# Level Degeneracy of the Generalized $N$-Particle Dual Amplitude with Satellites*

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We show that the level degeneracy at $\alpha(s)=n$ for the general $N$-point function with satellites is $d_n \sim e^{n\gamma}$ as $n\rightarrow\infty$ where $\gamma=1-1/(2k+1)$, $k=1, 2, \ldots$. The form of the single loop constructed from such amplitudes is briefly discussed.

WHEN the Venziano model[1] was originally introduced as a specific example of a dual crossing-symmetric four-point function, it was not apparent whether there was any fundamental difference between a single-term amplitude and a many-term amplitude ($B$ function with satellites).

After the general $N$-point function with satellites had been constructed,[2] it became apparent that the presence of satellite terms modified the properties of the $N$-point function considerably.

Examination of the *simplest* class of $N$-point functions with satellites showed

(i) that the mass degeneracy is changed [from $d_n \sim \exp n^{1/2}$ to $d_n \sim \exp n^{2/3}$] [3];

(ii) that the invariance of the amplitude under the one-parameter gauge group of projective transformations was lost,[4] depriving the naively factorized amplitude of ghost-eliminating mechanisms;

(iii) that it was impossible to eliminate the essential singularity in the factorized planar loop by a particular choice of satellite terms.[5]

In this paper we extend the analysis to the *general* class of $N$-point functions with satellites discussed in Ref. 2.

Our main result is that the $N$-point scalar dual amplitudes with satellites $B_N$ can be classified into sets $\{B_N{}^k\}$ $k=1, 2, \ldots$ (as was essentially done in Ref. 2) such that the level degeneracy at $\alpha(s)=n$ for the amplitude $B_N{}^k$ is

$$d_n{}^{(k)} \sim \exp(b_k n^{\gamma_k}) \quad \text{as} \quad n\rightarrow\infty\,, \tag{1}$$

where

$$\gamma_k = 1-1/(2k+1), \quad k\geqslant 1\,. \tag{2}$$

We examine the $N$-point function with satellites using the terminology of Ref. 2. The $N$-point scalar amplitude has the form

$$\begin{aligned} B_N(p_i) &= \int (\prod_i dx_i)\tilde{\phi}_N(x_i,p_i) \\ &= \int (\prod_i dx_i)\phi_N(x_i,p_i) \exp F_N(x_i)\,, \end{aligned} \tag{3}$$

where $F_N\equiv 0$ corresponds to the usual $N$-point function.

The satellite function $F_N(x_i)$ has the form

$$F_N(x_i) = \sum_k F_N{}^k(x_i)\,, \tag{4}$$

where

$$F_N{}^k(x_i) = \sum f_N{}^k(u_{l,l+1},\ldots,u_{l-2,l-1})\,, \tag{5}$$

the sum in Eq. (5) being taken over the $N$ sets of $u$'s at the vertices of the dual polygon. Only a finite number of the $f_N{}^k$ are nonzero, and they have the form

$$\begin{aligned} f_N{}^k(x_1,\ldots,x_{N-3}) &= \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant N-3} x_{i_1}x_{i_1+1}\cdots x_{i_k}(1-x_{i_1})(1-x_{i_2})\cdots \\ &\qquad \times(1-x_{i_k})\tilde{f}_{k+3}(x_{i_1},x_{i_2},\ldots,x_{i_k})\,. \end{aligned} \tag{6}$$

Let us factorize the $N=(r+s+4)$-point function given by Fig. 1 in the $s=(\sum p_i)^2=(\sum q_j)^2$ channel. If $z$ is the variable associated with this channel (Fig. 2), we expand $F_N{}^k$ as

$$F_N{}^k = \sum_{n=0}^{\infty} z^n D_n{}^k(x,y)\,, \tag{7}$$

associating $x_i$ with the incoming momenta $p_i$, $y_j$ with the outgoing momenta $q_j$. By construction,[2] $D_n{}^k$ is a finite sum of factorized terms. Suppose $D_n{}^k$ can be factorized in a *symmetric* way in the *least* number of factored terms as

$$D_n{}^k(x,y) = \sum_{\alpha=1}^{\delta_n{}^k} L_{n\alpha}{}^k(x)R_{n\alpha}{}^k(y)\,. \tag{8}$$

