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Bohr Quantization of Relativistic Bound States of Two Point Particles*

A. DEGASPERIS

Physics Department, Indiana University, Bloomington, Indiana 47401

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We consider the action-at-a-distance relativistic theory proposed by Van Dam and Wigner in its more general version discussed by Katz. Exact bound-state solutions for systems of two particles are given. Bohr quantization of these solutions is also suggested.

I. INTRODUCTION

CLASSICAL mechanics failed to provide stable bound-state solutions for sources of the radiation field. Yet, the first successful attempts to understand the stability of an atomic system and its energy spectrum made use of Newtonian mechanics by disregarding the source nature of the particles and by imposing simple quantum conditions on the classical motion. One reason for the striking success of this approach is surely the fact that the relativistic effects in atomic phenomena are small, so that the Newtonian mechanics is still a good approximation to the real situation. In other words, the mass defects in electrodynamic bound states are very small compared to the energies exchanged in the emission and absorption processes. This is not the case in hadron dynamics, where the strong attraction causes mass defects of the same order of magnitude as the rest energy of the interacting particles. This implies that Bohr's approach to electromagnetic interactions could be tentatively adopted for strong interactions only if relativistic effects can be calculated in classical dynamics. Unfortunately, despite the beautiful mass-energy relation in Einstein's kinematics,[1] relativistic dynamics has suffered difficulties which have prevented an exact dynamical derivation of the mass defect. The main reason for this difficulty is the traditional assumption that invariance under Lorentz transformations implies that the interaction must be described by a field carrying energy, linear momentum, and angular momentum. Of course, no stability is possible for a bound state in such a dynamics. This situation has been clarified by Van Dam and Wigner,[2] who proposed a relativistic dynamics which is analogous to Newton's action-at-a-distance theory. This theory is a generalization of the time-symmetric electrodynamics of Tetrode and Fokker.[3,4] The relationship of this theory to the previous relativistic dynamics is briefly reviewed in the paper by Van Dam and Wigner and is not repeated here. More recently, Katz[5,6] proposed several action functions from which relativistic equations of motion are derived by a variational principle. Using one of these, the Van Dam–Wigner equations of motion are recovered. The Katz equations of motion are coupled integrodifferential equations and coincide with the Newtonian equations of motion in the nonrelativistic limit.

The relevance of a dynamical calculation of the mass defect for a better understanding of nuclear structure and of the quark model of hadrons has been already pointed out by Katz, who recently[6] discussed the approximate equations of motion up to order $1/c^2$ in the case of strong binding on slow particles. However, the task of understanding the properties of bounded systems in this mechanics, both in its original scalar version[2] and in its generalizations,[6] still faced the problem of finding the very definition of the ten constants of motion corresponding to the generators of the Poincaré group. In this respect, the relativistic dynamics, as formulated by Van Dam and Wigner and reformulated by Katz, does not present an unified treatment of the scattering states and bound states. In fact, in the original paper[2] it was possible to prove, by using the time-symmetric nature of the interaction, that the sum of the four-

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[1] A. Einstein, Ann. Physik **17**, 891 (1905).

[2] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965).

[3] H. Tetrode, Z. Physik **10**, 317 (1922); A. D. Fokker, *ibid.* **58**, 386 (1929); Physica **9**, 33 (1929); **12**, 145 (1932).

[4] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

[5] A. Katz, J. Math. Phys. **10**, 1929 (1969).

[6] A. Katz, J. Math. Phys. **10**, 2215 (1969).

momenta of the asymptotically free particles is conserved after the collision takes place; no expression of the mass of a bounded system was presented. The derivation of the equations of motion from an action principle, as proposed by Katz, did not solve this problem because the suggested action is a functional defined on the space of the physical states, such that its free-motion term diverges everywhere and the interaction term diverges on the subset of states in which two or more particles are bounded. This last feature prevents a meaningful derivation of the constants of motion related to the invariance under the Poincaré group. The aim of this paper is to present a family of exact solutions for bound states of two particles, interacting via arbitrary forces. Furthermore, the expression of the ten constants of motion is given and the exact dynamical calculation of the mass defect and of the angular momentum is presented. Although I limit myself to a two-particle system, the method of obtaining the constants of motion apply to a system of any number of particles. Since the expression of the constants of motion turns out to be the same for collision states and for bound states, in Sec. II I formally derive the equations of motion and the constants of motion following Katz, while the method of avoiding the divergency of the action is sketched in the Appendix. In Sec. III the exact solutions are given and, following Bohr,[7] quantum conditions on the classical motion are suggested.

II. DYNAMICAL EQUATIONS AND CONSTANTS OF MOTION

As has been proposed by Katz, the equations of motion of a system of two classical relativistic point particles can be derived from the action obtained by adding to the free-particle action

$$A_0=m_1c\int_{-\infty}^{+\infty}[\dot{x}_1^2(\tau_1)]^{1/2}d\tau_1+m_2c\int_{-\infty}^{+\infty}[\dot{x}_2^2(\tau_2)]^{1/2}d\tau_2 \qquad (2.1)$$

the following terms which represent the interaction:

$$A(l)=\frac{1}{2c}\int_{-\infty}^{+\infty}d\tau_1\int_{-\infty}^{+\infty}d\tau_2V_l([x_1(\tau_1)-x_2(\tau_2)]^2)\times[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{1-l}[\dot{x}_1^2(\tau_1)]^{\frac{1}{2}l}[\dot{x}_2^2(\tau_2)]^{\frac{1}{2}l}. \qquad (2.2)$$

