TABLE I. Effective polytropic indices of degenerate matter for different equations of state. (i) An ideal, Fermi gas consisting of three noninteracting components: protons, neutrons, and electrons (see Harrison *et al.*, Ref. 6). (ii) A cold, degenerate mixture of neutrons, protons, leptons, hyperons, and massive baryons including the nucleon-nucleon interactions of Levinger and Simmons (see Langer *et al.*, Ref. 6). (iii) Equation of state for cold degenerate matter taking into account clustering and nuclear forces (see Cohen *et al.*, Ref. 6).

(i) ρ (g/cm^3)	P (dyne/cm^2)		n_{eff}
4.63×10^{12}	6.72×10^{30}		1.52
8.00×10^{12}	1.66×10^{31}		1.52
1.00×10^{13}	2.40×10^{31}		1.52
5.00×10^{13}	4.60×10^{32}		1.57
1.00×10^{14}	1.99×10^{33}		1.62
2.00×10^{14}	2.09×10^{34}		1.77
2.51×10^{15}	1.65×10^{35}		2.09
2.16×10^{16}	3.26×10^{36}		3.45
2.44×10^{17}	5.87×10^{37}		9.01
5.05×10^{17}	1.30×10^{38}		12.75
1.52×10^{18}	4.18×10^{38}		22.25
(ii)	A	B	n
V_α potential for nucleon-nucleon interaction	1.68×10^{-2}	2.46	0.68
V_γ potential	3.64×10^{-3}	2.52	0.66
(iii)	A	B	n
V_α potential	2.42	2.32	0.75

ρ given. The method is basically one of least-squares fit, and the results using different equations of state are summarized in Table I. We have included for purposes of comparison the case of an ideal Fermi gas consisting of three noninteracting components: neutrons, protons, and electrons (see Harrison, Thorne, Wakano, and Wheeler in Ref. 6). The effective indices there are determined for various densities by the equation $d\ln P/d\ln\rho=n+1$, and it can be seen that the index is always larger than 1, as expected. The other two equations are based on much more elaborate details of interaction in nuclear matter. Both make use of the V_α, V_γ nucleon-nucleon interaction potential of Levinger and Simmons,[7] and in addition, one equation takes into account the effects of nuclear clustering, and the other the existence of hyperons and massive baryons. The effective index n_{eff} is now lowered to a value of ~0.7. Qualitatively similar results are also arrived at by Tsuruta and Cameron,[8] who based their calculations on equations of state characterized just by the Levinger-Simmons potentials, and who assigned effective indices by comparing the ratio of the mean density to the central density. We believe our results are more reliable both because of the improved equations of state used and because of the better method for assigning indices.

To the extent that the classical criterion for bifurcation is applicable, we therefore conclude that a neutron star can deform into a triaxial, nonaxisymmetric Jacobi configuration.

We thank Professor R. N. Henriksen for many useful discussions in the course of this investigation.

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Divergence of the N-Loop Planar Graph in the Dual-Resonance Model*

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Many of the results of a previous paper on the divergence of the two-loop planar graph are generalized to the multiloop case. In particular, the N-loop amplitude is shown to diverge in at least 2^N-1 places, though no attempt is made to show that these are the only divergences. The implication of this result to the renormalization program is discussed.

I. INTRODUCTION

NOW that the connection between the multiloop dual-resonance amplitudes[1–3] and the analog model[2,3] has been made, we find compelling reasons to believe that the singularities of the N-loop integrand are associated with a change of the topology of the corresponding Riemann surface. In the single-loop case, for example, the associated surface is a disk with a hole; the singularity of the partition function arises when the hold shrinks to a point. Likewise, we predict that the

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double-loop amplitude contains three singularities, two associated with the individual holes shrinking to points, and one associated with the two circles touching each other. In general, we expect 2^N-1 divergent factors in the N-loop case, corresponding to the shrinking of the individual circles and the various ways in which these circles can touch.

Though this picture gives a simple and intuitive approach to the problem of isolating the divergences of the multiloop amplitude, the actual problem is quite tediously involved, because of the existence of an infinite number of potential singularities, each associated with a closed path of the Kikkawa-Sakita-Virasoro[4] (KSV) dual diagram.