Then, from Eq. (4),

$$\begin{aligned} F_N &= \sum_{n=0}^{\infty} z^n \sum_k D_n{}^k(x,y) \\ &= \sum_{n=0}^{\infty} z^n \sum_{\alpha=1}^{\delta_n} L_{n\alpha}(x)R_{n\alpha}(y) \end{aligned} \tag{9}$$

in an obvious notation, where $\delta_n=\sum_k \delta_n{}^k<\infty$.

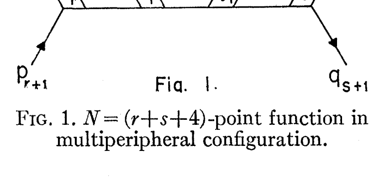


FIG. 1. $N=(r+s+4)$-point function in multiperipheral configuration.

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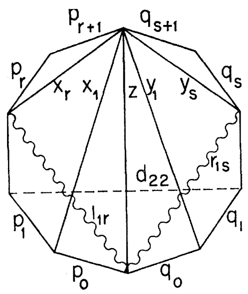

FIG. 2. Momentum-space dual diagram of Fig. 1.

Inserting Eq. (9) in Eq. (3) gives

$$B_N=\int[\prod_i dx_i\tilde{\phi}_{r+3}(x_i,p_i)][\prod_j dy_j\tilde{\phi}_{s+3}(y_j,q_j)]$$
$$\times\int_0^1 dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}$$
$$\times\exp\sum_{n=1}^{\infty} z^n\left[\frac{1}{n}P^{(n)}Q^{(n)}+\sum_\alpha L_{n\alpha}(x)R_{n\alpha}(y)\right], \quad (10)$$

in the usual notation.[6]

We see from Eq. (9) that we can perform the single factorization by introducing harmonic-oscillator operators $a_\mu{}^{(s)}$ ($\mu=0,1,2,3$), $A_\alpha{}^{(r)}$ ($\alpha=1,2,\ldots,\delta_r$) satisfying

$$[a_\mu{}^{(r)},a_\nu{}^{\dagger(s)}]=-\delta_{rs}g_{\mu\nu},\quad [A_\alpha{}^{(r)},A_\beta{}^{\dagger(s)}]=\delta_{rs}\eta_{\alpha\beta}, \quad (11)$$

with all other commutators zero. The $a^{(r)}$ are the usual four-dimensional operators[7] that couple to the external momenta, and the $A^{(r)}$ couple to the additional internal degrees of freedom. The metric $\eta_{\alpha\beta}$ is determined by the details of $F_N$.

Since single factorization is sufficient to excite all modes, Eq. (11) implies an internal Hamiltonian

$$H=-\sum_{r=1}^{\infty} r(a_\mu{}^{\dagger(r)}g_{\mu\nu}a_\nu{}^{(r)}-A_\alpha{}^{\dagger(r)}\eta_{\alpha\beta}A_\beta{}^{(r)}). \quad (12)$$

If $d_n$ is the degeneracy of $H$ with eigenvalue $n$, we have

$$P(x)=\sum_{n=0}^{\infty} d_n x^n=\mathrm{Tr}x^H=\prod_{r=1}^{\infty}(1-x^r)^{-4-\delta_r}. \quad (13)$$

We now compute $\delta_n{}^k$ [Eq. (8)] for large $n$.

