# Arguments against Anomalous Dimensions*

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(Received 21 December 1970)

Two-dimensional solvable field theories provide examples of anomalous dimensions and conserved dilatation current, while in four-dimensional calculations anomalous dimensions occur together with strongly broken dilatation invariance. We show that the situation in two dimensions arises from peculiar phase-space properties which do not generalize to four dimensions. It appears that two-dimensional physics is not a good "laboratory" for studying scale invariance: Even the free-field theory behaves anomalously. We argue that at the present time there is no theoretical or experimental evidence for the consistency of anomalous dimensions with conserved or weakly broken scale invariance in physically realistic models.

## I. INTRODUCTION

IN the contemporary development of the theory and application of scale symmetry in high-energy physics, it has been suggested by Wilson that scale dimensions of operators are dynamically determined quantities, not necessarily equal to their canonical values.[1] It is supposed that this can occur even in a *scale-invariant* theory. Evidence for this phenomenon derives from model calculations. Computations have been performed for exactly solvable, scale-invariant theories in two dimensions,[1,2] as well as for four-dimensional theories, in perturbation theory.[3] However, the latter results do not provide illuminating information about anomalous dimensions in *scale-invariant* theories, because it is found that scale invariance cannot be maintained in perturbative calculations. Thus in these theories the anomalous commutators and Ward identities, which lead to anomalous dimensions, are correlated with anomalous divergences of the dilatation current. This is in complete analogy with the state of affairs previously discovered in conventional current algebra: Commutators and Ward identities involving the axial-vector current are in general anomalous; correspondingly the divergence of the axial-vector current is anomalous.[4]

We have examined the remaining evidence for dynamical dimensions—the two-dimensional calculations. Our conclusion is that, although anomalous dimensions indisputably occur in these models, this happens for reasons which are peculiar to two dimensions, and there is no evidence that they generalize to the physically interesting case of four dimensions. Indeed it is likely that two-dimensional physics is not a good "laboratory" for the study of scale invariance; we demonstrate that even the *free-field theory* behaves anomalously with respect to scale transformations. The purpose of this paper is to report the investigation which has led us to these results.

In Sec. II we summarize the evidence for anomalous scale dimensionality in solvable models. The pathologies of a free, scale-invariant boson field $\varphi$ are explained in Sec. III. It is shown that contrary to canonical expectations, this field does not have zero dimension, but rather the commutator of $\varphi$ with the dilatation generator $D$ yields a singular operator. In Sec. IV we show how the singular behavior of $\varphi$ leads to well-defined, anomalous dimensions for fermion fields $\psi$. Finally, concluding remarks are made in Sec. V.

## II. ANOMALOUS SCALE DIMENSIONS IN TWO-DIMENSIONAL MODELS

Two models are known to exhibit anomalous scale dimension: the Thirring model and the closely related, but simpler, gradient-coupling model.[5] The respective Lagrangians are

$$\mathcal{L}_1 = i\bar{\psi}\gamma^\mu\partial_\mu\psi + g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)\,, \qquad (2.1a)$$

$$\mathcal{L}_2 = i\bar{\psi}\gamma^\mu\partial_\mu\psi + \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + g(\bar{\psi}\gamma^\mu\psi)\partial_\mu\varphi\,. \qquad (2.1b)$$

Both models are exactly solvable. The fermion two-point expectation value is found to be

$$S^{(+)}(x) = \langle 0|\psi(x)\bar{\psi}(0)|0\rangle = S_0^{(+)}(x)(-\epsilon^2x^2+i\eta x_0)^{-\gamma/4\pi}\,, \qquad (2.2a)$$

$$\gamma = (g^2/\pi)(1-g^2/4\pi^2)^{-1}\,, \quad \text{Thirring model} \qquad (2.2b)$$

$$\gamma = g^2\,, \quad \text{gradient-coupling model}. \qquad (2.2c)$$

Here $S_0^{(+)}$ is the free-field version of $S^{(+)}$, and $\epsilon$ is a

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(30-1)2098.