In (2.2), l is a continuous parameter. In the following, let us consider only actions of the form

$$A=A_0+A(l) \qquad (2.3)$$

because our result can be simply generalized to any action containing a finite number of terms (2.2). In the expressions (2.1) and (2.2), m_i is the mass of the particle i ($i=1, 2$), and x_i is its four-vector position depending on a parameter τ_i. The dot means the derivative with respect to the parameter τ. The notation is such that, for any four-vector,

$$Q^\mu\equiv\{Q^0,Q^1,Q^2,Q^3\}=\{Q^0,\mathbf{Q}\},$$

and the metric is $g_{00}=c^2$, $g_{ij}=-\delta_{ij}$ for $i, j=1, 2, 3$. Furthermore let us adopt the convention of indicating with the symbol of functional derivative the operation of taking the functional derivative and then letting the parameters τ_1 and τ_2 be the proper time of the corresponding particles. The relativistic potential $V_l(s)$ entering in the expression (2.2) is such that

$$V_l(s)=0 \quad \text{if} \quad s>0. \qquad (2.4)$$

The Van Dam–Wigner theory is obtained by setting $l=0$ in the action (2.3). We define the generalized linear momentum

$$p_i^\mu(\tau_i)\equiv\frac{\delta A}{\delta\dot{x}_{i\mu}(\tau_i)}=\left(m_i+\frac{1}{2c}lc^{2l-2}\int_{-\infty}^{+\infty}d\tau_jV_l([x_1(\tau_1)-x_2(\tau_2)]^2)\times[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{1-l}\right)\dot{x}_i^\mu(\tau_i)+\frac{1}{2c}(1-l)c^{2l}\times\int_{-\infty}^{+\infty}d\tau_jV_l([x_1(\tau_1)-x_2(\tau_2)]^2)\times[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{-l}\dot{x}_j^\mu(\tau_j), \quad i\neq j \qquad (2.5)$$

so that the equations of motion read

$$\frac{d}{d\tau_i}p_i^\mu(\tau_i)=\frac{\delta A}{\delta x_{i\mu}(\tau_i)}=c^{2l-1}\int_{-\infty}^{+\infty}d\tau_jV_l'([x_1(\tau_1)-x_2(\tau_2)]^2)\times[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{1-l}[x_i^\mu(\tau_i)-x_j^\mu(\tau_j)], \quad i\neq j \qquad (2.6)$$

where for $(d/ds)V_l(s)$ we write $V_l'(s)$. Both sides of Eq. (2.6) have a finite nonrelativistic limit $(c\to\infty)$ which vanishes for the time component, and for the space components yields

$$m_i\frac{d^2}{dt^2}\mathbf{x}_i=-\frac{\partial}{\partial\mathbf{x}_i}\Phi_l(|\mathbf{x}_1-\mathbf{x}_2|), \quad i=1,2 \qquad (2.7)$$

namely, the Newtonian equation of motion. The proof of this result is the same as that given in Ref. 5 for the cases $l=0$, 1. The relation between the two potentials is[5]

$$\Phi_l(r)=\frac{1}{2}\int_{-\infty}^{+\infty}dz\ V_l(z^2-r^2). \qquad (2.8)$$

[7] N. Bohr, Phil. Mag. 26, 1 (1913).

If the action contains more than one term (2.2), the Newtonian potential obtained in the nonrelativistic limit is simply the sum of the corresponding potentials (2.8). In order to discuss the relativistic properties of a bound state we need an expression for the total linear four-momentum P^μ of the system and for its antisymmetric total angular momentum tensor $\Omega^{\mu\nu}$. The total linear four-momentum is defined as

$$P^\mu(\tau_1,\tau_2)=P_1{}^\mu(\tau_1)+P_2{}^\mu(\tau_2)\,, \tag{2.9}$$

where the four-vector $P_i{}^\mu(\tau_i)$ is obtained by integrating the equation of motion

$$P_i{}^\mu(\tau_i)=p_i{}^\mu(\tau_i)-\int_{-\infty}^{\tau_i}d\tau_i'\frac{\delta A}{\delta x_{i\mu}(\tau_i')}\,,\quad i=1,2 \tag{2.10}$$

and imposing the asymptotic conditions for a collision state

$$\lim_{\tau_1\to\pm\infty;\,\tau_2\to\pm\infty}[P^\mu(\tau_1,\tau_2)-m_1\dot{x}_1{}^\mu(\tau_1)-m_2\dot{x}_2{}^\mu(\tau_2)]=0\,. \tag{2.11}$$

The four-vector (2.9) is, of course, a constant of motion and its nonrelativistic limit is

$$\lim_{c\to\infty}c(P^0-m_1c-m_2c)=\tfrac{1}{2}m_1\left(\frac{d\mathbf{x}_1}{dt}\right)^2+\tfrac{1}{2}m_2\left(\frac{d\mathbf{x}_2}{dt}\right)^2+\Phi(|\mathbf{x}_1-\mathbf{x}_2|)\,, \tag{2.12}$$

$$\lim_{c\to\infty}\mathbf{P}=m_1\frac{d\mathbf{x}_1}{dt}+m_2\frac{d\mathbf{x}_2}{dt}\,. \tag{2.13}$$