(a) $k=1$.[5] Expand $f_4(x)$ as

$$x(1-x)\tilde{f}_4(x)=\sum_{k\geqslant 1} e_k(1-x)^k, \quad (14)$$

where $\sum e_k=0$. The main contribution to $\delta_n{}^1$ comes from the sum

$$F_N{}'^1=\sum_{1\leqslant p\leqslant r+1;\ 1\leqslant q\leqslant s+1} d_{pq}(1-d_{pq})\tilde{f}_4(d_{pq}), \quad (15)$$

where $d_{pq}$ are the lines intersecting $z$. The $d_{pq}$ satisfy

$$(1-d_{pq})^k=\sum_{st\leqslant r=k}^{\infty} C_r{}^{st}(k)x_p{}^s(\rho_{p-1}z\sigma_{q-1})^r y_q{}^t, \quad (16)$$

where $\rho_p=x_1x_2\cdots x_p$, $\sigma_q=y_1y_2\cdots y_q$, and

$$C_r{}^{st}(k)=\sum_{p+q=r-k}(-1)^{s+t}\binom{-k}{p}\binom{-k}{q}\times\binom{k}{s-p}\binom{k}{t-q}. \quad (17)$$

Inserting Eq. (14) into Eq. (15), we have

$$F_N{}'^1=\sum_{r=1}^{\infty} z^r\sum_{st\leqslant r=1}^{\infty} L_{rs}(x)C_r{}^{st}R_{rt}(y), \quad (18)$$

where

$$C_r{}^{st}=\sum_{k\geqslant 1} C_r{}^{st}(k)e_k \quad (19)$$

and

$$L_{rs}(x)=\sum_{p=1}^{r+1} x_p{}^s\rho_{p-1}{}^r,\quad R_{rt}(y)=\sum_{q=1}^{s+1} y_q{}^t\sigma_{q-1}{}^r. \quad (20)$$

Diagonalizing $C_r$, we write

$$F_N{}'^1=\sum_{r=1}^{\infty} z^r\sum_{\alpha=0}^{r} L_{r\alpha}{}'(x)\lambda_{\alpha\beta}{}^r R_{r\beta}{}'(y), \quad (21)$$

where $\lambda_{\alpha\beta}{}^r$ are the signs of the eigenvalues of $C_r$.

The $z$-dependent terms in $F_N{}^1$ omitted in $F_N{}'^1$ add only a constant number of factorized terms to each mode. Thus

$$\delta_r{}^1\sim r \quad \text{as} \quad r\to\infty. \quad (22)$$

(b) $k\geqslant 1$. One of the terms giving the asymptotic forms of $\delta_n{}^k$ is $F_N{}'^k=F_N{}^{(1)k}+F_N{}^{(2)k}$, where

$$F_N{}^{(1)k}=\sum_{1\leqslant p\leqslant r+1}\ \sum_{1\leqslant q_1<\cdots<q_{k-1}\leqslant s+1} l_{1p}(1-l_{1p})d_{p1}\cdots$$
$$\times d_{pq_{k-1}}(1-d_{pq_1})(1-d_{pq_2})\cdots$$
$$\times(1-d_{pq_{k-1}})\tilde{f}_{k+3}(l_{1p},d_{pq_1},\ldots,d_{pq_{k-1}}), \quad (23)$$

$$F_N{}^{(2)k}=\sum_{1\leqslant q\leqslant s+1}\ \sum_{1\leqslant p_1<\cdots<p_{k-1}\leqslant r+1} r_{1q}(1-r_{1q})d_{1q}\cdots$$
$$\times d_{p_{k-1}q}(1-d_{p_1q})(1-d_{p_2q})\cdots$$
$$\times(1-d_{p_{k-1}q})\tilde{f}_{k+3}(r_{1q},d_{p_1q},\ldots,d_{p_{k-1}q}). \quad (24)$$

The $l_{1p}$, $r_{1q}$ (Fig. 2) satisfy

$$(1-l_{1p})^k=\sum_{st\leqslant r=k}^{\infty} C_r{}^{st}(k)x_p{}^s\rho_{p-1}{}^r z^t, \quad (25)$$

with a similar expression for $r_{1q}$.

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[7] Y. Nambu, University of Chicago Report No. EFI 69-44 (unpublished); S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969).

Expand $f_{k+3}$ as

$$x_1(1-x_2)\cdots(1-x_k)x_k\tilde{f}_{k+3}(x_1,x_2,\ldots,x_k)$$
$$=\sum_{l_1\geqslant 1;\; i=1,2,\ldots,k} e_{l_1l_2\cdots l_k}(1-x_1)^{l_1}\cdots(1-x_k)^{l_k}, \quad (26)$$

where $\sum_{l_1\geqslant 1} e_{l_1\cdots l_k}=\sum_{l_k\geqslant 1} e_{l_1\cdots l_k}=0$.