In a previous paper,[5] we rigorously proved that the double-loop amplitude diverged in only three places, corresponding to the set of closed paths which did not cross themselves. In this paper, we make no attempt to prove that all self-crossing lines do not diverge because the method of isometric circles, which was employed to get advantage in the double-loop case, fails for the arbitrary case. Because only two projective transformations can have isometric circles with unit radii at any given time, we can only generalize our results to the paths which encircle only two internal points of the dual diagram.

In Sec. II, we express the singularity in terms of projectively invariant cross ratios of invariant points. Particular emphasis is given to the complicated interrelations among these cross ratios. In Sec. III, we actually construct the 2^N-1 singularities found in the N-loop planar graph.

The cataloging of these divergences is critical to the renormalization program.[6] It has been shown that the additional divergence appearing in the double-loop case can be renormalized by introducing appropriate counterterms similar to the one defined in the single-loop case.[7]

II. MULTILOOP SINGULARITY

In our previous paper,[5] we showed that it is possible to express the multiplier of the product of two projective transformations in terms of the individual multipliers and invariant or fixed points. We found that if two transformations P_1 and P_2 have multipliers given by W_1 and W_2 and invariant points given by (x_1,x_1') and (x_2,x_2'), respectively, then the multiplier of the product $(W_{P_1P_2})$ is given by

$$W_{P_1P_2}{}^{1/2}+W_{P_1P_2}{}^{-1/2}=\mathrm{Tr}(P_1P_2)\,[\det(P_1P_2)]^{-1/2}=|W_1+W_2+\rho_{12}(1-W_1)(1-W_2)|\times(W_1W_2)^{-1/2}, \quad (2.1)$$

where

$$\rho_{12}=\frac{(x_1-x_2')(x_2-x_1')}{(x_1-x_1')(x_2-x_2')}.$$

The multiloop singularity, unfortunately, involves multipliers of large numbers of projective transformations; in fact, the singularity includes an arbitrarily large number of products, one for each closed path of the dual diagram:

$$\prod_{\{R\}}(1-W_R)^{-4}, \quad (2.2)$$

where $\{R\}$ is the group generated by N projective transformations P_i and their inverses, such that cyclic permutations and inverses of such permutations of previously included members of $\{R\}$ are omitted. Our first task, then, is to generalize (2.1) for the arbitrary case. Once the multiplier of the product of three projective transformations is analyzed in detail, it is not hard to generalize to the general case:

$$W_{P_1{}^{\alpha(1)}P_2{}^{\alpha(2)}\cdots P_N{}^{\alpha(N)}}{}^{1/2}+W_{P_1{}^{\alpha(1)}P_2{}^{\alpha(2)}\cdots P_N{}^{\alpha(N)}}{}^{-1/2}=\frac{\mathrm{Tr}(P_1{}^{\alpha(1)}P_2{}^{\alpha(2)}\cdots P_N{}^{\alpha(N)})}{[\det(P_1{}^{\alpha(1)}P_2{}^{\alpha(2)}\cdots P_N{}^{\alpha(N)})]^{1/2}}$$
$$=\Bigg|2-\sum_{i=1}^{N}(1-W_i{}^{\alpha(i)})+\sum_{i>j}^{N}\rho_{ij}(1-W_i{}^{\alpha(i)})(1-W_j{}^{\alpha(j)})-\sum_{i>j>k}^{N}\rho_{ijk}(1-W_i{}^{\alpha(i)})(1-W_j{}^{\alpha(j)})(1-W_k{}^{\alpha(k)})+\cdots+(-1)^N\rho_{123\cdots N}\prod_{i=1}^{N}(1-W_i{}^{\alpha(i)})\Bigg|\Big(\prod_{i=1}^{N}W_i{}^{\alpha(i)}\Big)^{-1/2}, \quad (2.3)$$

where

$$\rho_{ij\cdots m}=\frac{(x_i-x_j')(x_j-x_k')\cdots(x_m-x_i')}{(x_i-x_i')(x_j-x_j')\cdots(x_m-x_m')},\ \alpha(i)\text{ are integers.}$$

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Our task, then, is to find solutions of the equation

$$W_{P_1P_2\cdots P_N}=1\,, \tag{2.4}$$

subject to the conditions implied by (2.3).