The mass M of the bound state is given by the formula

$$M^2c^2=P^\mu P_\mu\,. \tag{2.14}$$

The expression for the total linear four-momentum coincides with that given in Ref. 5 for the cases $l=0$, 1. To obtain the total angular momentum tensor, we may first define the generalized angular momentum tensor of each particle,

$$L_i{}^{\mu\nu}(\tau_i)\equiv x_i{}^\mu(\tau_i)p_i{}^\nu(\tau_i)-x_i{}^\nu(\tau_i)p_i{}^\mu(\tau_i)\,,\quad i=1,2 \tag{2.15}$$

which reduces to the usual one if the particle is free. The antisymmetric tensor (2.15) satisfies the equation of motion

$$\begin{aligned}\frac{d}{d\tau_i}L_i{}^{\mu\nu}(\tau_i)=\frac{(1-l)}{2c}c^{2l}\int_{-\infty}^{+\infty}&d\tau_jV_l([x_1(\tau_1)-x_2(\tau_2)]^2)[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{-l}[\dot{x}_i{}^\mu(\tau_i)\dot{x}_j{}^\nu(\tau_j)-\dot{x}_i{}^\nu(\tau_i)\dot{x}_j{}^\mu(\tau_j)]\\ -c^{2l-1}\int_{-\infty}^{+\infty}&d\tau_jV_l'([x_1(\tau_1)-x_2(\tau_2)]^2)[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{1-l}[x_i{}^\mu(\tau_i)x_j{}^\nu(\tau_j)-x_i{}^\nu(\tau_i)x_j{}^\mu(\tau_j)]\,,\quad i=1,2;\,i\neq j\,.\end{aligned} \tag{2.16}$$

We may then define the total angular momentum of our system as the sum

$$\Omega^{\mu\nu}(\tau_1,\tau_2)=\Omega_1{}^{\mu\nu}(\tau_1)+\Omega_2{}^{\mu\nu}(\tau_2)\,, \tag{2.17}$$

where the tensor $\Omega_i{}^{\mu\nu}(\tau_i)$ is obtained by integrating Eq. (2.16) and imposing asymptotic conditions similar to the condition (2.11). The result is

$$\begin{aligned}\Omega_i{}^{\mu\nu}(\tau_i)=L_i{}^{\mu\nu}(\tau_i)-\frac{(1-l)}{2c}c^{2l}\int_{-\infty}^{\tau_i}d\tau_i'\int_{-\infty}^{+\infty}d\tau_j'V_l([x_1(\tau_1')-x_2(\tau_2')]^2)[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{-l}\\ \times[\dot{x}_i{}^\mu(\tau_i')\dot{x}_j{}^\nu(\tau_j')-\dot{x}_i{}^\nu(\tau_i')\dot{x}_j{}^\mu(\tau_j')]+c^{2l-1}\int_{-\infty}^{\tau_i}d\tau_i'\int_{-\infty}^{+\infty}d\tau_j'V_l'([x_1(\tau_1')-x_2(\tau_2')]^2)\\ \times[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{1-l}[x_i{}^\mu(\tau_i')x_j{}^\nu(\tau_j')-x_i{}^\nu(\tau_i')x_j{}^\mu(\tau_j')]\,,\quad i=1,2;\,i\neq j\,.\end{aligned} \tag{2.18}$$

In order to specify the physical meaning of the antisymmetric tensor $\Omega^{\mu\nu}(\tau_1,\tau_2)$, let us consider it in a particular reference frame; the properties of $\Omega^{\mu\nu}(\tau_1,\tau_2)$ in all the other reference frames can be obtained by using a Lorentz transformation. The total angular momentum tensor is characterized by the following physical interpretation in the reference frame in which $P^0=Mc$ and $P^1=P^2=P^3=0$: the time components are

$$\Omega^{i0}=McA^i\,,\quad i=1,2,3 \tag{2.19}$$

where the three-dimensional vector A^i is defined as the position vector of the center of mass of the system, and the space components are such that

$$N_h=\tfrac{1}{2}\epsilon_{hik}\Omega^{ik} \tag{2.20}$$

is the three-dimensional total angular momentum. We note that the definition (2.19) is justified by considering that the translation $x'^\mu=x^\mu+d^\mu$ induces on the total angular momentum tensor (2.17) the following transformation law:

$$\Omega'^{\mu\nu}=\Omega^{\mu\nu}+d^\mu P^\nu-d^\nu P^\mu\,, \tag{2.21}$$

where P^μ is the total linear four-momentum (2.9).

So far I have translated all the ten well-known Newtonian constants of motion into relativistic language; in fact, both the Newtonian and the present dynamics have a ten-parameter symmetry group, the Galilean group and the Poincaré group, respectively. In a collision process, the constants of motion are determined by the asymptotic positions and velocities of the particles, while the constants of motion corresponding to a bound state can be obtained only if the solution of the equations of motion is known. In Sec. III, I give a class of particular bound-state solutions and the explicit expression for the corresponding two main constants of motion, the mass and the modulus of the total angular momentum in the rest frame of the bound state.