Then, from Eqs. (16), (25), and

$$d_{p1}\cdots d_{pq_{k-1}-1}=1-\sum_{st\leqslant r=1}^{\infty} C_r^{st}(1)x_p{}^s\rho_{p-1}{}^r z^r\sigma_{q_{k-1}-1}{}^t, \quad (27)$$

we obtain

$$F_N{}^{(1)k}=\sum_{1\leqslant p\leqslant s+1}\;\sum_{1\leqslant q_1<\cdots<q_{k-1}\leqslant s+1}\Big\{\sum_{a_i,b_i\leqslant c_i=1;\; i=0,1,\ldots,k-1}^{\infty} E_{c_0,\ldots,c_{k-1}}{}^{a_0,\ldots,a_{k-1}:\; b_0,\ldots,b_{k-1}} x_p{}^{a+a_0}\rho_{p-1}{}^{c+c_0}$$
$$\times z^{c+b_0}\prod_{i=1}^{k-1}\sigma_{q_i-1}{}^{c_i}y_{q_i}{}^{b_i}\Big[1-\sum_{a_kb_k\leqslant c_k=1}^{\infty} C_{c_k}{}^{a_kb_k}(1)x_p{}^{a_k}\rho_{p-1}{}^{c_k}z^{c_k}\sigma_{q_{k-1}-1}{}^{b_k}\Big]\Big\}, \quad (28)$$

where

$$E_{c_0,\ldots,c_{k-1}}{}^{a_0,\ldots,a_{k-1}:\; b_0,\ldots,b_{k-1}}=\sum_{l_i\geqslant 0;\; i=0,1,\ldots,k-1} e_{l_0\cdots l_{k-1}}C_{c_0}{}^{a_0b_0}(l_0)C_{c_1}{}^{a_1b_1}(l_1)\cdots C_{c_{k-1}}{}^{a_{k-1};\; b_{k-1}}(l_{k-1}), \quad (29)$$

and where

$$a=\sum_{i=1}^{k-1}a_k \quad\text{and}\quad c=\sum_{i=1}^{k-1}c_k.$$

In an obvious notation, we can write Eq. (28) as

$$F_N{}^{(1)k}=\sum_{b_0}\sum_{b_i\leqslant c_i=1;\; i=1,\ldots,k-1}^{\infty} L_0{}^{c_1,\ldots,c_{k-1}:\; b_0,\ldots,b_{k-1}}(x)z^{c+b_0}R_2{}^{c_1,\ldots,c_{k-1}:\; b_1,\ldots,b_{k-1}}(y)$$
$$-\sum_{b_0}\sum_{b_i\leqslant c_i=1;\; i=1,\ldots,k} L_1{}^{c_1,\ldots,c_k:\; b_0,\ldots,b_k}(x)z^{c+c_k+b_0}R_2{}^{c_1,\ldots,c_{k-1}+b_k:\; b_1,\ldots,b_{k-1}}(y), \quad (30)$$

where

$$R_2{}^{c_1,\ldots,c_{k-1}:\; b_1,\ldots,b_{k-1}}(y)=\sum_{1\leqslant q_1<\cdots<q_{k-1}\leqslant N-3}\;\prod_{i=1}^{k-1}\sigma_{q_i-1}{}^{c_i}y_{q_i}{}^{b_i},$$
$$L_0{}^{c_1,\ldots,c_{k-1}:\; b_0,\ldots,b_{k-1}}(x)=\sum_{1\leqslant p\leqslant s+1}\sum_{c_0}\sum_{a_i\leqslant c_i;\; i=0,\ldots,k-1} E_{c_0,\ldots,c_{k-1}}{}^{a_0,\ldots,a_{k-1}:\; b_0,\ldots,b_{k-1}}x_p{}^{a+a_0}\rho_{p-1}{}^{c+c_0}, \quad (31)$$
$$L_1{}^{c_1,\ldots,c_k:\; b_0,\ldots,b_k}(x)=\sum_{1\leqslant p\leqslant s+1}\sum_{c_0}\sum_{a_i\leqslant c_i;\; i=0,\ldots,k-1} C_{c_k}{}^{a_kb_k}(1)$$
$$\times E_{c_0,\ldots,c_{k-1}}{}^{a_0,\ldots,a_{k-1}:\; b_0,\ldots,b_{k-1}}x_p{}^{a+a_0+a_k}\rho_{p-1}{}^{c+c_0+c_k}.$$