Since the proof of (2.3) proceeds easily by induction, we will sketch the proof. Because we know that (2.1) is valid, we will take $(P_1P_2\cdots P_{N-1})$ and P_N as our two projective transformations:

$$\begin{aligned}W_{(P_1P_2\cdots P_{N-1})P_N}{}^{1/2}&+W_{(P_1P_2\cdots P_{N-1})P_N}{}^{-1/2}\\&=|W_{12\cdots N-1}+W_N+\rho_{(12\cdots N-1)N}(1-W_N)\\&\qquad\times(1-W_{(12\cdots N-1)})|(W_NW_{(12\cdots N-1)})^{-1/2}\\&\equiv D_N(W_NW_{(12\cdots N-1)})^{-1/2}\,.\end{aligned} \tag{2.5}$$

We see immediately that D_N is linear in W_N. But we could equally well have chosen P_1 and $(P_2P_3\cdots P_N)$ as our two projective transformations, or any other cyclic permutation. Therefore, D_N must be linear in all W_i. We can always choose D_N to be of the form

$$\begin{aligned}D_N=A+\sum_{i=1}^N\alpha_iW_i+\sum_{i>j}^N\sigma_{ij}(1-W_i)(1-W_j)\\-\sum_{i>j>k}^N\sigma_{ijk}(1-W_i)(1-W_j)(1-W_k)+\cdots\\+(-1)^N\sigma_{12\cdots N}\prod_{i=1}^N(1-W_i)\,,\end{aligned} \tag{2.6}$$

where A, α_i, and the σ's are unspecified functions of the invariant points x_i and x_i'. We must prove the equivalence of these functions with the ρ's. Assume now that (2.6) is valid for all orders less than or equal to $N-1$. In the limit $W_i\to 1$, D_N (which is as yet undetermined) must reduce to D_{N-1}. It is easy to see that D_N reduces to D_{N-1} if we can equate the A, α_i, and σ's to the corresponding ρ's. The only undetermined function is $\sigma_{12\cdots N}$, because this factor vanishes whenever one of the W_i is set equal to 1. We obtain conditions on this last undertermined function if we let $x_N=x_{N-1}$ and $x_N'=x_{N-1}'$ (i.e., $W_{N,N-1}=W_NW_{N-1}$). In this limit, we have

$$D_N(W_1,W_2,\ldots,W_N)\to D_{N-1}(W_1,W_2,\ldots,W_{N-1}W_N)\,,$$

so therefore

$$\sigma_{12\cdots N}(x_N=x_{N-1},\,x_N'=x_{N-1}')\to\sigma_{12\cdots N-1}\,.$$

Likewise, in the case $x_N=x_{N-1}'$ and $x_N'=x_{N-1}$ (i.e., $W_{N-1,N}=W_{N-1}W_N{}^{-1}$) we have

$$D_N(W_1,W_2,\ldots,W_N)\to D_{N-1}(W_1,W_2,\ldots,W_{N-1}W_N{}^{-1})$$

and therefore

$$\sigma_{12\cdots N}(x_N=x_{N-1}',\,x_N'=x_{N-1})\to 0\,.$$

There is only one function that satisfies these two conditions along with cyclic symmetry and certain linearity conditions imposed by (2.5), and this function is the cross ratio. Q.E.D.

Now that (2.3) is proven we look for solutions of (2.4). Unfortunately, the problem is complicated because of certain interrelations that exist among ρ's of the same order and ρ's of different orders. Our goal is to express (2.3) entirely in terms of a fundamental set of variables, say, the set of all second-order ρ's.

We shall list some of these relations. It is not hard to show that both ρ_{123} and ρ_{321} are solutions of the quadratic equation

$$k^2-k(\rho_{12}+\rho_{23}+\rho_{31}-1)+\rho_{12}\rho_{23}\rho_{31}=0 \tag{2.7}$$

and that ρ_{1234} and ρ_{4321} are solutions of

$$\begin{aligned}k^2-k(\rho_{12}\rho_{34}+\rho_{23}\rho_{41}-\rho_{13}\rho_{24}+\rho_{24}+\rho_{13}-1)\\+\rho_{12}\rho_{23}\rho_{34}\rho_{41}=0\,.\end{aligned} \tag{2.8}$$

Also, we have the condition

$$\rho_{ij\cdots mn}\rho_{nm\cdots ji}=\rho_{ij}\rho_{jk}\cdots\rho_{mn}\rho_{ni}\,. \tag{2.9}$$

To further complicate the situation, it can be shown that the second-order cross ratios obey the KSV duality conditions. For example, let $\rho_{i,i+1}$ form the sides of an N-sided polygon, while the other $\rho_{i,j}$ form the various diagonals. The relationship between crossed lines of the polygon form the usual KSV relations. If we let $x_1=0$, $x_1'=\infty$, and $x_2=1$, we can explicitly calculate all ρ's of second (and higher) order in terms of a fundamental set of second-order ρ's.

