Locally Generated Dilatation Charge in the Thirring Model*

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By applying a "normal-product quantization" the energy-momentum tensor of the Thirring model is constructed as a local function of the Thirring fields. It is shown that the commutation relation of the dilatation current with the field is anomalous, and that the space integrals converge on a dense set of states to the group generators. The appearance of anomalous dimensions for local products of Thirring fields is discussed in the special case of a bilinear mass interaction.

I. INTRODUCTION

SCALE invariance and its relevance to the short-distance behavior of quantum fields has recently received considerable attention in connection with a conjecture of Wilson.[1] According to Wilson a field theory which has only dimensionless coupling constants, and which, therefore, at least on the classical level exhibits conformal invariance in the limit of vanishing masses, is likely to possess a unitary representation of the dilatation group $\rho \rightarrow U(\rho)$ with

$$U(\rho)|0\rangle = |0\rangle\,, \tag{1.1a}$$

$$U(\rho)\Phi(x)U(\rho)^{-1} = \rho^{d(g)}\Phi(\rho x)\,. \tag{1.1b}$$

Here the "dimension" $d(g)$ is the sum of the canonical dimension of the field [i.e., 1 for bosons and $\frac{3}{2}$ for fermions satisfying canonical (anti-) commutation relations in four-dimensional space-time] and a positive "anomalous dimension" which depends on the strength of the coupling. Furthermore, $U(\rho)$ is assumed to be "locally generated." This means that, as in the classical version of the theory, there exists a conserved, covariant current $D_\mu(x)$ which may be constructed from the energy-momentum tensor. $D_\mu(x)$ is a local field, relatively local with respect to the basic fields of the theory. The integral

$$D(t) = \int D_0(x)d^3x \tag{1.2}$$

converges on a suitable dense set of states[2] to the infinitesimal generator of $U(\rho)$. Although $D(t)$ is not expected to exist as an operator,[3] Wilson conjectures that the symmetry breaking is "mild," in the sense that $D(t)$ retains its equal-time commutation relation

$$i[D(t),\Phi(0)]_{\mathrm{ETL}} = d(g)\Phi(0)\,. \tag{1.3}$$

A characteristic feature of scale-invariant theories in the zero-mass limit is the exponential form of the two-point function. In the case of a scalar field, for example, Eq. (1.1) applied to the Källén-Lehmann representation

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int_0^\infty \rho(\kappa)i^{-1}\Delta_\kappa{}^{(-)}(x-y)d\kappa \tag{1.4}$$

yields

$$\langle 0|\phi(x)\phi(y)|0\rangle = C[(x-y)^2 - i\epsilon(x_0-y_0)]^{-d(g)}\,, \tag{1.5}$$

whereas in the spinor case, with the aid of the representation[4]

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int_{-\infty}^\infty \rho(\kappa)i^{-1}S_\kappa{}^{(-)}(x-y)d\kappa\,, \tag{1.6}$$

it gives

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = C[(x-y)^2 - i\epsilon(x_0-y_0)]^{3/2-d(g)}i^{-1}S_0{}^{(-)}(x-y)\,. \tag{1.7}$$

Various attempts have been made to test Wilson's conjecture in Lagrangian perturbation theory. It has been found[5] that the power law (1.5) is consistent with the requirements of renormalization group analysis[6] in the limit (assumed to exist) of vanishing masses only if the power $d(g)$ in (1.5) is independent of the renormalized coupling constant g. On the other hand, model calculations[5] indicate the validity of Eq. (1.5) to second order in g, with an anomalous dimension of the form $d(g) \approx a + bg^2$. One concludes that (1.5)—and hence, scale invariance—is not valid in the perturbation theory of these models (e.g., $\lambda_0\phi^4$ interaction). Wilson[7] has pointed out that in such models the zero-mass interaction Lagrangian acquires an anomalous dimension in second-order perturbation theory, and this is responsible for the breaking of scale invariance in higher orders.

It is not our intention to tackle the most serious of the questions raised by Wilson's speculations. Rather it is our aim to present a complete and rigorous study of scale invariance and related problems in the model which inspired many of Wilson's conjectures in the

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[1] K. G. Wilson, Phys. Rev. **179**, 1499 (1969). The conformal group without Wilson's picture of anomalous dimension of fields was already discussed in H. A. Kastrup, Nucl. Phys. **58**, 561 (1964); G. Mack, *ibid.* **B5**, 499 (1968).

[2] For an exhaustive discussion on the connection of conserved densities with global symmetries, see the review article by Claudio A. Orzalesi, Rev. Mod. Phys. **42**, 381 (1970).

[3] B. Schroer and P. Stichel, Commun. Math. Phys. **3**, 258 (1966).

[4] In this form the spectral representation was derived by W. Thirring, *Einführung in die Quantenelektrodynamik* (Wien, Deudicke, 1955).

[5] R. Jackiw, MIT report, 1970 (unpublished); C. G. Callan, Jr., Phys. Rev. D **2**, 1541 (1970).

[6] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[7] K. G. Wilson, SLAC Report No. SLAC-Pub.-737, 1970 (unpublished).

first place,[8] namely, the Thirring model of a massless spinor field in a two-dimensional space-time.[9] Since there are no essential mass parameters appearing in the model, one expects all aspects of Wilson's conjecture to be valid in their strongest form. That this is so is verified in the course of the present study.

Georgi[10] has recently given a discussion of certain time-ordered functions involving the energy-momentum tensor of the Thirring model. Using rather questionable methods of defining the tensor and computing equal-time limits, he arrives at results which indicate the existence of two energy-momentum tensors leading to different commutation relations (1.3). Our aim is to clarify the situation by explicitly presenting an energy-momentum tensor, suitably defined as a covariant, local field, from which conserved currents can be constructed which generate locally the Poincaré and dilatation invariances.

The material is organized as follows. Section II is a review of the principal invariance properties of the Thirring model. With the aid of relations of the form (1.1a) and (1.1b), we discuss the role of the dimensional infrared regularization parameter which appears in the operator solution of the model.[9] In Sec. III we construct the energy-momentum tensor with the aid of a normal-product[11] quantization of the corresponding classical expression, and show that this tensor leads to the Poincaré and dilatation generators defined in Sec. II. Throughout the article we use the operator formulation of the Thirring model due to Klaiber.[12] A summary of Klaiber's methods and notation, as well as a few words on normal products in the Thirring model, are presented in the Appendix.

