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Recalculation of the Crossed-Graph Contribution to the Fourth-Order Lamb Shift*

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This paper reports an analytic calculation of the "crossed"-graph contribution to the fourth-order radiative correction of the slope of the Dirac form factor of the free-electron vertex at zero momentum transfer. The result,

$$m^2F_1'(0)=\frac{\alpha^2}{\pi^2}[-\tfrac{13}{36}\ln\lambda^{-2}+\tfrac{411}{864}\pi^2+\tfrac{7}{12}\pi^2\ln2+\tfrac{313}{8}\zeta(3)-\tfrac{92189}{1728}]=\frac{\alpha^2}{\pi^2}(-\tfrac{13}{36}\ln\lambda^{-2}+2.3659)$$

[where $\zeta(3)$ is the Riemann ζ function of argument 3], confirms the result of Appelquist and Brodsky.

I. INTRODUCTION

Recently, a computer-assisted calculation by Appelquist and Brodsky[1] of the fourth-order electrodynamic corrections to the Lamb shift increased the $2S_{1/2}-2P_{1/2}$ separations by 0.35 ± 0.07 MHz and brought the theoretical values of the Lamb shift into good agreement with experiment.[2]

The results of Appelquist and Brodsky (hereafter referred to as AB) differ substantially from the original work of Weneser, Bersohn, and Kroll[3] and the more recent work of Soto.[4] In addition to an over-all sign discrepancy, there were significant differences in the noninfrared terms of two of the five contributions to the fourth-order corrections. This paper reports a recalculation of one of the contributions in favor of the AB result.

Rigorously, one must do this calculation within the framework of bound-state perturbation theory. But for the terms of interest here, those of order $\alpha^2(Z\alpha)^4m$, it suffices to use the so-called scattering approximation,[5] in which one first computes the radiative corrections to the scattering of a free electron and then infers from them a modified interaction potential from which the level shifts can be determined.

To compute the necessary radiative corrections, one needs to know the matrix element of the electromagnetic current between two free-electron states. It is[6]

$$J_\mu=-ie\bar{u}(p+\tfrac{1}{2}q)\left\{\gamma_\mu F_1(q^2)+\frac{1}{4m}[\not{q},\gamma_\mu]F_2(q^2)\right\}u(p-\tfrac{1}{2}q), \tag{1}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the Dirac form factors of free-electron vertices. The modification of the Coulomb interaction due to $F_1(q^2)$ contributes an energy shift

$$\Delta E(j,n,l)=\delta_{l0}\left[\frac{4(Z\alpha)^4mc^2}{n^3}\right]m^2\frac{\partial F_1(q^2)}{\partial q^2}\bigg|_{q^2=0} \tag{2}$$

(j, n, and l are the usual quantum numbers, and m the electron mass) plus terms of higher order in $Z\alpha$.

The fourth-order scattering diagrams used to calculate the contributions to $F_1'(0)$ are given in Fig. 1. Except for the sign discrepancy, agreement between AB and Soto was excellent for graphs (c)–(e). There was a large difference (~20%) in the noninfrared terms of graphs (b) and a slight but noticeable one (~3%) in (a). The so-called corner graphs, (b), have been checked twice: by de Rafael *et al.*[7] and by Barbieri *et al.*[8] Their results confirm those of AB.

This note describes an analytic calculation of the contribution to $F_1'(0)$ from the crossed graph (a) in Fig. 1. The result is

$$m^2F_1'(0)=\frac{\alpha^2}{\pi^2}[-\tfrac{13}{36}\ln\lambda^{-2}+\tfrac{411}{864}\pi^2+\tfrac{7}{12}\pi^2\ln2+\tfrac{313}{8}\zeta(3)-\tfrac{92189}{1728}] \tag{3a}$$