† Alfred P. Sloan Fellow.

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[2] B. Schroer, revised version of a report presented at the Conference on Special Topics in Quantum Field Theory, University of Missouri, St. Louis, Mo., 1970 (unpublished); and Pittsburgh University Report No. NYO-3829-56, 1970 (unpublished); J. H. Lowenstein and B. Schroer, this issue, Phys. Rev. D **3**, 1981 (1971).

[3] K. Wilson, Phys. Rev. D **2**, 1478 (1970); S. Coleman and R. Jackiw, Ann. Phys. (N. Y.) (to be published); R. Jackiw, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, to be published), Vol. XIII; C. G. Callan, Jr., Phys. Rev. D **2**, 1541 (1970); K. Symanzik, Commun. Math. Phys. **18**, 227 (1970).

[4] J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); S. L. Adler, Phys. Rev. **177**, 2426 (1969); R. Jackiw and K. Johnson, *ibid.* **182**, 1459 (1969); S. L. Adler and D. G. Boulware, *ibid.* **184**, 1740 (1968).

[5] The original calculations which at the present time are used to exhibit anomalous dimensions are due to K. Johnson, Nuovo Cimento **20**, 773 (1961) for the Thirring model; and to B. Schroer, Fortschr. Physik **11**, 1 (1963) for the gradient coupling model. In connection with the Thirring model, see also C. R. Hagen, Nuovo Cimento **51B**, 169 (1967).

cutoff which evidently must be introduced to avoid divergences.

In both theories it is possible to introduce a symmetric, traceless energy-momentum tensor $\Theta^{\mu\nu}$ which indicates that scale invariance is present.[6] In scale-invariant theories, two-point functions should behave as $x^{-2d}$, where $d$ is the scale dimension of the field. Examining (2.2), we find that for the fermion field $d$ takes on the noncanonical value

$$d=\tfrac{1}{2}+\gamma/4\pi. \tag{2.3}$$

(Canonical reasoning gives $d=\frac{1}{2}$ for the fermion field, and $d=0$ for the boson field.) Equation (2.3) may be further substantiated by explicitly constructing the dilatation generator $D=\int dx^1 x^1\Theta_{01}$. The commutator of $D$ with $\psi$ is now carefully evaluated; the result is

$$i[D,\psi(x)]=x^\mu\partial_\mu\psi(x)+d\psi(x), \tag{2.4}$$

with $d$ given by (2.3).[2]

One may inquire how it is that the Lagrangian continues to possess the scale-invariant dimension of 2, if $\psi$ no longer has dimension $\frac{1}{2}$. (It can be shown that $\partial^\mu\varphi$ continues to have the dimension 1; see below.) The answer lies in the fact that the anomalies are not distributative: The dimension of $:\psi\bar{\psi}:$, considered as a well-defined (normal-ordered) operator, is not necessarily $2d$. Indeed explicit calculation shows that $:\bar{\psi}\gamma^\mu\psi: = j^\mu$ has dimension 1.[1] This also follows from the commutator satisfied by $j^0$,

$$[\psi(t,x^1),j^0(t,y^1)]=\psi(x)\delta(x^1-y^1). \tag{2.5}$$

In the remainder of this paper we explain how these anomalies arise from the peculiar behavior of a free boson field under scale transformations. We shall concern ourselves only with the simple gradient-coupling model, (2.1b), where the field $\varphi$ is manifestly free. That a free spin-0 field is also present in the Thirring model, albeit in a recondite form, follows from the fact that the current $j^\mu$ is given by the divergence of such a field, since the dual current $j_5{}^\mu=\epsilon^{\mu\nu}j_\nu$ is also conserved:

$$\begin{gathered} j^\mu=\partial^\mu\varphi\,, \\ \partial_\mu j^\mu=\Box\varphi=0. \end{gathered} \tag{2.6}$$