II. INVARIANCE PROPERTIES OF THIRRING MODEL

In this section we review some of the basic invariance properties of the Thirring model. The most fundamental of those follow directly from an examination of the Wightman functions (we use the notation of Klaiber[12]):

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_m)|0\rangle &= \prod_{i<j} d^-(x_i-x_j)d^-(y_i-y_j)\prod_{k,l} d^-(x_k-y_l)\\ &\quad\times\langle 0|\psi(x_1)\cdots\psi(x_m)\psi^\dagger(y_1)\cdots\psi^\dagger(y_m)|0\rangle, \end{aligned}\tag{2.1}$$

where

$$d^-(x-y)=\exp i[(a+b\gamma_x{}^5\gamma_y{}^5)D^-(x-y)+\lambda(\gamma_x{}^5+\gamma_y{}^5)\tilde{D}^-(x-y)].$$

The parameters a, b, and λ are related parametrically to the coupling constant g and the "spin" S as follows:

$$\begin{aligned} g&=(\sqrt{\pi})(\alpha-\beta), & a&=\alpha^2-2\alpha\sqrt{\pi},\\ S&=\tfrac{1}{2}(1+\lambda/\pi), & b&=\beta^2-2\beta\sqrt{\pi},\\ && \lambda&=\alpha\beta-\alpha\sqrt{\pi}-\beta\sqrt{\pi}.\end{aligned}\tag{2.2}$$

One defines operators $U(a)$, $U(\chi)$, and $U(\rho)$ corresponding to the space-time transformations in the following way:

translation $\quad x\to x+a$;

Lorentz transformation

$$x\to\Lambda x,\quad \Lambda=\begin{pmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{pmatrix};\tag{2.3}$$

dilatation $\quad x\to\rho x,\quad \rho>0.$

This is done by specifying their action on states formed by applying products of Thirring model fields on the vacuum:

$$\begin{aligned} &U(a)|0\rangle=U(\chi)|0\rangle=U(\rho)|0\rangle=|0\rangle,\\ &U(a)\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)|0\rangle\\ &\qquad=\phi(x_1+a)\cdots\phi^\dagger(u_n+a)|0\rangle,\\ &U(\chi)\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)|0\rangle\\ &\qquad=\exp[-\tfrac{1}{2}\chi(\textstyle\sum_i\gamma_{x_i}{}^5+\sum_j\gamma_{y_j}{}^5)]\\ &\qquad\qquad\times\phi(\Lambda x_1)\cdots\phi^\dagger(\Lambda y_n)|0\rangle,\\ &U(\rho)\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)|0\rangle\\ &\qquad=\rho^{(m+n)(S+g^2/4\pi^2)}\phi(\rho x_1)\cdots\phi^\dagger(\rho y_n)|0\rangle.\end{aligned}\tag{2.4}$$

From the form of the Wightman functions (2.1) one readily verfies that $U(a)$, $U(\chi)$, and $U(\rho)$ are unitary operators, and that the field $\phi(x)$ has the covariance properties

$$U(a)\phi(x)U(a)^{-1}=\phi(x+a),\tag{2.5}$$

$$U(\chi)\phi(x)U(\chi)^{-1}=e^{-\frac{1}{2}\gamma^5\chi}\phi(\Lambda x),\tag{2.6}$$

$$U(\rho)\phi(x)U(\rho)^{-1}=\rho^{S+g^2/4\pi^2}\phi(\rho x).\tag{2.7}$$

The unitary operators $U(a)$ and $U(\chi)$ were discussed by Klaiber[11] and in fact played an important role in his construction. The implementation of scale transformations in the Thirring model was introduced by Wilson.[9] Expression (2.7) is a generalization to arbitrary spin of his formula for the spin-$\frac{1}{2}$ case.

It is interesting that the Thirring model should be scale invariant in spite of the presence of an infrared regularization parameter κ with the dimension of a mass. However, an examination of the Wightman functions (2.1) reveals that this parameter plays no essential role, the choice of κ merely fixing the normalization of $\phi(x)$. In Klaiber's formulation[12] the operator solutions labeled by different values of κ differ by a unitary transformation as well as the change of normalization. To see this, we consider the unitary representation $V(\rho)$ of the

[8] B. Schroer, University of Pittsburgh report, 1970 (unpublished). The local currents of dilatation and their commutation relations were discussed in the Thirring model and the simpler derivative coupling model.

[9] K. Wilson, Phys. Rev. D 2, 1473 (1970).

[10] Howard Georgi, Phys. Rev. D 2, 2908 (1970).

[11] J. H. Lowenstein, Commun. Math. Phys. 16, 265 (1970).

[12] B. Klaiber, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X A, p. 141.

dilatation group with respect to which the free auxiliary field $\psi(x)$ is covariant:

$$V(\rho)\psi(x)V(\rho)^{-1}=\rho^{1/2}\psi(\rho x)\,. \tag{2.8}$$

The action of these scale transformations on the current and pseudocurrent potentials $\tilde{j}^{(\kappa)}(x)$ and $\tilde{j}^{(\kappa)}(x)$, where we have indicated explicitly the dependence on κ, may now be calculated:

$$\begin{aligned} V(\rho)j^{(\kappa)\pm}(x)V(\rho)^{-1}&=j^{(\kappa/\rho)\pm}(\rho x)\,,\\ V(\rho)\tilde{j}^{(\kappa)\pm}(x)V(\rho)^{-1}&=\tilde{j}^{(\kappa/\rho)\pm}(\rho x)\,. \end{aligned} \tag{2.9}$$

Moreover,

$$\begin{gathered} \Delta^{(\kappa)\pm}(x)=\Delta^{(\kappa/\rho)\pm}(\rho x)\,,\\ V(\rho)QV(\rho)^{-1}=Q\,,\quad V(\rho)\tilde{Q}V(\rho)^{-1}=\tilde{Q}\,, \end{gathered} \tag{2.10}$$

so that

$$V(\rho)\phi^{(\kappa)}(x)V(\rho)^{-1}=\rho^{1/2}\phi^{(\kappa/\rho)}(\rho x)\,. \tag{2.11}$$

Hence, from (2.7) and (2.11)

$$U(\rho)^{-1}V(\rho)\phi^{(\kappa)}(x)V(\rho)^{-1}U(\rho) =\rho^{1/2-S-g^2/4\pi^2}\phi^{(\kappa/\rho)}(x)\,. \tag{2.12}$$

Thus the unitary operator $U(\rho)^{-1}V(\rho)$ carries the Klaiber solution with one value of κ into that with another, except for normalization.

