Relativistic Harmonic Oscillators and the Structure of Baryons*

G. Cocho†
International Centre for Theoretical Physics, Trieste, Italy

AND

J. Flores
Instituto de Física, Universidad de México, México 20 DF

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We propose a method, based on group theory, by which a relativistic wave equation for a few-particle system can be related to a Schrödinger-like equation in such a way that solving the latter is equivalent to obtaining the solution to the first problem. The method is applied to studying the structure of the baryons considered as built from three quarks. We calculate the relativistic form factor of the proton, and a simple two-quark effective interaction consistent with the form factor is obtained. Using this effective potential, an analysis of the baryon mass spectrum is then made.

I. INTRODUCTION

THE quark model for baryons has had several striking successes.[1,2] Among them, one may mention the prediction of the correct quantum numbers for the allowed two-body resonance channels, as well as a qualitative explanation of the low-energy part of the baryon spectrum. The model has several drawbacks, however; one of them is the fact that the motion is described nonrelativistically. Consequently, several of the predicted results may be seriously questioned; for example, the predicted scalar form factor of the proton is altered considerably when a fully relativistic treatment is given.

On the other hand, an altogether different approach to explain the low-energy baryon spectrum and the proton form factor has been proposed.[3] Using the method of noncompact groups, in particular the $O(4,2)$ group, relativistically invariant wave equations have been obtained. A reasonable baryonic spectrum and reasonable proton form factors are obtained, but the model contains no elements to explain the supermultiplets and the corresponding permutational symmetry quantum numbers appearing in the spectrum.

In this series of papers we consider a different approach to the problem. Using the methods of group theory, we propose a way in which relativistically invariant equations could be formulated for systems of several internal degrees of freedom—or quasiparticles, as we prefer to call them. Since the model consists of several particles interacting among themselves, permutational symmetry considerations can be introduced. Using the appropriate effective interaction among the particles, the model can, in principle, explain the baryon spectrum quantum numbers and excitation energies.

In order to formulate the relativistic wave equations for the system of interacting quasiparticles, we use the generators of the noncompact group $U(3,1)$ and the components of the total linear momentum P_μ. We have discussed recently[4] some mathematical aspects of $U(3,1)$ and some of its applications to nuclear physics, in particular to the few-nucleon problem. The $U(3,1)$ group is the dynamical group of a possible relativistic generalization of the harmonic oscillator. We use this fact to obtain a nonrelativistic problem isomorphic to the relativistic one, formulated in terms of $U(3,1)$ bases. The two problems are isomorphic in the sense that solving one is equivalent to solving the other. The techniques developed recently[5] for dealing with the few-body Schrödinger equation, using truncated harmonic-oscillator states, can then be applied. Relativistic form factors[6] and excitation energies are then obtained.

Since a truncated set of many-particle oscillator states is used, an effective interaction among the particles is introduced. From the experimental proton form factor,[7] known rather well up to momentum transfers of about 600 F^{-2}, we can determine several features of this effective interaction and from it obtain a baryonic spectrum. We shall do this in Sec. III.

II. RELATIVISTIC AND NONRELATIVISTIC WAVE EQUATIONS

In this section we shall indicate how to construct relativistic wave equations and their approximate solutions which describe the properties of extended composite systems. The systems are extended in the sense that their form factors decrease asymptotically with the

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† On leave of absence from Instituto de Física, Universidad de México, México DF.

[1] H. J. Lipkin, in *Proceedings of the International Conference on Elementary Particles, Heidelberg, 1967*, edited by H. Filthoth (North-Holland, Amsterdam, 1968); and invited paper, European Physical Society Meeting, Florence, 1969 (unpublished).

[2] G. Morpurgo, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[3] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968); A. O. Barut and K. C. Tripathy, *ibid.* **19**, 918 (1967); **19**, 1081 (1967).

[4] G. Cocho and J. Flores, Rev. Mex. Fis. **19**, 67 (1970); Phys. Letters **31B**, 639 (1970).