III. EXACT BOUND-STATE SOLUTIONS AND QUANTIZATION

The dynamical equations (2.6) are two coupled integrodifferential equations. Since this dynamics is invariant under the transformations of the Lorentz group and time-symmetric, the force acting on the particle 1 at the position $x_1(\tau_1)$ is a functional of the world line of the other particle which depends only on that part of the world line which runs in the spacelike region of $x_1(\tau_1)$ and vice versa. Those functionals are expressed by the integrals which enter in the dynamical equations. One can then try to solve these equations by iteration, starting with suitable zero-approximation trajectories. This approach has been used by Van Dam and Wigner[2] for large mass ratio and distant collisions; however, there is no proof of convergence. My exact solutions are circular orbits in the rest frame of the bound state. Furthermore the two particles have the same constant angular velocity ω. Let me first introduce a convenient notation. We define $\hat{u}(t)$ as a three-dimensional unit vector

$$\hat{u}(t)\cdot\hat{u}(t)=1 \tag{3.1}$$

rotating in a plane with constant angular velocity ω. Let $\hat{v}(t)$ be the unit vector such that

$$\frac{d}{dt}\hat{u}(t)=\omega\hat{v}(t); \tag{3.2}$$

then the unit vector

$$\hat{\omega}=\hat{u}(t)\Lambda\hat{v}(t) \tag{3.3}$$

is constant and

$$\frac{d}{dt}\hat{v}(t)=-\omega\hat{u}(t). \tag{3.4}$$

The positions of the two particles are given by

$$\mathbf{x}_2(t)=R_2\hat{u}(t),\quad \mathbf{x}_1(t)=-R_1\hat{u}(t), \tag{3.5}$$

where R_1 and R_2 are the radius of the orbits. Since the modulus of the velocity of the particles is constant, the time in this frame of reference is a simple function of the proper time.

The four-vector positions of the two particles are

$$x_2^\mu(\tau_2)=\{\gamma_2\tau_2+Q_2,\ R_2\hat{u}(\gamma_2\tau_2+Q_2)\}, \tag{3.6}$$

$$x_1^\mu(\tau_1)=\{\gamma_1\tau_1+Q_1,\ -R_1\hat{u}(\gamma_1\tau_1+Q_1)\}, \tag{3.7}$$

so that their four-velocity and four-acceleration read

$$\begin{aligned}\dot{x}_2^\mu(\tau_2)&=\gamma_2\{1,\ \beta_2c\hat{v}(\gamma_2\tau_2+Q_2)\},\\ \ddot{x}_2^\mu(\tau_2)&=\gamma_2^2\{0,\ -\beta_2\omega c\hat{u}(\gamma_2\tau_2+Q_2)\},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\dot{x}_1^\mu(\tau_1)&=\gamma_1\{1,\ -\beta_1c\hat{v}(\gamma_1\tau_1+Q_1)\},\\ \ddot{x}_2^\mu(\tau_1)&=\gamma_1^2\{0,\ \beta_1\omega c\hat{u}(\gamma_1\tau_1+Q_1)\},\end{aligned} \tag{3.9}$$

where

$$\beta_i=\omega R_i/c,\quad \gamma_i=(1-\beta_i^2)^{-1/2},\quad i=1,2 \tag{3.10}$$

and Q_i is a constant depending on the particle proper-time scale. Next we write the expression of the four-distance between the two particles:

$$\begin{gathered}[x_2(\tau_2)-x_1(\tau_1)]^2=(4c^2/\omega^2)(\theta^2-\beta^2\cos^2\theta-\delta^2),\\ \beta^2=\beta_1\beta_2,\quad \delta=\tfrac{1}{2}(\beta_2-\beta_1),\quad \theta=\tfrac{1}{2}\omega(t_2-t_1),\\ \gamma_i\tau_i+Q_i=t_i,\quad i=1,2.\end{gathered} \tag{3.11}$$

The function

$$D(\theta)=\theta^2-\beta^2\cos^2\theta-\delta^2 \tag{3.12}$$

for any $0\leq\beta_i\leq 1$ has two zeros; let $s(\beta,\delta)$ be the function of $\beta_i\in(0,1)$ defined as the solution of the transcendental equation[8]

$$s(\beta,\delta)=[\beta^2\cos^2 s(\beta,\delta)+\delta^2]^{1/2}; \tag{3.13}$$

then we have

$$D(\pm s(\beta,\delta))=0,\quad D(\theta)<0\quad\text{for}\quad -s(\beta,\delta)<\theta<s(\beta,\delta),\quad \beta_i\in(0,1). \tag{3.14}$$

For future convenience we write down the product of the four-velocities:

$$\dot{x}_1^\mu(\tau_1)\dot{x}_{2\mu}(\tau_2)=c^2\gamma_1\gamma_2(1+\beta^2\cos 2\theta), \tag{3.15}$$

where θ is given in terms of τ_1 and τ_2 as in Eq. (3.11). Furthermore, the following relations turn out to be useful:

$$\hat{u}(t_2)=\cos\omega(t_2-t_1)\hat{u}(t_1)+\sin\omega(t_2-t_1)\hat{v}(t_1), \tag{3.16}$$

$$\hat{v}(t_2)=\cos\omega(t_2-t_1)\hat{v}(t_1)-\sin\omega(t_2-t_1)\hat{u}(t_1). \tag{3.17}$$

We now substitute the world-line expressions (3.6) and (3.7) together with (3.11) and (3.15) into (2.5) and (2.6) which define the generalized momentum and the "force" $\delta A/\delta x_{i\mu}(\tau_i)$. In order to have a compact ex-

[8] In the case of equal masses, namely, $\delta=0$, this equation is a particular case of Kepler's equation $z(x)=Q+x\sin z(x)$, whose solution has been tabulated [A. de Gasparis, Astron. Nachr. 46, 18 (1857)] because of its use in astronomy.