Since the particular choice of infrared parameter κ is not of physical significance, it is tempting to try to formulate the Thirring model in such a way that κ never appears in any of the formulas. Such an approach is implicit in the work of Wilson.[9] Restricting ourselves now to $S=\frac{1}{2}$, we define

$$\phi_W^{(\kappa)}(x)=\mu^{g^2/4\pi^2}\phi_K^{(\kappa)}(x)\,, \tag{2.13}$$

where $\phi_K^{(\kappa)}(x)$ is Klaiber's field with infrared parameter κ. Clearly (2.13) does not alter the Wilson dimension $\frac{1}{2}+g^2/4\pi^2$ [defined by (2.7)] of $\phi^{(\kappa)}$, but now the ordinary dimension has the same value. $\phi_W^{(\kappa)}(x)$ depends on κ only through its representation in the Fock space of the free field $\psi(x)$. The Wightman functions of the different $\phi_W^{(\kappa)}$ do not depend on κ, deflecting the unitary equivalence

$$U(\rho)^{-1}V(\rho)\phi_W^{(\kappa)}(x)V(\rho)^{-1}U(\rho)=\phi_W^{(\kappa/\rho)}(x)\,. \tag{2.14}$$

We now turn to the problem of defining normal products in the above scheme in which the infrared parameter does not appear explicitly. In the case of the product of two factors a suitable definition would be [compare with (A7)]

$$\begin{aligned} N[\phi_W^{(\kappa)}(x)\phi_W^{(\kappa)\dagger}(y)] &=(-\xi^2+i\epsilon\xi_0)^{(a+b\gamma_x^5\gamma_y^5)/4\pi}[\phi_W^{(\kappa)}(x)\phi_W^{(\kappa)\dagger}(y)\\ &\quad -:e^{i[K(x)-K(y)]}:\langle 0|\phi_W^{(\kappa)}(x)\phi_W^{(\kappa)\dagger}(x)|0\rangle]\,. \end{aligned} \tag{2.15}$$

These objects possess the desired properties of normal products, and in addition the Wilson and ordinary dimensions are both equal to $1+\frac{1}{2}(1-\gamma_x^5\gamma_y^5)b/\pi$.

A possible drawback of using $\phi_W^{(\kappa)}$ instead of $\phi_K^{(\kappa)}$ arises if we attempt to do perturbation theory on the Thirring model. For instance, to give the fermion a mass, one would like to add a term to the Hamiltonian density of the form

$$\mathcal{H}_{\text{int}}(x)=mN[\bar{\phi}\phi](x)\,. \tag{2.16}$$

If the normal product (2.15) is used, this cannot be done since $\mathcal{H}_{\text{int}}$ would not have the correct dimension. One would have to use instead the rather strange-looking

$$\mathcal{H}_{\text{int}}(x)=m^{1-b/\pi}N[\bar{\phi}_W\phi_W](x)\,. \tag{2.17}$$

If, on the other hand, one uses the field ϕ_K and the normal products on Ref. 11, one does not encounter this problem. However, one now has two mass parameters κ and m which will eventually have to be related by means of a renormalization process.

To obtain a better insight into what might happen in a perturbation theory of the Thirring model, let us examine the reverse side of the coin—the passage of the "massive Thirring model" to its zero-mass limit. The perturbation theory (in the coupling constant) of the model with nonzero mass leads to renormalized expressions (and a nontrivial S matrix) which diverge in the limit $m\to 0$. Following a suggestion of Gell-Mann and Low,[6] one can now renormalize the Green functions at some fixed point λ *off the mass shell.* Then the limit of $m\to 0$ exists for each graph, but the result depends on λ. In the operator solution of the Thirring model (see the Appendix), the choice of the Gell-Mann–Low renormalization point manifests itself in Klaiber's[12] infrared regularization parameter κ (see the Appendix). It may thus be advantageous, in attempting to reintroduce nonzero mass, to retain the dependence of $\phi^{(\kappa)}$ on κ, i.e., to use the Klaiber field ϕ_K and interaction term (2.16) (with $\phi=\phi_K$) instead of ϕ_W and (2.17).

III. DERIVATION OF ENERGY-MOMENTUM TENSOR

In classical field theory the Lagrangian density and energy-momentum tensor of the Thirring model are given by

$$\begin{aligned} \mathcal{L}^C(x)&=\tfrac{1}{2}i\bar{\psi}_C(x)\gamma_\mu\overleftrightarrow{\partial}^\mu\psi_C(x)\\ &\quad+\tfrac{1}{2}g(\bar{\psi}_C(x)\gamma_\mu\psi_C(x))(\bar{\psi}_C(x)\gamma^\mu\psi_C(x))\,, \end{aligned} \tag{3.1}$$

$$\Theta_{\mu\nu}{}^C(x)=\tfrac{1}{2}i\bar{\psi}_C(x)\gamma_\mu\overleftrightarrow{\partial}_\nu\psi_C(x)-g_{\mu\nu}\mathcal{L}^C(x)\,. \tag{3.2}$$

As a first conjecture regarding the quantum-mechanical energy-momentum tensor, we quantize (3.2) by replacing all products of classical fields with normal products[11] of the corresponding quantum fields. Thus we define

$$\begin{aligned} \hat{\theta}_{\mu\nu}(x)&=\tfrac{1}{2}iN[\bar{\phi}\gamma_\mu\overleftrightarrow{\partial}_\nu\phi](x)-\tfrac{1}{2}ig_{\mu\nu}N[\bar{\phi}\gamma_\rho\overleftrightarrow{\partial}^\rho\phi](x)\\ &\quad-\tfrac{1}{2}g_{\mu\nu}N[(\bar{\phi}\gamma_\rho\phi)(\bar{\phi}\gamma^\rho\phi)](x)\,, \end{aligned} \tag{3.3}$$

where $\phi(x)$ is the spin-$\frac{1}{2}$, massless Thirring model field constructed explicitly by Klaiber[12] in the Fock space of an auxiliary free field $\psi(x)$. For a brief discussion of

Klaiber's construction and the normal products of Ref. 11, the reader is referred to the Appendix.

The first two terms of (3.3) may be evaluated using Eq. (4.14) of Ref. 11, and the third term using Eq. (5.5) of the same reference. Thus,

$$\hat{\theta}_{\mu\nu}(x)=\theta_{\mu\nu}{}^{(0)}(x)+(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\times\{\tfrac{1}{2}\pi\zeta^{\rho}(x)\zeta^{\sigma}(x)-\pi J^{\rho}(x)\zeta^{\sigma}(x)-(\tfrac{1}{2}\sqrt{\pi})(\alpha+\beta)N[J^{\rho}J^{\sigma}](x)\}\,, \qquad (3.4)$$

where $\theta_{\mu\nu}{}^{(0)}$ is the free energy-momentum tensor

$$\theta_{\mu\nu}{}^{(0)}(x)=\tfrac{1}{2}i\!:\!\bar{\psi}\gamma_{\mu}\overleftrightarrow{\partial}_{\nu}\psi\!:(x) \qquad (3.5)$$

and $\zeta_{\mu}(x)$ is the difference between the currents of the free and interacting theories (see the Appendix)

$$\zeta_{\mu}(x)=j_{\mu}(x)-J_{\mu}(x)=\alpha\pi^{-1/2}Q\Delta_{\mu}(x)+\beta\pi^{-1/2}\tilde{Q}\tilde{\Delta}_{\mu}\,. \qquad (3.6)$$