Fortunately, there are two important reasons why our goal is feasible.

(1) If we take the limit $W_i\to 1$ (i.e., $x_i\to x_i'$),[8] then all higher-order ρ's reduce to simple products of second-order ρ's. For example, multiply both sides of (2.9) by products of $1-W_i$ and then take the limit as all W_i's tend to 1:

$$\begin{aligned}\lim_{W\to1}\rho_{i_1i_2\cdots i_N}\prod_{j=1}^N(1-W_j)\lim_{W\to1}\rho_{i_Ni_{N-1}\cdots i_1}\prod_{k=1}^N(1-W_k)\\=(-1)^N\bar\rho_{i_1i_2\cdots i_N}{}^2=\bar\rho_{i_1i_2}\bar\rho_{i_2i_3}\cdots\bar\rho_{i_Ni_1}\,,\end{aligned} \tag{2.10}$$

where

$$\bar\rho_{i_1i_2\cdots i_N}=\lim_{W\to1}\rho_{i_1i_2\cdots i_N}\prod_{j=1}^N(1-W_j)\,.$$

In the limit $W_i\to 1$, we see that our problem is reduced to evaluating all second-order $\bar\rho$'s.

(2) The second reason why our task simplifies is because higher-order ρ's decompose into products of lesser ρ's if any of the indices are repeated:

$$\rho_{ijkl\cdots mnjpqr}=\rho_{ijpqr}\rho_{kl\cdots mnj}\,. \tag{2.11}$$

[8] If linear dependences are neglected, then the projective transformations reduce to the identity in the limit where the multiplier goes to 1. Because of linear dependences, however, the invariant points become equal in the limit where the multiplier goes to 1, in which case the transformation is said to be parabolic. Notice that in (2.3), if all multipliers are set to 1, then we always have a solution for (2.4), meaning that all closed paths of the dual diagram diverge in the case where linear dependences are ignored.

Since self-crossing paths involve repeating one or more of the N projective transformations in the singularity (2.2), we will be able to reduce the complexity of the problem once we know (2.11). In the triple-loop case, for example, a knowledge of ρ_{12}, ρ_{23}, ρ_{31}, ρ_{123}, and ρ_{321} completely determines the nature of all paths.

III. CONSTRUCTION OF SOLUTIONS OF (2.4)

Now that all the mathematical preliminaries have been presented, we will explicitly display solutions of (2.4) in terms of the original N-loop variables. We will refer extensively to the notation of our previous paper.

In that paper, we showed that $W_{12}\to 1$ independently of the specific values of W_1 and W_2, but that the equations simplified enormously if we took the limit W_1 and $W_2\to 1$. We will take the limit $W_i\to 1$ in order to take advantage of (2.10).

As before, we find that (see Fig. 1)

$$P_i=z_i\begin{pmatrix}1 & -E_i\\ 1 & -E_i(1-\bar{u}_i)\end{pmatrix}z_i^{-1},$$

$$x_i=\text{invariant point}=P_i(x_i)=z_iE_it_i^{-1},$$

$$x_i'=\text{invariant point}=P_i(x_i')=z_it_i,$$

$$E_i=t_i(1-t_i)/[1-t_i(1-\bar{u}_i)],$$

$$\bar{u}_i=u_i(1-y_i),$$

$$z_i=y_i\prod_{j=1}^{i-1}y_jE_j(1-u_j),$$

$$W_i=\text{multiplier}=\bar{u}_it_i(1-t_i)^{-1}[1-t_i(1-\bar{u}_i)]^{-1},$$

$$0\leq y_i,\ u_i,\ t_i\leq 1,\quad 0\leq W_i\leq 1$$

and

$$x_1\geq x_1'\geq x_2\geq x_2'\geq\cdots\geq x_N'\,. \tag{3.1}$$

In the limit $W_i\to 1$, we find that

$$\bar{\rho}_{ij}=\lim_{W\to 1}\rho_{ij}(1-W_i)(1-W_j)=-\frac{(1-v_{ij}t_it_j)^2}{(1-t_i)(1-t_j)v_{ij}}, \tag{3.2}$$