[5] M. Moshinsky, *The Harmonic Oscillator in Modern Physics* (Gordon and Breach, New York, 1969).

[6] G. Cocho and J. Flores, Nucl. Phys. **A143**, 529 (1970).

[7] D. H. Coward *et al.*, Phys. Rev. Letters **20**, 292 (1968).

momentum transfer. They are composite in the sense that they are built up from several interacting particles. From the point of view of applications, the relativistic wave equations should be constructed in such a way that they predict the correct mass spectrum of the system, as well as its Lorentz transformation properties. These properties, in particular, imply predicting the observed transition and form factors.

In Sec. III we apply the formalism developed here to explain some of the properties of the baryons, assuming that they are extended composite systems. The assumption is justified since, at least for low excitation energies, the baryonic mass spectrum and some other properties can be explained by considering that the baryons are formed by three particles (that could be, for example, quarks).

Let us denote by Ψ the state function that describes the system in a limited energy range; in other words, Ψ is an effective wave function. The relativistic wave function Ψ, expressed in momentum space, will be the solution of the wave equation[8]

$$W\Psi=0, \tag{2.1}$$

where the wave operator W is an invariant with respect to Lorentz transformations. Our task will be to find an operator W such that the solution to Eq. (2.1) is consistent with the above-mentioned experimental data.

The wave operator W, for an extended composite system, contains the information related to the c.m. motion as well as that related to the internal motion. Consequently, W will be a function of P_μ, the total linear momentum operator, and of the internal variables. Instead of these latter, we describe as in previous work[4] the internal structure in an algebraic way, using the generators of a group G. For a system consisting internally of only one quasiparticle, we need for this purpose the generators of a single group G. For a system containing s quasiparticles, we need s different groups $G^{(t)}$, $t=1, \ldots, s$; the generators of $G^{(t)}$ commute with those of $G^{(t')}$ for $t\neq t'$. For systems formed by r particles, one has $s=r-1$, since it is only the internal structure we want to describe using these groups, the c.m. motion being dealt with by the operator P_μ, in momentum space.

Since W should be a Lorentz-invariant operator, reflecting the isotropy in space-time, it follows that $G^{(t)}$ should be a noncompact group. This is so, since $G^{(t)}$ must contain a subgroup isomorphic to the Lorentz group. Several choices are possible for $G^{(t)}$, although the simplest one from the mathematical point of view is the unitary group $U(3,1)$. Since this will allow us to formulate the problem in a very convenient way, we make this choice.

[8] It is worthwhile to remark that we are considering effective wave equations. Therefore, the troubles arising when dealing with infinite-component field theory do not appear here. See I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

We shall now review some properties of the $U(3,1)$ group and of a possible chain of its subgroups. A more detailed treatment can be found in Refs. 4 and 9. The generators of $U^{(t)}(3,1)$ are denoted by

$$\mathcal{C}_{\mu\nu}{}^{(t)}, \quad \mu, \nu=0, 1, 2, 3.$$

They fulfill the commutation rules

$$[\mathcal{C}_{\mu\nu}{}^{(t)}, \mathcal{C}_{\lambda\rho}{}^{(t')}]=(-g_{\nu\lambda}\mathcal{C}_{\mu\rho}{}^{(t)}+g_{\mu\rho}\mathcal{C}_{\lambda\nu}{}^{(t)})\delta_{tt'}, \tag{2.2}$$

where the metric tensor $g_{\mu\nu}$ is diagonal and has components

$$g_{00}=1, \quad g_{ij}=-\delta_{ij}, \quad i, j=1, 2, 3.$$

Regarding the bases for irreducible representations of $U(3,1)$, we simply mention here that in the completely symmetrical case, corresponding to a one-row Young diagram, the bases are labeled by a single quantum number N, real and negative for unitary representations.[9]