From (3.3) we know that $\hat{\theta}_{\mu\nu}(x)$ is a local, covariant tensor field, and from (3.4) (using the known properties of $\theta_{\mu\nu}{}^{(0)}$ and the conservation of J_{μ}, $\epsilon_{\mu\rho}J^{\rho}$, ζ_{μ}, and $\epsilon_{\mu\rho}\zeta^{\rho}$), we see that $\hat{\theta}_{\mu\nu}$ is symmetric, traceless, and conserved. We must now check the commutation relations between $\hat{\theta}_{\mu\nu}$ and ϕ. The following formulas will prove useful in the computation:

$$[\theta_{\mu\nu}{}^{(0)}(x),\psi(y)]=-\tfrac{1}{2}i(D_{\mu}(x-y)+\gamma^{5}\tilde{D}_{\mu}(x-y))\overleftrightarrow{\partial}_{\nu}{}^{x}\psi(x)\,, \qquad (3.7a)$$

$$[j^{\pm}(x),\theta_{\mu\nu}{}^{(0)}(y)]=-i(\sqrt{\pi})(\partial_{\nu}{}^{y}D_{\kappa}{}^{\pm}(x,y)\,j_{\mu}(y)+\partial_{\nu}{}^{y}\tilde{D}_{\kappa}{}^{\pm}(x,y)\,\tilde{j}_{\mu}(y))\,, \qquad (3.7b)$$

$$[\tilde{j}^{\pm}(x),\theta_{\mu\nu}{}^{(0)}(y)]=-i(\sqrt{\pi})(\partial_{\nu}{}^{y}\tilde{D}_{\kappa}{}^{\pm}(x,y)\,j_{\mu}(y)+\partial_{\nu}{}^{y}D_{\kappa}{}^{\pm}(x,y)\,\tilde{j}_{\mu}(y))\,, \qquad (3.7c)$$

$$[\chi^{\pm}(y),\theta_{\mu\nu}{}^{(0)}(z)]=-i(\sqrt{\pi})(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})(\alpha\partial_{z}{}^{\sigma}D_{\kappa}{}^{\pm}(y,z)+\beta\gamma_{y}{}^{5}\partial_{z}{}^{\sigma}\tilde{D}_{\kappa}{}^{\pm}(y,z))\,j^{\rho}(z)\,, \qquad (3.7d)$$

$$[\chi^{\pm}(x),[\chi^{\pm}(y),\theta_{\mu\nu}{}^{(0)}(z)]]=-(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\times(\alpha\partial_{z}{}^{\rho}D_{\kappa}{}^{\pm}(x,z)+\beta\gamma_{x}{}^{5}\partial_{z}{}^{\rho}\tilde{D}_{\kappa}(x,z))(\alpha\partial_{z}{}^{\sigma}D_{\kappa}{}^{\pm}(y,z)+\beta\gamma_{y}{}^{5}\partial_{z}{}^{\sigma}\tilde{D}_{\kappa}{}^{\pm}(y,z))\,. \qquad (3.7e)$$

Equations (3.7a)–(3.7c) are simple consequences of (A5); (3.7d) and (3.7e) follow immediately from the others using definition (A6).

The commutator of $\theta_{\mu\nu}{}^{(0)}(z)$ with $\phi(y)$ may be computed using the following general commutator identity: If $C=[B,[A,B]]$ is a c number, then

$$[A,e^{B}]=[A,B]e^{B}+\tfrac{1}{2}Ce^{B}=e^{B}[A,B]-\tfrac{1}{2}Ce^{B}\,.$$

Thus,

$$\begin{aligned}[\theta_{\mu\nu}{}^{(0)}(z),\phi(y)]&=[\theta_{\mu\nu}{}^{(0)}(z),e^{i\chi^{+}(y)}\psi(y)e^{i\chi^{-}(y)}]=-\tfrac{1}{2}ie^{i\chi^{+}(y)}[D_{\mu}(z-y)+\gamma^{5}\tilde{D}_{\mu}(z-y)]\partial_{\nu}{}^{z}\psi(z)e^{i\chi^{-}(y)}\\&\quad-(\sqrt{\pi})(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})[d^{+\sigma}(y,z)\,j^{\rho}(z)\phi(y)+d^{-\sigma}(y,z)\phi(y)\,j^{\rho}(z)]\\&\quad-(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})[d^{+\rho}(y,z)d^{+\sigma}(y,z)-d^{-\rho}(y,z)d^{-\sigma}(y,z)]\phi(y)\,,\end{aligned} \qquad (3.8)$$

where

$$d_{\rho}{}^{\pm}(y,z)=\alpha\partial_{\rho}{}^{z}D_{\kappa}{}^{\pm}(y,z)+\beta\gamma_{y}{}^{5}\partial_{\rho}{}^{z}\tilde{D}_{\kappa}{}^{\pm}(y,z)\,.$$

Combining this result with [see (A5) and (A6)]

$$\begin{aligned}-\pi(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})[J^{\rho}(y)\zeta^{\sigma}(y),\phi(z)]&=(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\{(\sqrt{\pi})[\alpha\Delta^{\sigma}(y)+\beta\gamma^{5}\tilde{\Delta}^{\sigma}(y)][J^{+\rho}(y)\phi(z)+\phi(z)J^{-\rho}(y)]\\&\quad+\pi[(1-\alpha\pi^{-1/2})D^{\sigma}(y-z)+\gamma_{z}{}^{5}(1-\beta\pi^{-1/2})\tilde{D}^{\sigma}(y-z)][\zeta^{+\rho}(y)\phi(z)+\phi(z)\zeta^{-\rho}(y)]\\&\quad+\pi([(1-\alpha\pi^{-1/2})D^{+\sigma}(y-z)+\gamma_{z}{}^{5}(1-\beta\pi^{-1/2})\tilde{D}^{+\sigma}(y-z)][\alpha\Delta^{+\rho}(y)+\beta\gamma_{z}{}^{5}\tilde{\Delta}^{+\rho}(y)]-[+\rightarrow-])\phi(z)\}\,,\end{aligned} \qquad (3.9)$$