Equation (30) can be written formally as

$$F_N{}^{(1)k}=\sum_{n=k-1}^{\infty}z^n\sum_{\alpha=1}^{\delta_n{}^k(0)}L_{0\alpha}{}^{(n)}(x)R_{2\alpha}{}'^{(n)}(y)-\sum_{n=k}^{\infty}z^n\sum_{\alpha=1}^{\delta_n{}^k(1)}L_{1\alpha}{}^{(n)}(x)R_{2\alpha}{}^{(n)}(y). \quad (32)$$

$F_N{}'^k$ [Eq. (23)] then has the factorized form

$$F_N{}'^k=\sum_{n=k-1}^{\infty}z^n\sum_{\alpha=1}^{\delta_k{}^n(0)}[L_{0\alpha}{}^{(n)}(x)R_{2\alpha}{}'^{(n)}(y)+L_{2\alpha}{}'^{(n)}(x)R_{0\alpha}{}^{(n)}(y)]$$
$$-\sum_{n=k}^{\infty}z^n\sum_{\alpha=1}^{\delta_k{}^n(1)}[L_{1\alpha}{}^{(n)}(x)R_{2\alpha}{}^{(n)}(y)+L_{2\alpha}{}^{(n)}(x)R_{1\alpha}{}^{(n)}(y)], \quad (33)$$

where a similar expansion has been performed for $F_N{}^{(2)k}$.

To factorize $F_N{}'^k$ symmetrically, we have to make the substitution

$$-(L_1R_2+L_2R_1)=L_1R_1+L_2R_2-(L_1+L_2)(R_1+R_2). \quad (34)$$

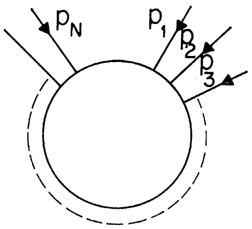


FIG. 3. $N$-particle planar loop.

This gives rise to a large number of components with negative metric which will give ghosts in the level degeneracy.

As in Ref. 4, the spectral function $e^{F_N}$ will not have any simple projective transformation properties, and there will be no mechanism for eliminating these ghosts (i.e., there will be no spurious states).

We observe immediately that $\delta_n{}^k(1) > \delta_n{}^k(0)$, so we restrict ourselves to examining the second term on the right-hand side of Eq. (30). In order that $\delta_n{}^k(1)$ be the *least* number of factored terms, there must be no linear dependences between the $R_\alpha{}^{(n)}$ or between the $L_\alpha{}^{(n)}$. That this is the case for the $R_\alpha{}^{(n)}(y)$ follows directly from Eq. (31). It is also apparent that for general values of the $e_{l_0\cdots l_{k-1}}$ the $L_{1\alpha}{}^{(n)}(x)$ will be linearly independent.[8]

Fixing the power of $z$ at $c+c_k+b_k=n$ in the second term of Eq. (30), we have

$$\delta_n{}^k(1) \sim b_k{}^{(1)} n^{2k-1}. \tag{35}$$

Since the other $z$-dependent terms in $F_N{}^k$ give similar power behavior [if they contain $l_{1p}$ and $d_{pq_i}$, $i=1, 2, \ldots$] or less, we have

$$\delta_n{}^k \sim b_k n^{2k-1} \quad \text{as} \quad n\to\infty,\ k>1. \tag{36}$$

Let $\{B_N{}^k\}$ be the set of $N$-point functions for which $f_N{}^k \neq 0$, $f_N{}^l \equiv 0$, $l>k$ [i.e., the simplest set of $N$-point functions $(N>k+3)$ compatible with arbitrary $(k+3)$- and lesser-point functions]. Then, for a particular $B_N{}^k$

$$\delta_n{}^{(k)} = \sum_l \delta_n{}^l \sim b_k n^{2k-1}, \quad k\geqslant 1 \tag{37}$$

where the sum is over the nonzero $\delta_n{}^l$, $l\leqslant k$.