We now introduce a chain of subgroups of $U(3,1)$; this will allow us to include in the formalism information related to some internal properties of the system. In particular, since the intrinsic spin is of physical interest, we include the angular momentum group $O(3)$. Another subgroup, that contains $O(3)$, the harmonic-oscillator symmetry group $U(3)$, is also included. This will lead to the simplest formulation possible. We are dealing, therefore, with the following chain of groups:

$$U(3,1)\supset U(3)\supset O(3). \tag{2.3}$$

As we have discussed previously,[6,9] the generators

$$\hat{\mathcal{C}}_{\mu\nu}=\sum_{\lambda\rho}\theta_\mu{}^\lambda\theta_\nu{}^\rho\mathcal{C}_{\lambda\rho}, \tag{2.4}$$

with

$$\theta_\mu{}^\nu=\delta_\mu{}^\nu-v_\mu v^\nu, \tag{2.5}$$

where $v_\mu=P_\mu/(|P|^2)^{1/2}$ is the four-velocity, are such that they reduce, in the rest frame, to the generators $\mathcal{C}_{ij}$, $i, j=1, 2, 3$ of the $U(3)$ group. The eigenstates of the Lorentz-invariant operator

$$\Gamma=\sum_\mu \hat{\mathcal{C}}_\mu{}^\mu \tag{2.6}$$

have equivalents, defined in an invariant way, which coincide with the number of quanta in the rest frame. We call $U_p(3)$ the group generated by Eq. (2.4). In an entirely analogous way one introduces the orthogonal subgroup $O_p(3)$.

The eigenvectors of the classification chain will now be denoted by

$$\psi_{Nnl}(p), \tag{2.7}$$

where N gives the irreducible representations of $U(3,1)$ and n and l those of $U_p(3)$ and of $O_p(3)$. In case one has only one quasiparticle $r=1$, the wave function $\Psi(p)$,

[9] G. Cocho, C. Fronsdal, I. T. Grodsky, and R. White, Phys. Rev. **162**, 1662 (1967); G. Cocho, J. Flores, and A. Mondragón, Nucl. Phys. **A128**, 110 (1969).

describing the intrinsic motion of the system that moves with four-momentum P_μ, can be expanded as

$$\Psi(p)=\sum_{Nnl} a_{Nnl}\psi_{Nnl}(p). \tag{2.8}$$

Determining the eigenvectors a_{Nnl} will now solve our problem; the internal structure is entirely described by them, in a completely algebraic way.

For a system of s quasiparticles, one introduces s groups $U^{(t)}(3,1)$, $t=1, s$, with generators $\mathcal{C}_{\mu\nu}{}^{(t)}$. The expansion corresponding to (2.8) is now

$$\Psi(p)=\sum_{N_in_il_i} a(N_in_il_i)\psi_{N_1n_1l_1}(p)\cdots\psi_{N_sn_sl_s}(p), \tag{2.9}$$

and our task is to determine the eigenvector $a(N_in_il_i)$. The summation in (2.9) extends, in principle, to infinity so, up to here, no approximation is made. If the quasiparticles are identical, all N_i can be taken to be equal.

We now turn to the wave operator. If one uses expansion (2.9) it is natural to assume that W is a function of P_μ and of $\mathcal{C}_{\mu\nu}{}^{(t)}$, $t=1, s$. Since these operators have definite Lorentz transformation properties, it is possible to construct an invariant operator function of them:

$$W=W(P_\mu,\mathcal{C}_{\mu\nu}{}^{(t)}), \quad t=1, s. \tag{2.10}$$

In order to clear up some of the above ideas, let us consider an example in which a specific form of W is assumed. Let us define, for a single quasiparticle,

$$W_1=P_\mu P^\mu-AP_\mu\mathcal{C}^\mu{}_\nu P^\nu-B,$$

where A and B are constants. The wave equation (2.1) has the solution

$$\psi=\psi_{Nnl}(p),$$

since W_1 is formed by the first-order Casimir operators of the chain (2.3). In the intrinsic frame, (2.1) reduces to

$$[m^2-Am^2(N-n)-B]\psi_{Nnl}(p)=0,$$

which leads to the mass spectrum

$$m^2=B/[1-A(N-n)]$$

that decreases with n. On the other hand, the form factor of the system is that obtained from $\psi_{Nnl}(p)$.[6] It is clear from this example what will happen in the general case: Mass spectrum and form factors are predicted. Of course, in any real situation, the form factors extracted from the $\psi_{Nnl}(p)$ of this example do not coincide with the experimental ones. A more general form of W is therefore needed, the wave function being then given as the linear combination (2.8) or, in the case of s quasiparticles, as in Eq. (2.9).