$$\begin{aligned}&\tfrac{1}{2}\pi(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})[\zeta^{\rho}(y)\zeta^{\sigma}(y),\phi(z)]\\&\quad=(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\{-(\sqrt{\pi})[\alpha\Delta^{\rho}(y)+\beta\gamma_{z}{}^{5}\tilde{\Delta}^{\rho}(y)][\zeta^{+\sigma}(y)\phi(z)+\phi(z)\zeta^{-\sigma}(y)]\\&\qquad-\tfrac{1}{2}[(\alpha\Delta^{+\sigma}(y)+\beta\gamma_{z}{}^{5}\tilde{\Delta}^{+\sigma}(y))(\alpha\Delta^{+\rho}(y)+\beta\gamma_{z}{}^{5}\tilde{\Delta}^{+\rho}(y))-(+\rightarrow-)]\phi(z)\}\,,\end{aligned}$$

and defining

$$\begin{aligned}\theta_{\mu\nu}&=\hat{\theta}_{\mu\nu}+(\tfrac{1}{2}\sqrt{\pi})(\alpha+\beta)(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})N[J^{\rho}J^{\sigma}]\\&=\tfrac{1}{2}iN[\bar{\phi}\gamma_{\mu}\overleftrightarrow{\partial}_{\nu}\phi]+(\sqrt{\pi})(\alpha+\beta)N[J_{\mu}J_{\nu}]-\beta(\sqrt{\pi})N[J_{\lambda}J^{\lambda}]g_{\mu\nu}\,,\end{aligned} \qquad (3.10)$$

we have

$$\begin{aligned}[\theta_{\mu\nu}(z),\phi(y)]&=\tfrac{1}{2}ie^{i\chi^{+}(y)}\{[D_{\mu}(z-y)+\gamma^{5}\tilde{D}_{\mu}(z-y)]\overleftrightarrow{\partial}_{\nu}{}^{y}\psi(y)\}e^{i\chi^{-}(y)}\\&\quad+(\sqrt{\pi})[\alpha D_{\nu}(z-y)+\beta\gamma^{5}\tilde{D}_{\nu}(z-y)]e^{i\chi^{+}(y)}\!:\!J_{\mu}(z)\psi(y)\!:e^{i\chi^{-}(y)}\\&\quad+(\sqrt{\pi})[\alpha\tilde{D}_{\nu}(z-y)+\beta\gamma^{5}D_{\nu}(z-y)]e^{i\chi^{+}(y)}\!:\!\tilde{J}_{\mu}(z)\psi(y)\!:e^{i\chi^{-}(y)}\\&\quad+\pi[D_{\nu}(z-y)+\gamma^{5}\tilde{D}(z-y)]e^{i\chi^{+}(y)}\!:\!(\alpha Q+\beta\gamma^{5}\tilde{Q})[\Delta_{\mu}(z)+\gamma^{5}\tilde{\Delta}_{\mu}(z)]\psi(y)\!:e^{i\chi^{-}(y)}\\&\quad-\tfrac{1}{2}(g^{2}/\pi)[D_{\mu}{}^{+}(z-y)D_{\nu}{}^{+}(z-y)+\tilde{D}_{\mu}{}^{+}(z-y)\tilde{D}_{\nu}{}^{+}(z-y)-(+\rightarrow-)]\phi(y)\,.\end{aligned} \qquad (3.11)$$

Finally, the contribution of the terms bilinear in the current is

$$[\hat{\theta}_{\mu\nu}(z)-\theta_{\mu\nu}(z),\phi(y)]=(\sqrt{\pi})(\alpha+\beta)\{(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})[(1-\alpha\pi^{-1/2})D^{\sigma}(z-y)+(1-\beta\pi^{-1/2})\tilde{D}^{\sigma}(z-y)\gamma^5]$$
$$\times[J^{+\rho}(z)\phi(y)+\phi(y)J^{-\rho}(z)]-\tfrac{1}{2}(1+g^2/\pi^2)[D_\mu{}^+(z-y)D_\nu{}^+(z-y)+\tilde{D}_\mu{}^+(z-y)\tilde{D}_\nu{}^+(z-y)-(+\to-)]\phi(y)$$
$$-\gamma^5[D_\mu{}^+(z-y)\tilde{D}_\nu{}^+(z-y)+D_\nu{}^+(z-y)\tilde{D}_\mu{}^+(z-y)-(+\to-)]\phi(y)\}. \quad (3.12)$$

For equal times, we obtain, using

$$D_0(0,x^1)=\delta(x^1),\quad D_1(0,x^1)=0,$$
$$D_0{}^+(0,x^1)D_1{}^+(0,x^1)-D_0{}^-(0,x^1)D_1{}^-(0,x^1)=(4\pi i)^{-1}\delta'(x^1), \quad (3.13)$$
$$[D_0{}^+(0,x^1)]^2+[D_1{}^+(0,x^1)]^2-(+\to-)=0,$$

$$[\theta_{00}(z),\phi(y)]_{z^0=y^0}=-i\delta(z^1-y^1)\partial_0\phi(y)-\tfrac{1}{2}i\delta'(z^1-y^1)\gamma^5\phi(y),$$
$$[\theta_{01}(z),\phi(y)]_{z^0=y^0}=-i\delta(z^1-y^1)\partial_1\phi(y)+i[\tfrac{1}{2}+g^2/4\pi^2]\delta'(z^1-y^1)\phi(y), \quad (3.14)$$

$$[\hat{\theta}_{00}(z)-\theta_{00}(z),\phi(y)]_{z^0=y^0}=\pi^{1/2}(\alpha+\beta)\{\delta(z^1-y^1)[(1-\alpha\pi^{-1/2})(J_0{}^+(y)\phi(y)+\phi(y)J_0{}^-(y))$$
$$-(1-\beta\pi^{-1/2})(J_1{}^+(y)\gamma^5\phi(y)+\gamma^5\phi(y)J_1{}^-(y))]-(2\pi i)^{-1}\delta'(z^1-y^1)\gamma^5\phi(y)\}, \quad (3.15)$$
$$[\hat{\theta}_{01}(z)-\theta_{01}(z),\phi(y)]_{z^0=y^0}=\pi^{1/2}(\alpha+\beta)\{\delta(z^1-y^1)[(1-\alpha\pi^{-1/2})(J_1{}^+(y)\phi(y)+\phi(y)J_1{}^-(y))$$
$$-(1-\beta\pi^{-1/2})(J_0{}^+(y)\gamma^5\phi(y)+\gamma^5\phi(y)J_0{}^-(y))]-(4\pi i)^{-1}(1+g^2/\pi^2)\delta'(z^1-y^1)\phi(y)\}.$$