The level degeneracy of $B_N{}^k$ at $\alpha(s)=n$ is $d_n{}^{(k)}$, where[9]

$$P^{(k)}(x) = \sum_{n=0}^{\infty} d_n{}^{(k)} x^n = \prod_{r=1}^{\infty} (1-x^r)^{-\delta_r{}^{(k)}-4}. \tag{38}$$

In Ref. 4 it has been shown that

$$\ln P^{(1)}(x) \sim b_1 \zeta(3)(1-x)^{-2} \quad \text{as} \quad x\to 1-. \tag{39}$$

By a straightforward generalization of this method, it can be shown that

$$\ln P^{(k)}(x) \sim b_k \Gamma(2k)\zeta(2k+1)(1-x)^{-2k} \quad \text{as} \quad x\to 1-. \tag{40}$$

We can evaluate the asymptotic form $d_n{}^{(k)}$ by applying the saddle-point method to

$$d_n{}^{(k)} = \frac{1}{2\pi i}\oint_C \frac{P^{(k)}(z)}{z^{n+1}}, \tag{41}$$

where the contour $C$ enclosing the origin lies within $|z|<1$. This gives

$$d_n{}^{(k)} \sim \exp\left[\left(\frac{2k+1}{2k}\right) \times [b_k\Gamma(2k)\zeta(2k+1)]^{1/(2k+1)} n^{2k/(2k+1)}\right]. \tag{42}$$

This is the result quoted in Eq. (1).

We conclude with a brief discussion of the single loop constructed from the $N$-point function with satellites.

First we consider the single planar loop which has been taken to give some guide to the magnitude of the unitarity corrections to the $N$-point function.

Because the trace factors into the usual trace (in the absence of satellites) and a momentum-independent term that depends on the satellite coefficients, the planar loop (Fig. 3) has the form

$$L_N(p_i) = \int (\prod_i dx_i) L_N(x_i,p_i)\Gamma_N(x_i), \tag{43}$$

where $\Gamma_N(x_i)=1$ corresponds to the absence of satellites. Near $w=x_1x_2\cdots x_N=1$,[10,11]

$$L_N(x_i,p_i) \sim P^{(0)}(w) = \prod_{r=1}^{\infty} (1-w^r)^{-4}$$

and

$$\Gamma_N(x_i) \sim P^{(k)}(w). \tag{44}$$

To examine this essential singularity, it is convenient to make the change of variables[11]

$$q = \exp(2\pi^2/\ln w). \tag{45}$$

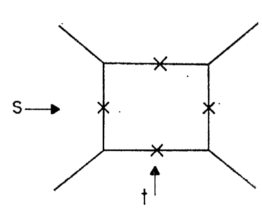


FIG. 4. An orientable nonplanar four-particle loop.

[8] One linear relation between the $L_{1\alpha}{}^{(n)}(x)$ implies a denumerable infinity of simultaneous equations to be satisfied by $\delta_n{}^k(1)$ coefficients. For general $e_{l_0\cdots l_{k-1}}$ these equations will not all be satisfied.

[9] We have omitted the small effect of the term $(1-x)^{\alpha(0)-1}$ in the propagator [Eq. (10)].

[10] D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters 2, 399 (1970).

[11] D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwartz, Phys. Rev. D 2, 697 (1970).

Near $q=0$,

$$P^{(0)}\sim q^{-1/3},$$
$$P^{(k)}\sim\exp[C_k \ln^{2k}q]. \tag{46}$$

In the absence of satellites it is possible to construct a dual counterterm[12] because of the power behavior of $P^{(0)}$. For $k\geqslant 1$ the singularity of $P^{(k)}$ dominates that of $P^{(0)}$.