pression for the final results, we introduce the following integrals:

$$U_K(\omega,\beta_1,\beta_2)=\int_{-\infty}^{+\infty}d\theta\, V_l\left(\left(\frac{2c}{\omega}\right)^2 D(\theta)\right)\times(1+\beta^2\cos 2\theta)^{1-K},\quad(3.18)$$

$$T_K(\omega,\beta_1,\beta_2)=\int_{-\infty}^{+\infty}d\theta\, V_l'\left(\left(\frac{2c}{\omega}\right)^2 D(\theta)\right)\times(1+\beta^2\cos 2\theta)^{1-K}.\quad(3.19)$$

The generalized linear momentum (2.5) is found to be

$$p_i^\mu(\tau_i)=\left[m_i+\frac{l\gamma_i(\gamma_1\gamma_2)^{-l}}{c\omega}U_l\right]\dot{x}_i^\mu(\tau_i)+\frac{(1-l)}{c\omega}(\gamma_1\gamma_2)^{-l}\times\left\{U_{1+l},\,(-)^{i-1}\frac{c}{\beta_i}(U_l-U_{1+l})\hat{v}(\gamma_i\tau_i+Q_i)\right\},\quad(3.20)$$

and the "force" (2.6) reads

$$\frac{\delta A}{\delta x_{i\mu}(\tau_i)}=\frac{2\gamma_i(\gamma_1\gamma_2)^{-l}c}{\omega}\times\left[T_l x_i^\mu(\tau_i)-\left\{T_l(\gamma_i\tau_i+Q_i),\,(-)^{i-1}\times\frac{c}{\omega\beta_i}(T_{l-1}-T_l)\hat{u}(\gamma_i\tau_i+Q_i)\right\}\right].\quad(3.21)$$

By taking the derivative with respect of the proper time of the linear momentum (3.20), we find that the equations of motion (2.6) are satisfied, provided the following equation holds:

$$\gamma_i\omega^2 R_i\left[m_i+\frac{l}{\omega c}\gamma_i(\gamma_1\gamma_2)^{-l}U_l\right]+\frac{(1-l)}{\beta_i}(\gamma_1\gamma_2)^{-l}(U_{1+l}-U_l)=\frac{2c^2}{\omega^2}(\gamma_1\gamma_2)^{-l}\times\left[T_l\left(\frac{1}{\beta_i}-\beta_i\right)-\frac{1}{\beta_i}T_{l-1}\right],\quad i=1,2.\quad(3.22)$$

Equation (3.22) gives the relationship between the angular velocity and the radius of the orbits corresponding to a given potential $V_l(s)$. In fact, Eq. (3.22) corresponds to the simpler equation valid in Newtonian mechanics

$$m_i\omega^2 R_i=\Phi_l'(R_1+R_2),\quad i=1,2\quad(3.23)$$

which is the nonrelativistic limit of our relativistic result (3.22). Now we evaluate the ten constants of motion in terms of the potential $V_l(s)$ and of the bound-state parameters m_i, R_i, and ω. Although the expressions of the constants of motion obtained in Sec. II have been derived only for collision states by using the asymptotic conditions of free motion, we will apply them also to our bound-state solutions. The justification for such a procedure is given in the Appendix. However, the application is not straightforward because the integral of the force in Eq. (2.10) as well as the double integral in Eq. (2.18) are divergent for a stable bounded system simply because the particles experience the interaction force for an infinite length of time. On the other hand, these infinities cancel each other in the sum which defines the total linear four-momentum, Eq. (2.9), and the total angular momentum tensor, Eq. (2.17). The resulting expressions are easily obtained; thus,

$$P^\mu(\tau_1,\tau_2)=p_1^\mu(\tau_1)+p_2^\mu(\tau_2)+c^{2l-1}\left(\int_{-\infty}^{\tau_1}d\tau_1'\int_{\tau_2}^{+\infty}d\tau_2'-\int_{-\infty}^{\tau_2}d\tau_2'\int_{\tau_1}^{+\infty}d\tau_1'\right)\times V_l'([x_1(\tau_1')-x_2(\tau_2')]^2)[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{1-l}[x_2^\mu(\tau_2')-x_1^\mu(\tau_1')],\quad(3.24)$$

$$\begin{aligned}\Omega^{\mu\nu}(\tau_1,\tau_2)=L_1^{\mu\nu}(\tau_1)+L_2^{\mu\nu}(\tau_2)+c^{2l-1}\left(\int_{-\infty}^{\tau_1}d\tau_1'\int_{\tau_2}^{+\infty}d\tau_2'-\int_{-\infty}^{\tau_2}d\tau_2'\int_{\tau_1}^{+\infty}d\tau_1'\right)\\ \times[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{-l}\{\tfrac{1}{2}(1-l)V_l([x_1(\tau_1')-x_2(\tau_2')]^2)[\dot{x}_2^\mu(\tau_2')\dot{x}_1^\nu(\tau_1')-\dot{x}_2^\nu(\tau_2')\dot{x}_1^\mu(\tau_1')]\\ -[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]V_l'([x_1(\tau_1')-x_2(\tau_2')]^2)[x_2^\mu(\tau_2')x_1^\nu(\tau_1')-x_2^\nu(\tau_2')x_1^\mu(\tau_1')]\}.\quad(3.25)\end{aligned}$$

The expression (3.24) was already given in Ref. 5 for the cases $l=0$, 1. The contribution to the constants of motion (3.24) and (3.25) due to the double integrals has no Newtonian counterpart since it is due to the retarded and advanced effects of the interaction, and it vanishes in the nonrelativistic limit.