It is clear from (3.14) and (3.15) that our candidate for the energy-momentum tensor should be $\theta_{\mu\nu}$ rather than $\hat{\theta}_{\mu\nu}$. In particular, $\theta_{\mu\nu}$ is symmetric, conserved, and traceless, and, defining formally the generators of translations, Lorentz transformations, and dilatations, respectively,

$$P_\mu=\int dz^1\theta_{0\mu}(z),$$
$$L=\int dz^1[z^0\theta_{01}(z)+z^1\theta_{00}(z)], \quad (3.16)$$
$$D=\int dz^1[z^0\theta_{00}(z)+z_1\theta_{01}(z)],$$

we easily confirm the correct commutation relations:

$$i[P_\mu,\phi(x)]=\partial_\mu\phi(x),$$
$$i[L,\phi(x)]=(x^0\partial_1+x^1\partial_0-\tfrac{1}{2}\gamma^5)\phi(x), \quad (3.17)$$
$$i[D,\phi(x)]=(x^0\partial_0+x^1\partial_1+\tfrac{1}{2}+g^2/4\pi^2)\phi(x).$$

We must now show that expressions (3.16) have more than formal meaning. To do this, we define

$$P_\mu{}^R(t)=\int_{z^0=t} dz^1 f_R(z^1)\theta_{0\mu}(z), \quad (3.18)$$

where

$$f_R\in\mathfrak{D},\quad f_R(z^1)=\begin{cases}1, & |z^1|\le R\\ 0, & |z^1|\ge R+\delta,\end{cases}$$

with analogous expressions for $L^R(t)$ and $D^R(t)$. As we shall see, the usual time-smearing[2] is not necessary here. Clearly the commutators of $P_\mu{}^R(t)$, $L^R(t)$, and $D^R(t)$ with $\phi(x)$ are identical to those of P_μ, L, and D for sufficiently large R (dependent on t). We now wish to show that as R tends to infinity, the matrix elements of $P_\mu-P_\mu{}^R(t)$, $L-L^R(t)$, and $D-D^R(t)$ between members of a dense set of quasilocal states tend to zero. Clearly it will be sufficient to show the convergence to zero of all expressions of the type

$$\int d^{2m}xd^{2m}y g(x_1,\ldots,x_m;\,y_1,\ldots,y_m)\langle 0|\phi(z_1)\cdots$$
$$\times\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_m)\begin{Bmatrix}P_\mu{}^R(t)\\ L^R(t)\\ D^R(t)\end{Bmatrix}|0\rangle, \quad (3.19)$$
$$g\in\mathcal{S}(R^{4m}).$$

First we must calculate

$$\langle 0|\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_m)\theta_{\mu\nu}(z)|0\rangle$$
$$=f(x_1,\ldots,x_m;\,y_1,\ldots,x_m)$$
$$\times\langle 0|\psi(x_1)\cdots\psi(x_m)\psi^\dagger(y_1)\cdots\psi^\dagger(y_m)$$
$$\times\exp\{i\sum_{i=1}^{m}[\chi^-(x_i)-\chi^-(y_i)]\}\theta_{\mu\nu}{}^{(0)}(z)|0\rangle, \quad (3.20)$$

where the c-number factor is defined in (A8). The exponential may be commuted through using (3.7d) and (3.7e) exactly as in the derivation of Eq. (3.8). The result is

$$\langle 0|\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_m)\theta_{\mu\nu}(z)|0\rangle = f(x_1,\ldots,x_m;\, y_1,\ldots,y_m)$$

$$\times\{i\sum_{j=1}^{m}[(D_\mu{}^+(z-x_j)+\gamma^5{}_{x_j}\tilde{D}_\mu{}^+(z-x_j))\partial_\nu{}^{x_j}-\tfrac{1}{2}(D_{\mu\nu}{}^+(z-x_j)+\gamma_{x_j}{}^5\tilde{D}_{\mu\nu}{}^+(z-x_j))+(x_j\to y_j)]$$

$$+\tfrac{1}{2}(g_{\mu\rho}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\sum_{i,j=1}^{m}[(a+b\gamma^5{}_{x_i}\gamma_{x_j}{}^5)D^{+\rho}(z-x_i)D^{+\sigma}(z-x_j)-(a+b\gamma_{y_i}{}^5\gamma_{x_j}{}^5)D^{+\rho}(z-y_i)D^{+\sigma}(z-x_j)+(x_j\to y_j)]\}$$

$$\times\langle 0|\psi(x_1)\cdots\psi(x_m)\psi^\dagger(y_1)\cdots\psi^\dagger(y_m)|0\rangle. \quad (3.21)$$

We must show that expression (3.21), when smeared with either $f^R(z^1)$ or $z^1f^R(z^1)$, vanishes in the limit $R\to\infty$. Typical contributions are of the sort

$$\int dz^1[D_0{}^+(z-\xi)D_0{}^+(z-\eta)+D_1{}^+(z-\xi)D_1{}^+(z-\eta)]f^R(z^1)=O(R^{-1}),$$

$$\int dz^1[D_0{}^+(z-\xi)D_1{}^+(z-\eta)+D_1{}^+(z-\xi)D_0{}^+(z-\eta)]f^R(z^1)=O(R^{-1}),$$

$$\int dz^1[D_{00}{}^+(z-\xi)\pm D_{01}{}^+(z-\xi)]f^R(z^1)=O(R^{-1}), \quad (3.22)$$

$$\int dz^1z^1[D_0{}^+(z-\xi)D_0{}^+(z-\eta)+D_1{}^+(z-\xi)D_1{}^+(z-\eta)]f^R(z^1)=O(R^{-1}),$$

and

$$\int dz^1[D_0{}^+(z-\xi)\pm D_1{}^+(z-\xi)]f^R(z^1)=\tfrac{1}{2}+O(R^{-1}),$$

$$\int dz^1z^1[D_0{}^+(z-\xi)\pm D_1{}^+(z-\xi)]f^R(z^1)=\mp\frac{1}{2\pi i}\int dz^1f^R(z^1)\mp\tfrac{1}{2}(z^0-\xi^0\mp\xi^1)+O(R^{-1}),$$

$$\int dz^1z^1[D_{00}{}^+(z-\xi)\pm D_{10}{}^+(z-\xi)]f^R(z^1)=\pm\tfrac{1}{2}+O(R^{-1}), \quad (3.23)$$

$$\int dz^1z^1[D_0{}^+(z-\xi)D_1{}^+(z-\eta)+D_1{}^+(z-\xi)D_0{}^+(z-\eta)]f^R(z^1)=-\frac{1}{4\pi i}+O(R^{-1}).$$

From (3.21) one verifies that the contributions of type (3.23) cancel, thus giving the desired result.

One might ask whether it is possible to find some conserved current $\hat{D}_\mu(x)\neq x^\nu\theta_{\mu\nu}(x)$ which leads to a commutation relation without anomalous dimension

$$i[\hat{D},\phi(x)]=(x^0\partial_0+x^1\partial_1+\tfrac{1}{2})\phi(x). \quad (3.24)$$

If one further demands that $\hat{D}^R(t)|0\rangle$ [see (3.18)] converges "weakly" to zero in the sense discussed above, then one obtains an infinitesimal invariance of the Wightman functions, which is in flat contradiction with the explicit form of these expectation values (2.1). We expect that even without assuming the annihilation of the vacuum (absence of spontaneous symmetry breaking) it will not be possible to find a current $\hat{D}_\mu(x)$ satisfying (3.24).