## III. SPIN-0 FIELD IN TWO DIMENSIONS

Consider the Lagrangian

$$\mathcal{L}_3=\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi. \tag{3.1}$$

The theory is formally scale invariant, the boson field carrying the canonical dimension 0. In the usual way, we may expand $\varphi$ in terms of creation and annihilation operators:

$$\varphi(x)=\varphi^+(x)+\varphi^-(x), \tag{3.2a}$$

$$\varphi^\pm(x)=\frac{1}{(2\pi)^{1/2}}\int\frac{dk^1}{(2k_0)^{1/2}}e^{\pm ikx}a^\pm(k^1), \tag{3.2b}$$

$$\begin{gathered} k^2=(k_0)^2-(k_1)^2=0, \\ [a^-(k^1),a^+(l^1)]=\delta(k^1-l^1). \end{gathered} \tag{3.2c}$$

It is trivial to form the two-point function

$$\begin{aligned} D^{(+)}(x)&=\langle 0|\varphi(x)\varphi(0)|0\rangle=\langle 0|\varphi^-(x)\varphi^+(0)|0\rangle \\ &=\frac{1}{4\pi}\int\frac{dk^1}{k_0}e^{-ikx}. \end{aligned} \tag{3.3}$$

However, it is now seen that it is a nontrivial task to perform the indicated integration, since the integral is logarithmically divergent at $k^1=0$. *It is important to notice that this divergence is peculiar to two dimensions. If we were dealing with a four-dimensional, massless boson field, the corresponding integral would converge.*

To make progress, the theory must be regulated. Since we are studying scale-invariance properties, it is necessary to regulate in a scale-invariant fashion. This may be achieved by replacing (3.2b) by a regulated formula:

$$\varphi^\pm(x)=\frac{1}{(2\pi)^{1/2}}\int\frac{dk^1}{(2k_0)^{1/2}}\rho(k_0)e^{\pm ikx}a^\pm(k^1). \tag{3.4}$$

Here $\rho(k_0)$ is a regulator, vanishing at the origin. At the end of the calculation $\rho$ is set to unity whenever possible. The two-point function has now the following form:

$$D^{(+)}(x)=\frac{1}{4\pi}\int\frac{dk^1}{k_0}\rho^2(k_0)e^{-ikx}. \tag{3.5a}$$

If $\rho(k_0)$ is chosen to be $\theta(k_0-\epsilon)$ and $\epsilon\to 0$, one finds

$$D^{(+)}(x)=-\frac{1}{4\pi}\ln(-\epsilon^2x^2+i\eta x_0). \tag{3.5b}$$

In presenting (3.5b), we have dropped unimportant constant terms; i.e., the normalization of $D^{(+)}(x)$ is arbitrary. The same structure appears in $S^{(+)}(x)$, Eq. (2.2).

The regulated field continues to satisfy $\Box\varphi=0$; hence the energy-momentum tensor

$$\Theta^{\mu\nu}=\partial^\mu\varphi\partial^\nu\varphi-\tfrac{1}{2}g^{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi \tag{3.6}$$

remains symmetric, conserved, and traceless. Therefore, a conserved dilatation charge can be constructed. However, one can no longer say that this free field transforms canonically under scale transformations:

$$i[D,\varphi(x)]=x^\alpha\partial_\alpha\varphi(x). \tag{3.7}$$

For if (3.7) were true, the conventional argument would

[6] C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) **59**, 42 (1970).