By substituting the solutions (3.6) and (3.7), together with the expression of the generalized linear momentum (3.20), into the Eq. (3.24) for the total energy-momentum four-vector, and into the Eq. (3.25) for the total angular momentum tensor, we get the following result:

$$P^\mu=\{Mc,0,0,0\},\quad\Omega^{\mu\nu}=N\begin{pmatrix}0&0&0&0\\0&0&\omega_3&-\omega_2\\0&-\omega_3&0&\omega_1\\0&\omega_2&-\omega_1&0\end{pmatrix},\quad(3.26)$$

where the mass of the bound state is

$$M=m_1\gamma_1+m_2\gamma_2+\frac{l}{\omega c}(\gamma_1\gamma_2)^{-l}U_l(\gamma_1^2+\gamma_2^2)$$
$$+\frac{2(1-l)}{c\omega}(\gamma_1\gamma_2)^{-l}U_{1+l}+\frac{8c}{\omega^3}(\gamma_1\gamma_2)^{-l}$$
$$\times\int_{-\infty}^{+\infty}d\theta\,\theta^2(1+\beta^2\cos 2\theta)V_l'\left(\left(\frac{2c}{\omega}\right)^2D(\theta)\right) \quad (3.27)$$

and the modulus of its total angular momentum is

$$N=m_1\gamma_1R_1^2\omega+m_2\gamma_2R_2^2\omega+\frac{l}{c}(\gamma_1\gamma_2)^{-l}U_l(\gamma_1^2R_1^2+\gamma_2^2R_2^2)$$
$$+\frac{2(1-l)c}{\omega^2}(\gamma_1\gamma_2)^{-l}(U_{1+l}-U_l)$$
$$+\frac{4R_1R_2}{\omega^2c}(\gamma_1\gamma_2)^{-l}\int_{-\infty}^{+\infty}d\theta\,\theta\sin 2\theta(1+\beta^2\cos 2\theta)^{-l}$$
$$\times\left[\tfrac{1}{2}(1-l)\omega^2V_l\left(\left(\frac{2c}{\omega}\right)^2D(\theta)\right)\right.$$
$$\left.-c^2(1+\beta^2\cos 2\theta)V_l'\left(\left(\frac{2c}{\omega}\right)^2D(\theta)\right)\right]. \quad (3.28)$$

Formula (3.26) means that the bound state is in its rest frame of reference and its center of mass lies at the origin of the space coordinates.

Previous results give an exact calculation of the parameters of a relativistic bounded system, namely, the mass and the angular momentum. From this point of view, it is clear that the advantage of the relativistic dynamics sketched in Sec. II, with respect to other classical and quantum theories is due to the absence of radiation effects and self-energy difficulties. However, our bound-state solution suffers from a major illness, namely, it does not describe a quantum system. However, this situation is similar to the early stage of the quantum theory of the atom when the first speculations by Bohr[7] made use of the circular orbits solutions in a Coulomb potential. This similarity suggests that we can still make use of our bound-state solution, provided certain conditions on the classical motion are introduced to take into account quantum behavior. A safe way of introducing these quantum conditions is the adoption of Ehrenfest's prescription, namely, to equate the adiabatic invariants to integer multiples of the Planck constant. Unfortunately, the proof that a constant of the motion is an adiabatic invariant has its natural place in the Jacobi-Hamilton formulation of the dynamical problem and such an approach is not available in the dynamics we are considering. The quantum conditions then have to be found by requiring that the two-particles system behave as suggested by physical considerations. If we ask that the set of all the stationary states of the two particles are bosons, then we should impose on the angular momentum the quantum condition

$$N=(\tfrac{1}{2}\Omega^{\mu\nu}\Omega_{\mu\nu})^{1/2}=nh/2\pi\,,\quad n=1,2,\ldots \quad (3.29)$$

where h is the Planck constant. Let M_n and M_k be the masses of two different stationary states, with $M_n>M_k$. The transition from the state n to the state k is then given by the emission of a particle with linear momentum

$$p=\frac{c}{2M_n}\{[(M_n+M_k)^2-\mu^2][(M_n-M_k)^2-\mu^2]\}^{1/2} \quad (3.30)$$

if μ is its mass, and with spin $(n-k)$. In the case $\mu=0$ and $n=k+1$, the process describes the electromagnetic emission of one photon.

Although I am aware that these dynamical solutions, together with the suggested quantum condition (3.29), describe an oversimplified world, it seems to me that this theory has, nevertheless, the virtue of giving an exact quantitative description of relativistic and quantum effects combined. In a following paper I will discuss the energy spectrum corresponding to simple interactions.