Although it led us eventually to the correct energy-momentum tensor, the quantization (3.3) of the classical expression was not entirely effective owing to the presence of terms bilinear in the currents in (3.4). We now investigate the possibility of redefining the normal products in such a way that expression (3.3) yields the correct tensor field immediately. Such a prescription is indeed conceivable, but only if we restrict ourselves to quantities which are invariant under both gauge and γ^5 transformations of the first kind (i.e., which commute with both Q and $\tilde{Q}$). Then one may multiply the normal product (suitably ordering creation and annihilation factors) by a scalar field which tends to unity as the arguments coincide. This would clearly leave unaffected such objects as $N[\bar{\phi}\gamma_\mu\phi](x)$ and $N[(\bar{\phi}\gamma_\mu\phi)\times(\bar{\phi}\gamma^\mu\phi)](x)$, but would alter $N[\bar{\phi}\gamma_\mu\overleftrightarrow{\partial}_\nu\phi](x)$ and other normal products involving derivatives. In particular, defining

$$N^*[\phi_\alpha{}^+(x)\phi_\alpha(y)] = \exp\{ir[J^+(x)-J^+(y)]+is\gamma_\alpha{}^5[\tilde{J}^+(x)-\tilde{J}^+(y)]\} \times N[\phi_\alpha{}^+(x)\phi_\alpha(y)]\exp\{ir[J^-(x)-J^-(y)] + is\gamma_\alpha{}^5[\tilde{J}^-(x)-\tilde{J}^-(y)]\}, \quad (3.25)$$

we see that

$$J_\mu = N[\bar{\phi}\gamma_\mu\phi](x) = N^*[\bar{\phi}\gamma_\mu\phi](x), \quad (3.26)$$

but

$$\tfrac{1}{2}iN^*[\bar{\phi}\gamma_\mu\overleftrightarrow{\partial}_\nu\phi](x) = \tfrac{1}{2}iN[\bar{\phi}\gamma_\mu\overleftrightarrow{\partial}_\nu\phi](x) + rN[J_\nu J_\mu](x) + sN[\tilde{J}_\nu\tilde{J}_\mu](x). \quad (3.27)$$

Choosing $r=s=\frac{1}{2}\pi^{1/2}(\alpha+\beta)$, we see that

$$\theta_{\mu\nu}(x) = \tfrac{1}{2}iN^*[\bar{\phi}\gamma_\mu\overleftrightarrow{\partial}_\nu\phi](x) - \tfrac{1}{2}ig_{\mu\nu}N^*[\bar{\phi}\gamma_\rho\partial^\rho\phi](x) - \tfrac{1}{2}gg_{\mu\nu}N^*[(\bar{\phi}\gamma_\rho\phi)(\bar{\phi}\gamma^\rho\phi)](x). \quad (3.28)$$

The requirement of correct quantization of the energy-momentum tensor has picked out one of the many possible normal-product prescriptions for gauge- and γ^5-invariant quantities.

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APPENDIX: OPERATOR SOLUTION OF THIRRING MODEL

The operator solution of the Thirring model given in Ref. 12 is represented in the Fock space of a free, massless spin-$\frac{1}{2}$ field in two dimensions:

$$\psi(x) = (2\pi)^{-1/2}\int dp^1[a^*(p^1)e^{ipx}+b(p^1)e^{-ipx}]u(p^1), \quad p^0=|p^1|, \quad (\mathrm{A1})$$

where $a(p^1)$, $b(p^1)$, and their conjugates satisfy canonical anticommutation relations and

$$u(p^1) = \begin{pmatrix}\theta(-p^1)\\ \theta(p^1)\end{pmatrix}.$$

This free field satisfies the zero-mass Dirac equation

$$\gamma^\mu\partial_\mu\psi = 0, \quad \gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \quad (\mathrm{A2})$$

The free current and pseudocurrent

$$j_\mu(x) = :\gamma(x)\gamma_\mu\psi(x):, \quad \tilde{j}_\mu(x) = :\gamma(x)\gamma_\mu\gamma^5\psi(x): = \epsilon_{\mu\nu}j^\nu, \quad (\mathrm{A3})$$

$$\gamma^5=\gamma^0\gamma^1, \quad (\epsilon_{\mu\nu}) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

are both conserved zero-mass free fields, and it is possible to write

$$j_\mu(x) = \pi^{-1/2}\partial_\mu j(x), \quad \tilde{j}_\mu(x) = \pi^{-1/2}\partial_\mu\tilde{j}(x), \quad (\mathrm{A4})$$

where $j(x)$ and $\tilde{j}(x)$ are infrared regularized scalar free fields whose positive- and negative-frequency parts satisfy the following commutation relations:

$$[j^-(x),j^+(y)] = [\tilde{j}^-(x),\tilde{j}^+(y)] = i^{-1}D_{\kappa\kappa}{}^-(x,y),$$
$$[j^-(x),\tilde{j}^+(y)] = [\tilde{j}^-(x),j^+(y)] = i^{-1}\tilde{D}_{\kappa\kappa}{}^-(x,y),$$
$$[j^\pm(x),\psi(y)] = -\pi^{1/2}[D_\kappa{}^\pm(x,y)+\gamma^5 D_\kappa{}^\pm(x,y)]\psi(x), \quad (\mathrm{A5})$$
$$[\tilde{j}^\pm(x),\psi(y)] = -\pi^{1/2}[\tilde{D}_\kappa{}^\pm(x,y)+\gamma^5\tilde{D}_\kappa{}^\pm(x,y)]\psi(y),$$
$$[Q,\psi(x)] = -\psi(x), \quad [\tilde{Q},\psi(x)] = -\gamma^5\psi(x),$$

where

$$Q=\int j_0(x)d^1x, \quad \tilde{Q}=\int\tilde{j}_0(x)d^1x,$$

$$D_{\kappa\kappa}{}^\pm(x,y) = D^\pm(x-y)-\Delta^\pm(x)+\Delta^\mp(y),$$
$$D_\kappa{}^\pm(x,y) = D^\pm(x-y)+\Delta^\mp(y),$$
$$D^\pm(\xi) = \mp\frac{i}{2\pi}\int\frac{dp^1}{2p^0}[e^{\pm ip\xi}-\theta(\kappa-p^0)] = \mp(4\pi i)^{-1}\ln(-\mu^2\xi^2\mp i\epsilon\xi^0),$$
$$\mu = e^{-\Gamma'(1)}\kappa,$$
$$\Delta^\mp(x) = \pm\frac{i}{4\pi}\int\frac{dk^1}{k^0}\theta(\kappa-k')(e^{\mp ikx}-1).$$