Returning to the general case, let us describe the internal motion in the rest frame. The wave equation then becomes

$$W(M,\mathcal{C}_{00}{}^{(t)},\mathcal{C}_{0i}{}^{(t)},\mathcal{C}_{i0}{}^{(t)},\mathcal{C}_{ij}{}^{(t)})\Psi(0)=0. \tag{2.11}$$

In Eq. (2.11), M is the mass operator. [Acting on states at rest, $P_\mu\Rightarrow(M,0,0,0)$.] Let us analyze how the generators $\mathcal{C}_{\mu\nu}{}^{(t)}$ operate on the wave function Ψ, as given by (2.9). In order to do this, we consider an explicit realization of both the generators $\mathcal{C}_{\mu\nu}{}^{(t)}$ and the wave functions ψ_{Nnl} in the rest frame. To obtain it we introduce four complex variables z_μ, $\mu=0, 1, 2, 3$, and define

$$\mathcal{C}_\mu{}^\nu\equiv -z_\mu\partial/\partial z_\nu, \tag{2.12}$$

as well as the operator

$$-\sum_{\mu\nu} g^{\mu\nu}\mathcal{C}_{\mu\nu}=\sum_\mu z_\mu\frac{\partial}{\partial z_\mu}, \tag{2.13}$$

which can be used as an invariant operator to characterize the irreducible representation of $U(3,1)$. The irreducible basis is obtained from the equation

$$\sum_\mu z_\mu\frac{\partial}{\partial z_\mu}f(z)=Nf(z),$$

where N is the index of the representation. The functions $f(z)$ can be expanded in terms of monomials of type

$$z_1{}^{n_1}z_2{}^{n_2}z_3{}^{n_3}z_0{}^\alpha, \quad n_1+n_2+n_3+\mathrm{g}=N. \tag{2.14}$$

For n_1, n_2, and n_3 positive integers, these monomials span a representation of $U(3)$, $\nu=n_1+n_2+n_3$ being the total number of quanta. The spherical basis for the chain of Eq. (2.3) may be obtained as a linear superposition of monomials of the basis of Eq. (2.14).

If we define the normalized Cartesian basis,

$$\psi_{Nn_1n_2n_3}(0)\equiv\frac{z_1{}^{n_1}z_2{}^{n_2}z_3{}^{n_3}z_0{}^{N-\nu}}{[n_1!n_2!n_3!(N-\nu)!]^{1/2}},$$

then using (2.12) and (2.14) one readily obtains

$$\mathcal{C}_{00}\psi_{Nn_1n_2n_3}(0)=(N-\nu)\psi_{Nn_1n_2n_3}(0), \tag{2.15}$$

$$\mathcal{C}_{01}\psi_{Nn_1n_2n_3}(0)=[n_1(N-\nu+1)]^{1/2}\times\psi_{N\,n_1-1\,n_2\,n_3}(0), \tag{2.16}$$

$$\mathcal{C}_{10}\psi_{Nn_1n_2n_3}(0)=[(n_1+1)(N-\nu)]^{1/2}\times\psi_{N\,n_1+1\,n_2\,n_3}(0), \tag{2.17}$$

$$\sum_{i=1}^{3}\mathcal{C}_{ii}\psi_{Nn_1n_2n_3}(0)=\nu\psi_{Nn_1n_2n_3}(0), \tag{2.18}$$

with similar expressions for $\mathcal{C}_{0i}$ and $\mathcal{C}_{i0}$ if $i\neq 1$.

Using these equations, one can establish an isomorphism between Eq. (2.11) and a Schrödinger-like equation of the type found in nonrelativistic quantum mechanics.