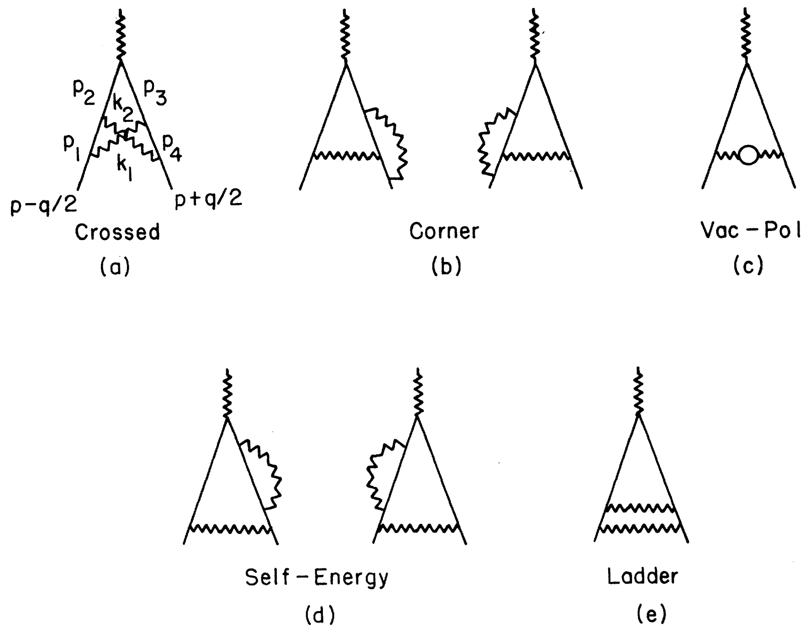

FIG. 1. Scattering diagrams used to compute the fourth-order contributions to $F_1(q^2)$.

$$= \frac{\alpha^2}{\pi^2}\left(-\tfrac{13}{36}\ln\lambda^{-2} + 2.3659\right) \tag{3b}$$

[$\zeta(3)$ is the Riemann ζ function of argument 3], which agrees well with AB's

$$m^2 F_1'(0) = \frac{\alpha^2}{\pi^2}\left(-\tfrac{13}{36}\ln\lambda^{-2} + 2.37\right), \tag{3c}$$

but disagrees with Soto's

$$m^2 F_1'(0) = \frac{\alpha^2}{\pi^2}\left[-\tfrac{13}{36}\ln\lambda^{-2} + \tfrac{1}{32}\pi^2 + \tfrac{7}{12}\pi^2\ln 2 - \tfrac{7}{8}\zeta(3) - \tfrac{1613}{1728}\right] \tag{3d}$$

$$= \frac{\alpha^2}{\pi^2}\left(-\tfrac{13}{36}\ln\lambda^{-2} + 2.31\right). \tag{3e}$$

II. DESCRIPTION OF CALCULATION

The techniques used were a combination of the approaches mentioned earlier.[3,4] As in AB, the Dirac algebra needed to find the contribution of (a) to $F_1'(0)$ was calculated (by computer) from

$$F_1(q^2) = \tfrac{1}{4}\mathrm{Tr}(\not{p} + \tfrac{1}{2}\not{q} + m)J_\mu(\not{p} - \tfrac{1}{2}\not{q} + m)\Lambda^\mu, \tag{4}$$

with

$$p^2 + \tfrac{1}{4}q^2 = m^2 \quad \text{and} \quad p\cdot q = 0,$$
$$\Lambda^\mu = -\frac{\gamma^\mu}{4p^2} + 3m\frac{p^\mu}{4p^4}. \tag{5}$$

All the computer traces were done using REDUCE,[9] a program of algebraic reduction written by Hearn. The denominator diagonalization[4] was also done by REDUCE, but it was Soto's transformations[4] that were employed rather than the more elegant methods (discussed in Bjorken and Drell[10]) used by AB. The integrations over the virtual photon momenta were also "done" by REDUCE. These integrations were "symbolic" only, and were possible because, for example, all convergent integrals of the form

$$\int \frac{d^4k\,(k^2)^{m-2}}{(k^2+a^2)^n} \quad (n>m>0) \tag{6}$$

could be replaced by

$$\frac{i\pi^2}{(a^2)^{n-m}}B(m, n-m), \tag{7a}$$

$$B(m, n-m) = \Gamma(m)\Gamma(n-m)/\Gamma(n), \tag{7b}$$

and REDUCE easily effects this type of substitution. Next, REDUCE performed the necessary differentiations with respect to q^2, set q^2 equal to zero, and performed the one trivial parametric integral much the same as (6) and (7).

At this stage of the calculation, the results could be directly compared to those of Soto. The terms were classified into two categories, depending on whether or not they exhibited uniform convergence with respect to λ, the photon mass. The terms which diverged were those which had no powers of the virtual photon momenta in them [see Eq. (10)]. Fortunately there were only five of them.