lead to the conclusion that $D^{(+)}(x)$ is a constant, which is manifestly false. Indeed by scaling the formula for $D^{(+)}(x)$, we find from (3.5a) that

$$x^\alpha\partial_\alpha D^{(+)}(x)=-\frac{1}{4\pi}\int dk^1 e^{-ikx}\frac{\partial}{\partial k_0}\rho^2(k_0)\,, \quad (3.8a)$$

while the specific choice $\rho(k_0)=\rho^2(k_0)=\theta(k_0-\epsilon)$ yields in the limit $\epsilon\to 0$:

$$x^\alpha\partial_\alpha D^{(+)}(x)=-1/2\pi\,. \quad (3.8b)$$

Consequently, the usual application of scale-invariance arguments must fail: Either $D$ does not annihilate the vacuum, or else its commutator with $\varphi$ is noncanonical (or both). We shall establish that it is the latter state of affairs that is true. It should be clear that it is not sufficient to assign a noncanonical dimension to $\varphi$ since $L^{(+)}(x)$ does not exhibit power behavior.

To exhibit the peculiar workings of scale transformations in this free-field theory, we build the dilatation generator. Also we shall need the Poincaré generators $P^\mu$ and $M^{\mu\nu}$. These objects may be constructed in the usual way from (3.4) and (3.6). Evidently we obtain regulated quantities. However, it turns out that the corresponding unregulated formulas are also well defined, so we prefer to work with these simpler expressions. [We emphasize that the subsequent development may be equally well carried out with the regulated generators, provided $\rho(k_0)$ is chosen to be $\theta(k_0-\epsilon)$.]

$$P^\mu=\int dk^1 k^\mu a^+(k^1)a^-(k^1)\,, \quad (3.9a)$$

$$M^{01}=\frac{i}{2}\int dk^1\bigg[\bigg(k^0\frac{\partial}{\partial k^1}a^+(k^1)\bigg)a^-(k^1) -a^+(k^1)\bigg(k^0\frac{\partial}{\partial k^1}a^-(k^1)\bigg)\bigg], \quad (3.9b)$$

$$D=\frac{i}{2}\int dk^1\bigg[\bigg(k^1\frac{\partial}{\partial k^1}a^+(k^1)\bigg)a^-(k^1) -a^+(k^1)\bigg(k^1\frac{\partial}{\partial k^1}a^-(k^1)\bigg)\bigg]. \quad (3.9c)$$

It is clear that the generators annihilate the vacuum; also the appropriate Lie algebra is satisfied.

We now inquire how the field $\varphi$ transforms. The translation generators provide no surprises, but the Lorentz and dilatation generators produce extra terms:

$$i[P^\mu,\varphi^\pm(x)]=\partial^\mu\varphi^\pm(x)\,, \quad (3.10a)$$

$$i[M^{01},\varphi^\pm(x)]=(x^0\partial^1-x^1\partial^0)\varphi^\pm(x) +\int\frac{dk^1}{(4\pi k_0)^{1/2}}a^\pm(k^1)e^{\pm ikx}k^1\frac{\partial}{\partial k_0}\rho(k_0)\,, \quad (3.10b)$$

$$i[D,\varphi^\pm(x)]=x^\alpha\partial_\alpha\varphi^\pm(x) +\int\frac{dk^1}{(4\pi k_0)^{1/2}}a^\pm(k^1)e^{\pm ikx}k^0\frac{\partial}{\partial k_0}\rho(k_0)\,. \quad (3.10c)$$

The additional terms in the commutators with $M^{01}$ and $D$ seem to vanish as $\rho\to 1$. Indeed if we take $\rho(k_0)$ to be $\theta(k_0-\epsilon)$ we find, as $\epsilon\to 0$,

$$i[M^{01},\varphi^\pm(x)] =(x^0\partial^1-x^1\partial^0)\varphi^\pm(x) +(\epsilon/4\pi)^{1/2}[a^\pm(\epsilon)-a^\pm(-\epsilon)]\,, \quad (3.11a)$$

$$i[D,\varphi^\pm(x)] =x^\alpha\partial_\alpha\varphi^\pm(x)+(\epsilon/4\pi)^{1/2}[a^\pm(\epsilon)+a^\pm(-\epsilon)]\,. \quad (3.11b)$$

However, the limit $\epsilon\to 0$ cannot be taken with impunity; a nonvanishing contribution may remain if $a^\pm(\epsilon)\pm a^\pm(-\epsilon)$ possesses matrix elements which behave as $\epsilon^{-1/2}$. We now show that the Lorentz generator anomaly vanishes with $\epsilon$; however, the dilatation anomaly survives in this limit. This is not surprising, since it is seen from (3.11) that the former vanishes more rapidly than the latter.