As can be seen from Eqs. (2.16) and (2.17), $\mathcal{C}_{0i}$, $i=1, 2, 3$, acts on the $U(3,1)$ basis in the same form as

the nonlocal operator

$$a_i(N-\sum_{i=1}^{3} a_i^\dagger a_i+1)^{1/2} \tag{2.19}$$

on the $U(3)$ basis, i.e., the harmonic-oscillator states. Here $a_i^\dagger$ and a_i are the usual creation and annihilation operators. $\mathcal{C}_{i0}$ acts on the $U(3,1)$ basis in the same form as

$$a_i^\dagger(N-\sum_{i=1}^{3} a_i^\dagger a_i)^{1/2} \tag{2.20}$$

on the $U(3)$ basis. Conversely, $\mathcal{C}_{0i}(\mathcal{C}_{00}+1)^{-1/2}$ acts identically on $\psi_{Nn_1n_2n_3}(0)$ as a_i operates on the $U(3)$ basis, with a similar expression for $a_i^\dagger$.

In other words, given a wave operator $W(m,\mathcal{C}_{\mu\nu}{}^{(t)})$, one can always find another operator W', function of $a_i{}^{(t)\dagger}$ and $a_i{}^{(t)}$ isomorphic to it. Solving the problem with W' in the $U(3)$ basis is tantamount to solving it for W in the $U(3,1)$ basis. Although the isomorphism has been proved in the rest frame, since W is a Lorentz-invariant operator, we have a covariant formulation. We are in a position to solve a relativistic many-particle problem by solving a corresponding nonrelativistic one.

As can be seen from (2.16) and (2.17), when W is simple, W' has, in general, a very complicated structure. On the other hand, since rather powerful techniques for dealing with the few-particle Schrödinger equation using oscillator states have been developed,[5] one can formulate the problem such that W' corresponds to a Schrödinger-like equation which can then be solved approximately.

Owing to this and also to the fact that nonrelativistic intuition can be used to formulate the appropriate dynamics, we first postulate an explicit form for W' and then solve the problem with the known nonrelativistic techniques, having always in mind that there exists a relativistic problem being solved simultaneously.

When W' defines a Schrödinger-like equation, one is introducing the dynamics through an effective potential V_{eff}. The effective potential will depend on the interaction mechanism among the constituents of the system. It will be different if the interaction occurs via the exchange of mesons or if it occurs via the exchange of Regge poles, or any other mechanisms. In any case, it should be chosen in such a way that the observed mass spectrum and form factors are obtained simultaneously.

In order to complete the formulation presented here, we now indicate the main features of the solution technique. Using harmonic-oscillator functions, Moshinsky[5] has developed a method for constructing few-particle wave functions in which the c.m. motion has been eliminated. Besides this, the state functions are classified by permutational symmetry and angular momentum quantum numbers. Furthermore, they obtain matrix elements of one- and two-body operators with respect to these functions. By expanding the wave function Ψ in terms of these functions and restricting the expansion to a finite number of them, one obtains a matrix problem solvable by conventional methods, eigenvalues and eigenvectors being consequently obtained. Clearly, this is precisely the nonrelativistic isomorphic problem one needs for our application.

III. BARYON FORM FACTORS AND MASS SPECTRUM

In Sec. II we have discussed a method, based on noncompact group theory, useful in constructing relativistically invariant wave equations for a few-particle system. We have shown that the relativistic wave equation is formally equivalent to a nonrelativistic one, which can then be solved approximately using well-known techniques.[5] In this section we apply the method to calculate the low-energy baryon spectrum and the proton form factor, assuming that baryons are composite systems.