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APPENDIX

For the reasons mentioned in Sec. I, the expression of the constants of motion used in Sec. III to derive the mass and the angular momentum of the bound states still needs a justification. In this Appendix the conserved quantities P^μ and $\Omega^{\mu\nu}$, namely, the total linear four-momentum and the total angular momentum antisymmetric tensor, will be derived from the relativistic invariance of the theory. The standard approach to this problem is based on the scalar nature of the action with respect to the Poincaré group. However, the action given in Sec. II, Eq. (2.3), is a divergent functional and its meaning is only formal. The integrals defining the action are divergent simply because the integration runs on the entire world line of the particles. The fact that we cannot cut the integration interval to a finite portion of the world line of the particles is a peculiarity of any system of relativistic particles, for the interaction is nonlocal in time. In other words, the dynamics of a bounded, isolated system cannot be fully described in a finite interval of time because the force acting on each particle in a neighborhood of the ends of this interval depends on the other particles' positions and velocities evaluated outside the considered time interval. Let us first note that because of the nonlocality of the interaction the action functional has the following mathematical structure: The free-motion term is given by an integral on the line, and the interaction term is an integral on the plane (τ_1,τ_2). The region of this plane in which the relativistic potential $V_l([x_1(\tau_1)-x_2(\tau_2)]^2)$ is different from zero is always unbounded. As an example, in the case of the bound states given in Sec. III, this region is the strip confined between the two parallel straight lines

$$\tau_2=\gamma_1/\gamma_2(\tau_1-A_+)\,,\quad \tau_2=\gamma_1/\gamma_2(\tau_1-A_-)\,,$$
$$A\pm=\pm\frac{2}{\omega\gamma_1}s(\beta,\delta)+\frac{1}{\gamma_1}(Q_2-Q_1)\,. \tag{A1}$$

The middle straight line is the simultaneity line in the frame of reference in which the bound state is at rest. Let us now introduce a finite region in the plane (τ_1,τ_2), for sake of simplicity, the rectangular

$$\rho(\alpha)\equiv\{\tau_1,\,\tau_2{:}-\alpha_i\leq\tau_i\leq\alpha_i,\,i=1,\,2\}\,,\quad \alpha_i>0 \tag{A2}$$

and let us define the functional

$$A_\alpha=\sum_{i=1}^{2}m_ic\int_{-\alpha i}^{\alpha i}d\tau_i[\dot{x}_i^2(\tau_i)]^{1/2}+\frac{1}{2c}\int_{\rho(\alpha)}d\tau_1d\tau_2V_l([x_1(\tau_1)-x_2(\tau_2)]^2)\times[\dot{x}_1(\tau_1)\cdot\dot{x}_2(\tau_2)]^{1-l}[\dot{x}_1^2(\tau_1)]^{\frac{1}{2}l}[\dot{x}_2^2(\tau_2)]^{\frac{1}{2}l}\,. \tag{A3}$$

This functional is different from the formal expression (2.3) only with respect to the region of integration. By applying the variational principle, we obtain the coupled integrodifferential equations

$$\frac{d}{d\tau_i}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\tau_i)}\right)=\left(\frac{\delta A_\alpha}{\delta x_{i\mu}(\tau_i)}\right),\quad i=1,\,2 \tag{A4}$$

which could be interpreted as the equations of motion of a fictitious system of two particles. However, the forces acting on the particles are incomplete, and the solutions of Eq. (A4) could be very different from the physical solutions. These last solutions are obtained by letting α_i go to $+\infty$, since in this limit Eq. (A4) coincide with the equations of motion (2.6). Although Eq. (A4) describe a nonphysical system, the functional (A3) is a meaningful scalar quantity, and we can derive the conservation laws from the invariance under the Poincaré group using standard techniques. Let us start with the general variation of the functional (A3)

$$\delta A_\alpha=\sum_{i=1}^{2}\left[\int_{-\alpha i}^{\alpha i}d\tau_i\delta x_i^\mu(\tau_i)\left(\frac{\delta A_\alpha}{\delta x_i^\mu(\tau_i)}\right)+\int_{-\alpha i}^{\alpha i}d\tau_i\delta\dot{x}_i^\mu(\tau_i)\left(\frac{\delta A_\alpha}{\delta\dot{x}_i^\mu(\tau_i)}\right)\right], \tag{A5}$$

corresponding to the transformation

$$x_i'^\mu(\tau_i)=x_i^\mu(\tau_i)+\delta x_i^\mu(\tau_i)\,,\qquad \dot{x}_i'^\mu(\tau_i)=\dot{x}_i^\mu(\tau_i)+\delta\dot{x}_i^\mu(\tau_i)\,. \tag{A6}$$

Consider now the infinitesimal transformations belonging to the Poincaré group so that $\delta A_\alpha=0$. For a space-time translation

$$\delta x_i^\mu(\tau_i)=d^\mu,\quad \delta\dot{x}_i^\mu(\tau_i)=0\,, \tag{A7}$$

Eq. (A5) together with the equations of motion (A4) leads to

$$\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\alpha_i)}\right)=\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(-\alpha_i)}\right). \tag{A8}$$

On the other hand, the infinitesimal Lorentz transformation

$$\delta x_i^\mu(\tau_i)=g^{\mu\nu}\epsilon_{\alpha\beta}G_{\nu\rho}{}^{\alpha\beta}x_i^\rho(\tau_i)\,,\qquad \delta\dot{x}_i^\mu(\tau_i)=g^{\mu\nu}\epsilon_{\alpha\beta}G_{\nu\rho}{}^{\alpha\beta}\dot{x}_i^\rho(\tau_i)\,, \tag{A9}$$

where $\epsilon_{\alpha\beta}$ is an arbitrary infinitesimal antisymmetric matrix and $G_{\nu\rho}{}^{\alpha\beta}$ are the generators of the Lorentz group, leads to the formula

$$\sum_{i=1}^{2}G_{\nu\rho}{}^{\alpha\beta}x_i^\nu(\alpha_i)\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\rho}(\alpha_i)}\right)=\sum_{i=1}^{2}G_{\nu\rho}{}^{\alpha\beta}x_i^\nu(-\alpha_i)\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\rho}(-\alpha_i)}\right). \tag{A10}$$