Consider the two-point function again. From (3.10b) we have

$$\begin{aligned}0&=i\langle 0|[M^{01},\varphi^-(x)\varphi^+(0)]|0\rangle=(x^0\partial^1-x^1\partial^0)D^{(+)}(x)\\ &+\int\frac{dk^2}{(4\pi k_0)^{1/2}}e^{-ikx}k^1\frac{\partial}{\partial k_0}\rho(k_0)\langle 0|a^-(k^1)\varphi^+(0)|0\rangle\\ &+\int\frac{dk^1}{(4\pi k_0)^{1/2}}k^1\frac{\partial\rho(k_0)}{\partial k_0}\langle 0|\varphi^-(x)a^+(k^1)|0\rangle\\ &=(x^0\partial^1-x^1\partial^0)D^{(+)}(x)+\int\frac{dk^1}{4\pi k_0}e^{-ikx}k^1\frac{\partial}{\partial k_0}\rho^2(k_0)\,. \quad (3.12)\end{aligned}$$

The derivative of the regulator has support only for small $k_0$. Hence $e^{-ikx}$ may be replaced by 1, and the integral then vanishes by symmetry. This general argument may be verified by setting $\rho(k_0)=\rho^2(k_0)=\theta(k_0-\epsilon)$, and letting $\epsilon\to 0$. Of course we are merely confirming the fact that (3.5b) is covariant.

The analogous computation with $D$ yields

$$\begin{aligned}0&=i\langle 0|[D,\varphi^-(x)\varphi^+(0)]|0\rangle=x^\alpha\partial_\alpha D^{(+)}(x)\\ &+\int\frac{dk^1}{(4\pi k_0)^{1/2}}e^{-ikx}k^0\frac{\partial}{\partial k^0}\rho(k_0)\langle 0|a^-(k^1)\varphi^+(0)|0\rangle\\ &+\int\frac{dk^1}{(4\pi k_0)^{1/2}}k^0\frac{\partial}{\partial k^0}\rho(k_0)\langle 0|\varphi^-(x)a^+(k)|0\rangle \quad (3.13a)\end{aligned}$$

or

$$x^\alpha\partial_\alpha D^{(+)}(x)=-\frac{1}{4\pi}\int dk^1 e^{-ikx}\frac{\partial}{\partial k_0}\rho^2(k_0)\,. \quad (3.13b)$$

This verifies (3.8a). A nonvanishing result remains even in the limit $\rho\to 1$, as is seen from (3.8b). We may also compute the commutator of $D$ with $\partial^\mu\varphi$. Here everything works out canonically, as can already be seen from the two-point function for $\partial^\mu\varphi$:

$$\langle 0|\partial^\mu\varphi^-(x)\partial^\nu\varphi^+(0)|0\rangle = -\partial^\mu\partial^\nu D^{(+)}(x)$$
$$= \frac{1}{\pi}\left(\frac{x^\mu x^\nu}{(x^2-i\eta x_0)^2} - \frac{g^{\mu\nu}}{2(x^2-i\eta x_0)}\right) \sim (x)^{-2}. \quad (3.14)$$

We may therefore replace the commutators (3.10) by their effective values as $\rho \to 1$:

$$i[P^\mu,\varphi^\pm(x)] = \partial^\mu\varphi^\pm(x), \quad (3.15a)$$
$$i[M^{01},\varphi^\pm(x)] = (x^0\partial^1 - x^1\partial^0)\varphi^\pm(x), \quad (3.15b)$$
$$i[D,\varphi^\pm(x)] = x^\alpha\partial_\alpha\varphi^\pm(x) + A^\pm, \quad (3.15c)$$
$$i[D,\partial^\mu\varphi^\pm(x)] = \partial^\mu\varphi^\pm(x) + x^\alpha\partial_\alpha\partial^\mu\varphi^\pm(x). \quad (3.15d)$$