where

$$v_{ij}=z_iz_j^{-1}E_j^{-1}\quad(\text{i}>j)\,. \tag{3.3}$$

In the double-loop case, we have a solution of (2.4) if we let $v_{12}\to 1$; then $t_1=t_2\to 1$ (i.e., $\bar{\rho}_{12}=-4$). In the triple-loop case, we must perform a more complicated limiting procedure:

$$\begin{aligned}t_1=t_2=t_3\cong E_1^{1/2}=E_2^{1/2}=E_3^{1/2}\cong 1-\bar{u}_1^{1/2}=1-\bar{u}_2^{1/2}\\=1-\bar{u}_3^{1/2}=1-u_2=y_2\to 1\,,\end{aligned} \tag{3.4}$$

$$1-y_2\gg u_1\,,\quad 1-y_3\ll u_2\,.$$

In this limit, we find that $\bar{\rho}_{12}=-9$, $\bar{\rho}_{23}=-9$, and $\bar{\rho}_{31}=-36$.

It is not hard to generalize from the triple-loop case

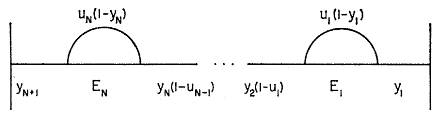

FIG. 1. Variables found in the multiloop calculation.

to the general case:

$$\begin{aligned}t_i=t_{i+1}\cong E_i^{1/2}=E_{i+1}^{1/2}\cong 1-\bar{u}_i^{1/2}=1-\bar{u}_{1+1}^{1/2}\cong y_j\\=y_{j+1}=1-u_k=1-u_{k+1}\to 1\end{aligned} \tag{3.5}$$

for $i=1,\ \ldots,\ N-1$, $j=2,\ \ldots,\ N-2$, $k=2,\ \ldots,\ N-1$, and $1-y_2\gg u_1$ and $1-y_N\ll u_{N-1}$.

With this particular choice of limits, we easily find

$$\begin{aligned}\bar{\rho}_{ij}&=-[4(i-j)]^2\,, && i,j\neq 1 \text{ or } N\\ \bar{\rho}_{Ni}&=-[4(N-i)-1]^2\,, && i\neq 1\\ \bar{\rho}_{i1}&=-[4(i-1)-1]^2\,, && i\neq N\\ \bar{\rho}_{N1}&=-[4(N-1)-2]^2\,.\end{aligned} \tag{3.6}$$

The proof that (3.6) satisfies (2.4) is long and tedious, so only the highlights of the proof will be presented. In general, we wish to perform summations like the following:

$$\begin{aligned}\lim_{W\to 1}\sum_{i>j>k=2}^{N-1}&\rho_{ijk}(1-W_i)(1-W_j)(1-W_k)\\&=\sum_{i>j>k=2}^{N-1}-(\bar{\rho}_{ij}\bar{\rho}_{jk}\bar{\rho}_{ki})^{1/2}\\&=\sum_{i>j>k=2}^{N-1}4^3(i-j)(j-k)(k-i)\,.\end{aligned} \tag{3.7}$$

Clearly, our task is to find a closed expression for

$$\sum_{i_1>i_2>\cdots>i_n=2}^{N-1}(i_1-i_2)(i_2-i_3)\cdots(i_{n-1}-i_n)(i_n-i_1)\,. \tag{3.8}$$

As a first step, it is not hard to show that

$$f_1(p)\equiv\sum_{j=k+1}^{i-1}(i-j)(j-k)=\frac{(p+1)p(p-1)}{2\times 3},\qquad p=i-k\,. \tag{3.9}$$

If we define

$$f_n(i_1-i_{n+2})\equiv\sum_{\substack{i_2>i_3>\cdots>i_{n+1}\\(i_1>i_2,\ i_{n+1}>i_{n+2})}}(i_1-i_2)(i_2-i_3)\cdots(i_{n+1}-i_{n+2}) \tag{3.10}$$

and

$$g_n(p)\equiv\begin{pmatrix}p+n\\2n+1\end{pmatrix}, \tag{3.11}$$

then we suspect that $f_n=g_n$ for all n. The proof of this last statement is easy once we establish two more identities:

$$\sum_{p=n+1}^{N} g_n(p)=g_{n+1}(N+1)(2n+3)(N+n+2)^{-1} \quad (3.12)$$

and

$$\sum_{p=n+1}^{N} pg_n(p)=g_{n+1}(N)\times(2N-1)(n+1)(N+n+1)^{-1}. \quad (3.13)$$

(These last two statements are easily proven by induction.)