Except for an over-all sign difference, direct comparison of the infrared (ID) terms with Soto's terms revealed complete agreement both as regards the numerical coefficients of the terms and the values of the parametric integrals associated with them. The sum of these terms is

$$m^2F_1'(0)_{\mathrm{ID}} = \frac{\alpha^2}{\pi^2}\left(-\tfrac{13}{36}\ln\lambda^{-2} + \tfrac{1}{6}\pi^2 - \tfrac{29}{27}\right) \tag{8a}$$

$$= \frac{\alpha^2}{\pi^2}\left(-\tfrac{13}{36}\ln\lambda^{-2} + 0.5709\right). \tag{8b}$$

For the noninfrared (NI) terms a different procedure was used. First, the parametric integrals were evaluated directly from Soto's[4] tables. These tables were extensively spot-checked and the recursion relations and equivalences used to derive them were also verified. The result of this calculation was

$$m^2F_1'(0)_{\mathrm{NI}} = \frac{\alpha^2}{\pi^2}(1.7951). \tag{9a}$$

Next, the analytic forms of the integrals were worked out, yielding

$$m^2F_1'(0)_{\mathrm{NI}} = \frac{\alpha^2}{\pi^2}\left[\tfrac{267}{864}\pi^2 + \tfrac{7}{12}\pi^2\ln 2 + \tfrac{313}{8}\zeta(3) - \tfrac{40148}{768}\right] \tag{9b}$$

$$= \frac{\alpha^2}{\pi^2}(1.79505). \tag{9c}$$

Soto's[4] result was

$$m^2F_1'(0)_{\mathrm{NI}} = \frac{\alpha^2}{\pi^2}\left[-\tfrac{13}{96}\pi^2 + \tfrac{7}{12}\pi^2\ln 2 - \tfrac{7}{8}\zeta(3) + \tfrac{9}{64}\right] \tag{9d}$$

$$= \frac{\alpha^2}{\pi^2}(1.74925). \tag{9e}$$

III. DETAILS OF THE CALCULATION

The unrenormalized amplitude for the crossed graph is (see Fig. 1)

$$\begin{aligned} M_\mu = ie\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\bar{u}(p+\tfrac{1}{2}q) &\times \gamma_\beta(\not{p}_4+m)\gamma_\alpha(\not{p}_3+m)\gamma_\mu(\not{p}_2+m)\gamma^\beta(\not{p}_1+m)\gamma^\alpha \\ &\times \frac{1}{k_1^2-\lambda^2}\frac{1}{k_2^2-\lambda^2}\prod_{i=1}^{4}\frac{1}{p_i^2-m^2} \end{aligned} \tag{10}$$

$$\equiv -ie\bar{u}(p+\tfrac{1}{2}q)J_\mu^{\text{crossed}}u(p-\tfrac{1}{2}q), \tag{11}$$

where

$$\begin{aligned} p_4 &= p+\tfrac{1}{2}q-k_2, \\ p_3 &= p+\tfrac{1}{2}q-k_2-k_1, \\ p_2 &= p-\tfrac{1}{2}q-k_2-k_1, \\ p_1 &= p-\tfrac{1}{2}q-k_1. \end{aligned} \tag{12}$$

The denominators of (10) were combined using

$$\frac{1}{abcdef} = 5!\int_0^1 \frac{du\, w\,dw\, z^2\,dz\, x^3\,dx\, y^4\,dy}{[(a-b)uwzxy+(b-c)wzxy+(c-d)zxy+(d-e)xy+(e-f)y+f]^6}, \tag{13}$$

where

$$\begin{aligned} a &= (p-\tfrac{1}{2}q-k_1-k_2)^2-m^2, \\ b &= (p+\tfrac{1}{2}q-k_1-k_2)^2-m^2, \\ c &= (p+\tfrac{1}{2}q-k_2)^2-m^2, \\ d &= k_2^2-\lambda^2, \\ e &= (p-\tfrac{1}{2}q-k_1)^2-m^2, \\ f &= k_1^2-\lambda^2. \end{aligned} \tag{14}$$