The anomaly is

$$A^\pm = \int \frac{dk^1}{(4\pi k_0)^{1/2}} a^\pm(k^1) e^{\pm ikx} k^0 \frac{\partial}{\partial k_0}\rho(k_0)$$
$$\to \int \frac{dk^1}{(4\pi k_0)^{1/2}} a^\pm(k^1) k^0 \frac{\partial}{\partial k_0}\rho(k_0)$$
$$\to (\epsilon/4\pi)^{1/2}[a^\pm(\epsilon) + a^\pm(-\epsilon)]. \quad (3.16)$$

Our conclusion therefore is that the scale-invariant boson free field possess a singular commutator with the dilatation generator. This singular operator has the appropriate matrix elements, so that the scaling equation for the $n$-point function agrees with explicit computation. The phenomenon derives from special phase-space properties of two-dimensional physics, and does not occur in higher dimensions.

## IV. DIMENSION OF FERMION FIELD

The Lagrangian (2.1b) possesses the formal solution $\psi = e^{ig\varphi}\psi_0$, where $\varphi$ is a free boson field and $\psi_0$ is a free fermion field. The well-defined quantum-mechanical solution is obtained by normal ordering,

$$\psi(x) = e^{ig\varphi^+(x)}\psi_0(x)e^{ig\varphi^-(x)}. \quad (4.1)$$

Using the results of Sec. III, we may compute the scale dimension of $\psi$. The dilatation generator $D$ now has two pieces, one generating scale transformations on the free fermion field $\psi_0$, the other performing the same task for $\varphi$. There are no scale anomalies associated with $\psi_0$; the scale dimension remains $\frac{1}{2}$, as is seen from the two-point function

$$\langle 0|\psi_0(x)\bar\psi_0(0)|0\rangle = S_0^{(+)}(x)$$
$$= i\gamma_\alpha\partial^\alpha D^{(+)}(x)$$
$$= -\frac{1}{2\pi}\frac{\gamma_\alpha x^\alpha}{x^2-i\eta x_0} \sim x^{-1}. \quad (4.2)$$

Therefore, we find

$$i[D,\psi(x)] = (\tfrac{1}{2} + x^\alpha\partial_\alpha)\psi(x) - igx^\alpha\partial_\alpha\varphi^+(x)\psi(x)$$
$$-ig\psi(x)x^\alpha\partial_\alpha\varphi^-(x)$$
$$+i[D,e^{ig\varphi^+(x)}]e^{-ig\varphi^+(x)}\psi(x)$$
$$+\psi(x)e^{-ig\varphi^-(x)}i[D,e^{ig\varphi^-(x)}]. \quad (4.3a)$$

The remaining commutators with $D$ are evaluated from (3.15c):

$$i[D,\psi(x)] = (\tfrac{1}{2} + x^\alpha\partial_\alpha)\psi(x) + igA^+\psi(x) + ig\psi(x)A^-. \quad (4.3b)$$

The singular contribution to (4.3b) may be extracted by scaling $\psi(x)\bar\psi(0)$:

$$i[D,\psi(x)\bar\psi(0)]$$
$$= (1 + x^\alpha\partial_\alpha)\psi(x)\bar\psi(0) + ig[A^+,\psi(x)]\bar\psi(0) + ig\psi(x)[A^-,\bar\psi(0)]$$
$$= (1 + x^\alpha\partial_\alpha)\psi(x)\bar\psi(0) + ig\psi(x)e^{-ig\varphi^-(x)}[A^+,e^{ig\varphi^-(x)}]\bar\psi(0)$$
$$+ig\psi(x)[A^-,e^{-ig\varphi^+(0)}]e^{ig\varphi^+(0)}\bar\psi(0). \quad (4.4)$$

