1}{(1-z_{n})^{2}}\underset{z_{n}\uparrow1;\ n\rightarrow\infty}{\sim}n\frac{1}{z_{n}}\,,$$

[7] R. Ayoub, *An Introduction to the Analytic Theory of Numbers* (American Mathematical Society, Providence, R. I., 1963), Chap. III; G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. **17**, 75 (1917).

[8] See, for example, J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Benjamin, New York, 1964), Chap. 3.6.

i.e.,

$$12\zeta(3)\frac{1}{(1-z_n)^3}\sim\frac{n}{z_n}\sim n.$$

Therefore,

$$1-z_n\sim\left[\frac{12\xi(3)}{n}\right]^{1/3},$$

and therefore

$$f(z_n)\equiv 6\ln G(z_n)-(n+1)\ln z_n,$$
$$\underset{n\to\infty}{\sim}\tfrac{3}{2}[12\xi(3)]^{1/3}n^{2/3},$$

whence

$$d(n)\underset{n\to\infty}{\sim}e^{f(z_n)}\frac{1}{2\pi i}\oint\exp[\tfrac{1}{2}f''(z_n)(z-z_n)^2+\cdots]dz$$
$$\sim e^{f(z_n)}\frac{1}{2\pi}\int_{-\infty}^{\infty}dy[-\tfrac{1}{2}f''(z_n)(y-y_n)^2]$$
$$\underset{n\to\infty}{\sim}\frac{e^{f(z_n)}}{\sqrt{2\pi f''(z_n)}},$$

since $f''(z_n)>0$, and $f''(z_n)\to\infty$ as $n\to\infty$. Thus we have

$$d(n)\underset{n\to\infty}{\sim}\exp\{\tfrac{3}{2}[12\xi(3)]^{1/3}n^{2/3}+O(n^{1/3})\},\quad(5.7)$$

compared with

$$d(n)\underset{n\to\infty}{\sim}\exp\left[\frac{4\pi}{\sqrt{6}}n^{1/2}+O(\ln n)\right]\quad(5.8)$$

for the beta-function model.[4]

In the simplest example of the Gross model[9] [Eq. (3.11) of Gross's paper], it is easy to see by inspection that the counting is similar; in fact, for this example

$$d(n)\underset{n\to\infty}{\sim}\exp\{3[\tfrac{1}{2}\xi(3)]^{1/3}n^{2/3}\}.\quad(5.9)$$

The more general forms of the Gross model, treated for example by Frampton and Rivers,[10] are of two types depending on whether a certain series they define has a finite or infinite number of terms. For the finite case, we will still have $d(n)\sim\exp(\text{const } n^{2/3})$ for the infinite case $d(n)\sim\exp(\text{const } n^{3/4})$. The fact that a state of angular momentum l is counted $2l+1$ times is unimportant, since it merely adds a term of order $\ln n$ to the exponent, and so does not affect the result.

It is obvious from Eq. (1.1) that there exist classes of ω's which admit continuous and even analytic deformation of the x_{ij}'s into the u_{ij}'s. Under such a deformation the integral representation for the Balachandran function becomes that for the beta function in a smooth limit to within a constant factor. We have studied the behavior of the harmonic oscillator states in this limit and find, after performing a unitary transformation, that the states separate into two classes. One class of states is in one-to-one correspondence with the beta-function states found in the original Fubini-Gordon-Veneziano treatment. The other class of states is not present in the beta-function model. Most of these excess states decouple from the beta-function states in the smooth limit mentioned above. The others make an unobservable contribution, analogous to that of allowed timelike and longitudinal photons in quantum electrodynamics.[11,12] On the subspace of beta-function states, the limiting forms of all operators assume the beta-function form.

VI. CONCLUSION

We have shown that the Balachandran N-point functions possess consistent factorization and level structure properties, and that a harmonic oscillator notation can be introduced to describe these properties. This establishes that the Balachandran functions constitute a fully viable alternative to the Veneziano N-point functions on which much current theoretical effort and hopes have been placed.

The Balachandran model is, in general, inequivalent to either the Veneziano model, or the Veneziano model with satellites, although it has been shown that the Balachandran model reduces to the Veneziano model in an appropriate limit.

The main difference between the models lies in the asymptotic degeneracy of levels. This has been found to be $\exp(\text{const } n^{2/3})$ for the Balachandran model compared to $\exp(\text{const } n^{1/2})$ for the Veneziano model without satellites. For two types of Veneziano models with satellites studied by Gross,[9,10] the asymptotic degeneracy of states is either $\exp(\text{const } n^{2/3})$ or $\exp(\text{const } n^{3/4})$.

The Balachandran model is plagued by negative-metric ghosts, similar to the beta-function model. The relations which exist between modes because of certain symmetries of the amplitude (the equivalents of the Ward identities of the N-point beta-function model[3]), and which might eliminate some of the ghosts, have not been investigated.

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