Since the conservation laws (A8) and (A10) are a direct consequence of the relativistic invariance, it is natural to derive from them the constants of motion of the physical system in the limit in which α_1 and α_2 go to infinity. Therefore we define the total linear momentum as

$$P^\mu\equiv\lim_{\alpha_1\to\infty;\,\alpha_2\to\infty}\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\alpha_i)}\right) \tag{A11}$$

and the total angular momentum tensor as

$$\Omega^{\mu\nu}\equiv\lim_{\alpha_1\to\infty;\,\alpha_2\to\infty}\sum_{i=1}^{2}G_{\lambda\rho}{}^{\mu\nu}x_i^\lambda(\alpha_i)\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\rho}(\alpha_i)}\right), \tag{A12}$$

where the generators are

$$G_{\lambda\rho}{}^{\mu\nu}=g^\mu{}_\lambda g^\nu{}_\rho-g^\nu{}_\lambda g^\mu{}_\rho\,. \tag{A13}$$

The last step consists in showing that the expressions (3.24) and (3.25) of the constants of motion coincide, respectively, with the definitions (A11) and (A12). Let us consider first the definition (A11). In order to perform the limit operation, it is convenient to use Eq. (A4) to rewrite the sum in the form

$$\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\alpha_i)}\right)=\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\tau_i)}\right)+\sum_{i=1}^{2}\int_{\tau i}^{\alpha i}d\tau_i'\left(\frac{\delta A_\alpha}{\delta x_{i\mu}(\tau_i')}\right),\quad -\alpha_i<\tau_i<\alpha_i; \tag{A14}$$

then one has to calculate explicitly the second sum of the right-hand side of this equality by substituting in it the expression of the functional derivative. Thus one gets

$$\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\alpha_i)}\right)=\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\tau_i)}\right)$$
$$+c^{2l-1}\left(\int_{\tau_1}^{\alpha_1}d\tau_1'\int_{-\alpha_2}^{\tau_2}d\tau_2'-\int_{\tau_2}^{\alpha_2}d\tau_2'\int_{-\alpha_1}^{\tau_1}d\tau_1'\right)$$
$$\times V_l'([x_1(\tau_1')-x_2(\tau_2')]^2)[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{1-l}$$
$$\times[x_1^\mu(\tau_1')-x_2^\mu(\tau_2')]. \quad \text{(A15)}$$

We can now take the limit (A11) of this last expression by considering that the limit holds

$$\lim_{\alpha_1\to\infty;\,\alpha_2\to\infty}\left(\frac{\delta A_\alpha}{\delta\dot{x}_{i\mu}(\tau_i)}\right)=p_i^\mu(\tau_i)\,, \quad \text{(A16)}$$

where $p_i^\mu(\tau_i)$ is defined by the formula (2.5); substituting the expression (A15) in the definition (A11) and making the limit, we obtain the expression of $P^\mu(\tau_1,\tau_2)$ given in Sec. III. By applying this same procedure, one can show that the definition (A12) of $\Omega^{\mu\nu}$ leads to the expression (3.25) of Sec. III. We note that the derivation of the previous results has been only sketched; in fact, the precise definition of the solutions of the equations (A4) has not been discussed, and their dependence on the parameters α_i has been omitted. This method applies to a system of any number of particles.

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Quantization of Evanescent Electromagnetic Waves*

C. K. CARNIGLIA† AND L. MANDEL
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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The problem of the quantization of evanescent waves, which appear in the angular spectrum representation of the electromagnetic field in a half-space, is discussed. Although evanescent waves are associated with material sources, scatterers, etc., we are able to treat the electromagnetic field, including the evanescent waves, effectively as a free field, by making use of the idea of the refractive index of a passive, macroscopically continuous medium. We consider a space which is filled with a homogeneous dielectric to the left of the plane $z=0$, and is empty to the right of the plane. Triplets of incident, reflected, and transmitted waves at the interface form the fundamental orthogonal modes of the space. By expanding the field in terms of these triplet modes, we show that the field Hamiltonian reduces to the sum of independent harmonic-oscillator Hamiltonians. The quantization is therefore straightforward. We introduce the creation and annihilation operators for the triplet wave modes, and encounter Fock states, coherent states, etc., for a field having evanescent wave components. The field commutator at two space-time points in the right half-space is shown to have an explicit contribution from evanescent waves, characterized by an exponential decay to the right and a propagation parallel to the interface. We also examine the problem of atomic excitation by quantized evanescent waves, and show that the results are of the form given by semiclassical treatments.

I. INTRODUCTION

ALTHOUGH evanescent electromagnetic waves have been well known in optics and in the microwave domain for many years, they have tended to be something of a curiosity. They are perhaps most familiar in connection with the total internal reflection of light at a glass-to-air interface, and quantitative features of the evanescent waves produced under these conditions were studied experimentally by Quincke[1] and by Hall[2] as long ago as 1866 and 1902, respectively. In recent years they have been frequently encountered in the context of diffraction, particularly in the angular spectrum representation of the electromagnetic field,[3–9] where the evanescent waves appear as a natural adjunct to the spectrum of homogeneous plane waves. Expansions involving evanescent waves have also proved valuable recently in the treatment of radiation from moving

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† Also at the Institute of Optics, University of Rochester, Rochester, N. Y.
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