From (3.16) it follows that

$$e^{-ig\varphi^-(x)}i[A^+,e^{ig\varphi^-(x)}] = \frac{g}{8\pi}\int dk^1 \frac{\partial}{\partial k_0}\rho^2(k_0)e^{-ikx}$$
$$\to \frac{g}{8\pi}\int dk^1\frac{\partial}{\partial k_0}\rho^2(k_0), \quad (4.5a)$$

$$i[A^-,e^{-ig\varphi^+(0)}]e^{ig\varphi^+(0)} = \frac{g}{8\pi}\int dk^1\frac{\partial}{\partial k_0}\rho^2(k_0). \quad (4.5b)$$

Therefore, (4.4) becomes

$$i[D,\psi(x)\bar\psi(0)]$$
$$= \left(1 + \frac{g^2}{4\pi}\int dk^1 \frac{\partial}{\partial k_0}\rho^2(k_0) + x^\alpha\partial_\alpha\right)\psi(x)\bar\psi(0) \quad (4.6a)$$
$$\to (1 + g^2/2\pi + x^\alpha\partial_\alpha)\psi(x)\bar\psi(0). \quad (4.6b)$$

In (4.6b) we have set $\rho(k_0) = \rho^2(k_0) = \theta(k_0 - \epsilon)$ and let $\epsilon \to 0$. Note that the dynamical dimension of $\psi(x)\bar\psi(0)$ involves exactly the same expression,

$$\frac{1}{4\pi}\int dk^1\frac{\partial}{\partial k_0}\rho^2(k_0),$$

as the anomalous scaling law for $D^{(+)}(x)$. This, of course, is as it should be; the fermion anomaly is a consequence of the free boson anomaly. Evidently (4.3b) may be replaced by the effective transformation law, which verifies (2.3):

$$i[D,\psi(x)] = (\tfrac{1}{2} + g^2/4\pi + x^\alpha\partial_\alpha)\psi(x). \quad (4.7)$$

In terms of our formalism, we may also understand the nondistributive property of dynamical scale dimensions. Although

$$\bar{\psi}(x)\gamma^\mu\psi(x)=e^{-ig\varphi^+(x)}\bar{\psi}_0(x)e^{-ig\varphi^-(x)}\gamma^\mu \times e^{ig\varphi^+(x)}\psi_0(x)e^{ig\varphi^-(x)} \quad (4.8a)$$

has anomalous dimensions, the normal-ordered product

$$J^\mu(x)=:\bar{\psi}(x)\gamma^\mu\psi(x):=:\bar{\psi}_0(x)\gamma^\mu\psi_0(x): \quad (4.8b)$$

is independent of the boson field, and carries the canonical dimension 1.

## V. CONCLUSION

Our purpose here is to demonstrate that the scaling behavior of two-dimensional models is not a useful "laboratory" for discovering the nature of dilatation symmetry in realistic, four-dimensional field theories. Thus we conclude that at the present time there is no reason to believe that anomalous, noncanonical scale dimensions are consistent with a *conserved* or *weakly broken* scale-invariance current. Perturbative calculations in four dimensions, which indicate that anomalous dimensions exist, also show that scale invariance is *strongly broken*: The divergence of the dilatation current is not a "gentle" operator involving mass terms, but rather it is a complicated $q$-number operator, whose reduced matrix elements diverge quadratically at large momentum.