We can now prove that $f_n=g_n$ by induction. Assume that the statement holds for all orders less than or equal to n. Then we have

$$\begin{aligned} f_{n+1}(i-k) &\equiv \sum_{j=k+n+1}^{i-1}(i-j)f_n(j-k)=\sum_{p=n+1}^{i-k-1}(i-k-p)g_n(p) \\ &=(i-k)g_{n+1}(i-k)(2n+3)(i-k+n+1)^{-1} \\ &\quad -g_{n+1}(i-k)(2n+2)(i-k-\tfrac{1}{2})(i-k+n+1)^{-1} \quad (3.14) \\ &=g_{n+1}(i-k). \quad \text{Q.E.D.} \end{aligned}$$

Unfortunately, because of the presence of the 1 and the 2 in (3.6), we find a large number of terms to evaluate (all of which may ultimately be reduced to simple functions of f_n):

$$\sum_{i_1>i_2>\cdots>i_{n+2}=2}^{N-1}(i_1-i_2)(i_2-i_3)\cdots(i_{n+1}-i_{n+2})=f_{n+1}(N-2) \quad (3.15)$$

and

$$\begin{aligned} \sum_{i_1>i_2>\cdots>i_{n+2}=2}^{N-1} &(i_1-i_2)(i_2-i_3)\cdots \\ &\times(i_{n+1}-i_{n+2})(i_{n+2}-i_1) \\ &=2f_{n+2}(N-1)(2n+5)(N+n+1)^{-1} \\ &\quad +(1-N)f_{n+1}(N-2). \quad (3.16) \end{aligned}$$

With the use of (3.12), (3.13), (3.15), and (3.16), it is now possible to perform all the summations necessary within each order in (2.3). The summations between different orders becomes easy once we recognize the hypergeometric function emerging from the series:

$$\sum_{n=0}^{p-1} f_n(p)(-4)^n=p\;{}_2F_1(p+1,1-p;\tfrac{3}{2};1)=p(-1)^{p+1},$$

$$\sum_{n=0}^{N-1}\sum_{p=n}^{N-1} f_{n-1}(p)(-4)^n={}_2F_1(N,1-N;\tfrac{1}{2};1)=(2N-1)(-1)^{N+1}.$$

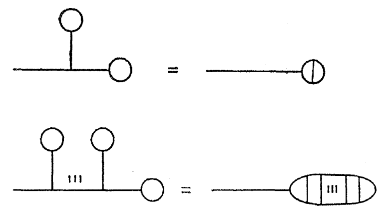

FIG. 2. Duality transformations which display the plausible divergence structure for multiloop amplitudes.

With these last two equations, it is now possible to perform all summations necessary to prove that (3.6) satisfies (2.4). The details themselves are quite tedious and will be omitted.

So far, we have only proven that $W_{P_1P_2\cdots P_N}$ can reach 1 in the N-loop amplitude; only a slight alteration, however, allows us to find the other primitive set of divergences. For example, if we let $\bar{u}_i$ go to zero much faster than the other variables, then it is easy to show that

$$W_{P_1P_2\cdots P_{i-1}P_{i+1}\cdots P_N}=1\,.$$

In other words, we can omit projective transformations in (2.4) at will by a simple change of limits. Since there are 2^N-1 ways in which projective operators may be omitted from (2.4), we find exactly that many divergences of the N-loop amplitude.

IV. CONCLUSION

In summary, we have proven that the 2^N-1 non-self-crossing paths mentioned earlier contribute divergences to the N-loop amplitude. The presence of these additional divergences could be expected both from the analog model and from the duality transformations exhibited in Fig. 2. (These figures seem to indicate that, depending on how one looks at a multiloop diagram, there are separate divergences associated with different clusters of loops.)

The absence of divergences other than the expected ones has not been demonstrated in the general case, because the geometric approach used in our previous paper cannot easily be extended to the N-loop case.

Because a detailed study of the $N=2$ case revealed no other divergence, it is likely that there are no additional divergences in the N-loop amplitude. If this last statement is true, then the renormalization program of Cremmer and Neveu seems likely to succeed.

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