The Thirring-model field, which depends explicitly on the infrared regularization parameter, is defined as

$$\phi(x) = e^{i\chi^+(y)}\psi(x)e^{i\chi^-(x)}, \quad (\mathrm{A6})$$

where

$$\chi^\pm(x) = K^\pm(x)+\Omega^\pm(x),$$
$$K^\pm(x) = \alpha J^\pm(x)+\beta\gamma^5\tilde{J}^\pm(x),$$
$$\Omega^\pm(x) = \pi^{1/2}(\alpha Q+\beta\gamma^5\tilde{Q})[\Delta^\pm(x)+\gamma^5\Delta^\pm(x)],$$
$$J^\pm(x) = j^\pm(x)-\alpha Q\Delta^\pm(x)-\beta\tilde{Q}\tilde{\Delta}^\pm(x).$$

The parameters α and β may be fixed by specifying the coupling constant $g=\pi^{1/2}(\alpha-\beta)$ and the spin

$$S=\tfrac{1}{2}+\tfrac{1}{2}(\alpha\beta-\alpha\pi^{1/2}-\beta\pi^{1/2})/\pi.$$

The field thus defined is an operator-valued tempered distribution with dense domain generated by acting on the vacuum state with Wightman polynomials in $\phi(x)$ and $\phi^\dagger(x)$. The usual axioms of quantum field theory may be verified by examining the Wightman functions (2.1), which one easily derives using definition (A6) and commutation relations (A5).

Normal products[11] of Thirring model fields are defined for $S=\frac{1}{2}$ as

$$N[\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]=\sum_S (-1)^{\delta_S+m_S}\langle 0|\phi(x_{S1})\cdots\phi(x_{Sm_S})\phi^\dagger(y_{S1})\cdots\phi^\dagger(y_{Sm_S})|0\rangle f(S)^{-1}f(T)^{-1}$$

$$\times\exp\{i\sum_{j=1}^{m_S}[K^+(x_{Sj})-K^+(y_{Sj})]\}\phi(x_{T1})\cdots\phi(x_{Tm_T})\phi^\dagger(y_{T1})\cdots\phi^\dagger(y_{Tn_T})\exp\{i\sum_{j=1}^{m_S}[K^-(x_{Sj})-K^-(y_{Sj})]\}, \quad \text{(A7)}$$

where

$$S=\{x_{S1},\ldots,x_{Sm_S};\, y_{S1},\ldots,y_{Sm_S}\},\quad S\cup T=\{x_1,\ldots,x_m;\, y_1,\ldots,y_n\}$$

$$T=\{x_{T1},\ldots,x_{Tm_T};\, y_{T1},\ldots,y_{Tn_T}\},\quad S\cap T=\phi$$

$$(-1)^{\delta_S}=(\text{sign of relative permutation of spinor factors})$$

$$f(x_1,\ldots,x_m;\, y_1,\ldots,y_m)=\prod_{i<j} d^-(x_i-x_j)d^-(y_i-y_j)\prod_{k,l}[d^-(x_k-y_l)]^{-1},$$

$$d^-(x-y)=\exp[i(a+b\gamma_x{}^5\gamma_y{}^5)D^-(x-y)].$$

In terms of the auxiliary free fields, these may be written

$$N[\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]=\exp\{i[\sum_{i=1}^{m}K^+(x_i)-\sum_{j=1}^{n}K^+(y_j)]\}\sum_P(-1)^{\delta_P+m_P}\sum_{k=1}^{m_P}\langle 0|\psi(x_{Pk})\psi^\dagger(y_{Pk})|0\rangle$$

$$\times\{\exp[i\Omega(x_{Pk})-\Omega(y_{Pk})]-1\}\exp\{i[\sum_{l=1}^{m_Q}\Omega^+(x_{Ql})-\sum_{r=1}^{n_Q}\Omega^+(y_{Qr})]\}:\psi(x_{Q1})\cdots\psi(x_{Qm_Q})\psi^\dagger(y_{Q1})\cdots\psi^\dagger(y_{Qn_Q}):$$

$$\times\exp\{i[\sum_{l=1}^{m_Q}\Omega^-(x_{Ql})-\sum_{r=1}^{n_Q}\Omega^-(y_{Qr})]\}\exp i[\sum_{i=1}^{m}K^-(x_i)-\sum_{j=1}^{n}K^-(y_j)], \quad \text{(A8)}$$

where the summation is over all sets of pairs

$$P=\{(x_{P1},y_{P1}),\ldots,(x_{Pm_P},x_{Pm_P})\}$$

and Q is the set of all x_i, y_j not included in P. These objects are covariant and local in their separate arguments and approach covariant local fields in the limit of coincident arguments. The same is true of arbitrary derivatives, which gives a natural definition of quantities such as $N[\phi^\dagger(x)\partial_\nu\phi(u)]$.

Normal products of $J_\mu(x)$ and $\tilde{J}_\mu(x)$ are conveniently defined by a Wick ordering of their positive- and negative-frequency parts.

The normal products of fields may be used to rewrite the current and equation of motion as

$$J_\mu(x)=\pi^{-1/2}\partial_\mu J(x)=N[\bar\phi\gamma_\mu\phi](x),$$
$$-i\gamma^\mu\partial_\mu\phi(x)=g[J_\mu{}^+(x)\gamma^\mu\phi(x)+\gamma^\mu\phi(x)J_\mu{}^-(x)] \quad \text{(A9)}$$
$$=gN[(\bar\phi\gamma_\mu\phi)\gamma^\mu\phi](x).$$

Note added in proof. In this issue[13] Jackiw criticizes the relevance of two-dimensional soluble models, such as the Thirring model, to realistic discussions of scale invariance. He claims that (1) anomalous dimension arises solely from peculiarities of scalar fields in two dimensions and (2) the commutator of the dilatation generator with an interacting Fermi field is singular in such theories, another consequence of the two-dimensional pathologies. Jackiw's pessimistic assessment of two-dimensional models may yet prove to be accurate, but we do not find his present arguments convincing. In particular, any argument against anomalous dimension is also an argument against the very concept of operator dimension in a scale-invariant theory describing interaction, as follows from a theorem of Pohlmeyer.[14] Moreover, the singular commutation relations which Jackiw displays in the case of the derivative coupling model[8] are due to his using an inappropriate energy-momentum tensor (the sum of free-field tensors, which is not local with respect to the interacting field). Clearly no singularities arise if one proceeds as in Ref. 8 or as in Sec. III above.

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