Although we are unable to show that counterterms do not exist, we consider their existence to be highly unlikely.

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[13] P. G. O. Freund and R. J. Rivers, Phys. Letters **29B**, 510 (1969); P. G. O. Freund, Nuovo Cimento Letters **4**, 147 (1970).

Finally, we consider the single nonplanar orientable loop of Fig. 4, which gives the unrenormalized two-Reggeon cuts and may give some indication of the nature of the Pomeranchuk singularity.[13] In the absence of satellites the loop converges for $u<-\frac{4}{3}$ because of the power behavior of $P^{(0)}$. With satellites the singularity of $P^{(k)}$ [Eq. (46)] would cause the loop to diverge for all values of $u$. As a corollary, the $s^{1/3}$ behavior[11,12] associated with the branchpoints will be destroyed.

In summary, if the one-loop diagrams have any meaning at all, it is only within the restricted context of the $N$-point function without satellites. Increasing the level degeneracy as in Eq. (1) by including satellites seems to remove any chance of renormalization (in the planar loop) or convergence (in the nonplanar loop).

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# Generalizations of the Dirac Representation

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Representation spaces of the relativistic symmetry are investigated, which are "infinite-dimensional" generalizations of the space of solutions of the Dirac equation. The representations are extended by the discrete operators $C$, $P$, and $T$. Application of these representations to the description of baryons and mesons is discussed.

## I. INTRODUCTION

Judging from the experience of the past few years, it appears that Dirac's $\gamma$'s are only some special cases of more general quantities with physical significance. As is well known, the usual $\gamma_\mu$ and $\sigma_{\mu\nu}$ are an irreducible matrix representation of the generators of $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$,[1] and the space of solutions of the Dirac equation is an irreducible representation space of the relativistic symmetry[2] $\mathfrak{S}=\mathcal{P}_{P_\mu,L_{\mu\nu}}\vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$,[1,3] where $\vdash$ denotes semidirect sum. In the connection with infinite multiplets the applicability of several unitary representations of $SO(3,2)$ or $\mathfrak{S}$ has been investigated, e.g., the four Majorana representations[4] or the oscillatorlike representations of $SO(4,2)$,[5,6] which are in fact singleton representations[7] of $SO(3,2)$.[8]

The Dirac representation of $\mathfrak{S}$ has a great advantage as compared to these representations: It is not only an irreducible representation of $\mathfrak{S}$, but it is also an irreducible representation of the full quantum-mechanical Poincaré group, including charge conjugation, and also an irreducible representation of $\mathfrak{S}$ extended by the discrete operations, space inversion $U_P$, time inversion $A_T$, and charge conjugation $U_C$. In analogy to the notation for the Poincaré group, we want to call $\mathfrak{S}$ extended by $U_P$, $A_T$, and $U_C$ the full relativistic symmetry $\mathfrak{S}^F$. The infinite-dimensional irreducible representations of $\mathfrak{S}$ considered so far are not irreducible representations of $\mathfrak{S}^F$; the discrete operations will transform out of an irreducible representation space. Thus the following question arises: Are there infinite-dimensional generalizations of the Dirac representation, i.e., are there infinite-dimensional representations of the full relativistic symmetry that remain irreducible when restricted to $\mathfrak{S}$? The answer to this question will be the subject of the present paper. It will turn out that there are two classes of infinite-

[1] The subscripts $X_i$ on the symbol for the group $G_{X_i}$ indicate that $X_i$ are the generators of $G$. This notation is necessary to allow us to distinguish between mathematically isomorphic groups, which have different physical observables.

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[8] These are the singleton representations with the multiplicity pattern given in Figs. 7-5 and 7-14 of Ref. 7(b). For $j_{\min}=0$ the $SO(4,2)$ irreducible representation reduces to a sum of two inequivalent irreducible representations of $SO(3,2)$ with $n_{\min}=1$ and $n_{\min}=2$.