Neither do the renormalization-group arguments appear to provide useful information. In the $\lambda\varphi^4$ theory, for example, one can derive the following formula[7]:

$$[p^\alpha\partial_\alpha+4-2d(\lambda)]G_\infty(p)=\psi(\lambda)\frac{\partial}{\partial\lambda}G_\infty(p). \quad (5.1)$$

Here $G_\infty(p)$ is the asymptotic form for the boson field propagator, $d(\lambda)$ is the dynamically dependent scale dimension of the boson field, and $\psi(\lambda)$ is the Gell-Mann–Low eigenvalue function. For arbitrary $\lambda$, the nonvanishing of the right-hand side indicates that scale invariance is strongly broken. Moreover, even if it is supposed that there exists a $\lambda_0$ such that $\psi(\lambda_0)$ vanishes, no information about $d(\lambda_0)$ is available. In the first place, it is not impossible that $d(\lambda_0)$ is equal to its canonical value. More importantly, it is not clear, that even if $\psi(\lambda_0)=0$, it follows that weakly broken scale invariance can be implemented in an operator fashion.

It is hoped the nature of the scale dimension can be studied experimentally. In this connection, it is useful to remark that the MIT-SLAC electroproduction experiments,[8] which gave the initial impetus to the study of scale symmetry, provide no support for the hypothesis of dynamical dimensionality; indeed the hypothesis appears troublesome in that context. The reason for this is the following.[9] The electroproduction cross sections are parametrized by two scaling functions $F_L(\omega)$ and $F_2(\omega)$ ($\omega=-q^2/2pq$; $q$=photon four-momentum, $p$=target four-momentum). Experimentally, it is indicated that moments of the $F_i(\omega)$ are finite:

$$0\leqslant\int_0^1 d\omega\,\omega^{2n}F_i(\omega)<\infty\,,\quad n=0,1,\ldots.$$

On the other hand, it is also possible to relate these moments to forward matrix elements of equal-time commutators of time derivatives of the electromagnetic current $J^j$:

$$\int_0^1 d\omega\,\omega^{2n}F_i(\omega)\propto\lim_{p_0\to\infty}\frac{1}{(p_0{}^2)^{n+1}} \times\int d^3x\langle p|\left[\frac{\partial^{2n+1}}{\partial t^{2n+1}}J^j(t,\mathbf{x}),J^k(t,\mathbf{0})\right]|p\rangle. \quad (5.2)$$

Therefore, the finiteness of the left-hand side of (5.2) imposes constraints on the coefficient of the $\delta$ function in the equal-time commutator on the right-hand side of (5.2). This constraint may be summarized by the statement that there *must* exist operators of spin $S$ and scale dimensionality $S+2$, $S=2n$, whose forward matrix element are nonvanishing. Furthermore, since the proton data appear to differ from the neutron data, these operators are not isoscalars. Therefore, if scale dimensionality is a dynamical quantity, it is difficult to understand why it nevertheless happens that an infinite set of operators exists with integer dimensionality.

To summarize, we believe that at the present time there is no theoretical or experimental evidence for anomalous scale dimensions. It should also be noted that recent studies of the scale dimension of the $SU(3)\times SU(3)$ symmetry breaking of the Hamiltonian indicate that if that quantity is 1, the behavior of certain amplitudes in the low-energy domain is maximally smooth.[10]

## ACKNOWLEDGMENTS

This investigation arose from conversations with Professor B. Schroer and Dr. J. Lowenstein, which I am pleased to acknowledge.

[7] S. Coleman and R. Jackiw (Ref. 3).

[8] E. D. Bloom *et al.*, Phys. Rev. Letters **23**, 930 (1969) and in *Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, 1970* (Academy of Science, IUPAP, Moscow, 1971).

[9] The argument presented here is well known. In the literature it appears implicitly in the work of Wilson, Ref. 1; S. Ciccariello, R. Gatto, G. Sartori, and M. Tonin, Phys. Letters **30B**, 546 (1969); G. Mack, Phys. Rev. Letters **25**, 400 (1970); and explicitly in G. Mack, ICTP Report No. IC 70 95, 1970 (unpublished). I learned about it through conversations with Professor David J. Gross.

[10] R. Jackiw, Phys. Rev. D **3**, 1351 (1971); G. Segrè, *ibid.* D **3**, 1303 (1971).