The transformation which diagonalized this denominator was

$$\begin{aligned} k_1 &= k_1' - wzk_2' + C_1q + 2C_2p, \\ k_2 &= k_2' + C_3q + 2C_4p, \end{aligned} \tag{15}$$

and brought it into the form

$$D_1k_1'^2 + D_2k_2'^2 - D_3m^2 - D_4\lambda^2 + D_5q^2, \tag{16}$$

where

$$\begin{aligned} C_1 &= \tfrac{1}{2}w - wzu + \frac{ywz}{2D_2}\{D+2xuz[u(1-wz)-(1-w)]\}, \\ C_2 &= \tfrac{1}{2}w - \frac{ywz}{2D_2}D, \\ C_3 &= -\frac{y}{2D_2}\{D+2xwz[u(1-wz)-(1-w)]\}, \\ C_4 &= \tfrac{1}{2}y\frac{D}{D_2}, \\ D &= 1-x(1-wz+w^2z), \\ D_1 &= xy, \\ D_2 &= 1-xy(1-wz+w^2z^2), \\ D_3 &= w^2xy+y^2[1-x+xwz(1-w)]^2/D_2, \\ D_4 &= 1-y+xy, \\ D_5 &= w^2xyuz(1-uz) + \frac{y^2xuz[1-x+uxwz(1-w)][1-w-u(1-wz)]}{[1-xy(1-wz+w^2z^2)]}. \end{aligned} \tag{17}$$

Next, Eq. (4) was used to project out $F_1(q^2)$. Unfortunately, even REDUCE could not handle all the necessary index contractions and traces at once. The numerator of (10) was thus factored according to the powers of m each term contained, and the F_1 projection operator was split into two pieces:

$$P_{1\mu} = -\gamma_\mu/4p^2, \tag{18a}$$

$$P_{2\mu} = 3mp_\mu/4p^4. \tag{18b}$$

After the traces and contractions were done, the transformation to the new photon variables, k_1' and k_2', was effected in such a way that odd powers of k_1' and k_2' were automatically eliminated (since they vanished on integration anyway). Next, the integrations over k_1' and k_2' were done according to the procedure mentioned earlier. [See Eqs. (6) and (7).] At this stage $F_1(q^2)$ looked like

$$F_1(q^2) = \int_0^1 dudwdzdxdy \sum_i \frac{P_i(q^2, u, w, z, x, y)}{R_i(q^2, u, w, z, x, y)}, \tag{19}$$

with P_i and R_i polynomials in the Feynman parameters. Also, REDUCE differentiates, and so $\partial F_1(q^2)/\partial q^2$ was computed and q^2 set equal to zero. Finally, the substitutions for the various coefficients were implemented [see Eq. (17)] and the trivial integration over u was accomplished. This integration was possible because simple powers of u appeared in the numerator only, and thus all integrations were of the form

$$\int_0^1 du\, u^n = \frac{1}{n+1}. \tag{20}$$

The result was

$$F_1'(0) = \int_0^1 dwdzdydx \sum_i \frac{P_i'(w, x, y, z)}{R_i'(w, x, y, z)} \tag{21}$$

(again, with P_i' and R_i' polynomials in the Feynman parameters) and could be compared directly to Soto's[4] tables.

As mentioned earlier, the terms could be classified according to their infrared behavior. The divergent contributions to $F_1'(0)$ all had the structure

$$I(k, m, n, r, s, \lambda) = \int_0^1 dwdzdxdy\, C_{rs}^{kmn} \frac{z^r w^s x^{n+2k-3-m} y^{n+2k-2} D^m}{D_2^n(BD_3 + BD_4\lambda^2/m^2)^k} \tag{22}$$

(C_{rs}^{kmn} being numerical coefficients).

As Soto[4] noted, in the noninfrared terms there was uniform convergence with respect to λ, and so λ could be set to zero before the parametric integrations. The finite integrals were thus of the form

$$J(k, m, n, r, s) = \int_0^1 dwdxdydz\, D_{rs}^{kmn} \frac{z^r w^s x^{n+2k-3-m} y^{n+k-2}}{D_2^n G^k}, \tag{23}$$

with $G = w^2xB + yD^2$, and D_{rs}^{kmn} being numerical coefficients. The methods used for evaluating these integrals as well as complete tables for them can be found in Soto.[4]

It has not been possible to discover the origin of the discrepancy between the results of Soto and the present work. However, it is highly plausible that the error is a simple computational one. As noted earlier, the analysis of this graph requires the multidimensional integration and addition of more than 300 terms. Furthermore, investigation of random finite terms shows that the omission of a single one of them from the final sum, while drastically altering the coefficients of π^2, ln2, $\zeta(3)$, and the constant term in (3a), has little effect (1–3%) on the final numerical answer.

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