Before proceeding, let us discuss to what model of the baryons as composite systems we should apply the techniques developed in Sec. II. In the simplest version of the quark model, one assumes that baryons are formed by three fractionally charged particles, having spin $\frac{1}{2}$ and normal magnetic moments.[10] This model has been useful in explaining qualitatively some of the features of the baryon structure. However, one could think of more complicated models. In particular, as some high-energy scattering and photoproduction experiments might suggest, one could assume the existence of an internal region, perhaps built up of several quark-antiquark pairs.[11]

These considerations lead us to analyze a more complicated model for baryons, assuming that they are formed by four particles, interacting among themselves, of which three have mass m and one has mass M. Let us denote by V_{mm} the interaction between any two particles of mass m and by V_{mM} the interaction of one of these with the fourth mass M. We can then distinguish among three different situations, according to the relative magnitudes of V_{mm} and V_{mM}. (a) $|V_{mm}|\gg|V_{mM}|$. One could think of this case as analogous to the adiabatic approximation in molecular physics. In this case, the energy due to excitation of the relative modes of the three-quark system is much larger than that due to the motion of particles of mass m relative to the fourth body. We then have two alternatives: Either we are detecting, when looking at the baryon spectrum, excitations of the first type, in which case the proton should have fine structure, or we observe excitations of the second type, in which case the three-quark system would always have the same orbital quantum numbers for all the observed levels. Both alternatives seem to be contradicted experimentally. (b) $|V_{mm}|\simeq|V_{mM}|$. This case is rather difficult to analyze qualitatively, since it

[10] J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969).

[11] H. Harari, Phys. Rev. Letters **24**, 286 (1970).

is the full four-body problem. However, as can be seen assuming harmonic potentials, one expects many more levels than in the three-quark case. (c) $|V_{mm}| \ll |V_{mM}|$. The low-energy spectrum is that obtained with three particles in a common well. Again, as in case (a), we have two possibilities and either too few levels or too few degenerate states are predicted. None of them agrees with the experimental facts.

The above analysis seems to indicate that, at least in a first approach to the problem, one should apply the techniques of Sec. II to the usual quark model, leaving the discussion of the four-body model for a future work.

Using the approach discussed in Sec. II we describe the three-particle system by separating the c.m. motion from the internal degrees of freedom (introducing, therefore, two quasiparticles). The internal motion of quasiparticle t ($t=1, 2$) is then represented by using the generators of the noncompact group $U^{(t)}(3,1)$ and the whole motion is then described in a relativistically invariant way.

In this section we consider the state function expanded in terms of a finite set of $U(3,1)$ irreducible states. The coefficients of the linear combination are extracted from the experimental form factor of the system ground state, i.e., the proton. Knowing the ground state, we deduce a simple two-body effective interaction from which we obtain the low-energy baryon spectrum.

A. Relativistic Form Factors

As discussed in Sec. II, we consider an effective wave equation

$$W(P_\mu, \mathcal{C}_{\mu\nu}{}^{(t)})\Psi(p_\mu)=0 \quad (\mu, \nu=0, 1, 2, 3; t=1, 2), \tag{3.1}$$

where P_μ is the c.m. four-momenta and $\mathcal{C}_{\mu\nu}{}^{(t)}$ are the generators of $U^{(t)}(3,1)$, the index t distinguishing between the two internal quasiparticles. In the rest frame, as we showed in Sec. II, one can relate the wave operator W to another wave operator W', a function of the usual creation and annihilation operators of the harmonic oscillator, $a_i{}^{(t)\dagger}$ and $a_i{}^{(t)}$ ($i=1, 2, 3$), in such a way that solving Eq. (3.1) with an expansion of the type

$$\Psi(p_\mu)=\sum_{n_i l_i} a(n_i l_i)\psi_{Nn_1l_1}(p_\mu)\psi_{Nn_2l_2}(p_\mu) \tag{3.2}$$

is equivalent to solving the equation with the wave operator W' and a wave function Ψ':

$$\Psi'=\sum_{n_i l_i} a(n_i l_i)\psi_{n_1l_1}\psi_{n_2l_2}. \tag{3.3}$$

In (3.2) $\psi_{Nnl}(p_\mu)$ stands for the irreducible basis of the chain of groups (2.3) and in (3.3) ψ_{nl} is the harmonic-oscillator wave function, with principal quantum number n and orbital angular momentum l, corresponding to a total number of quanta ν given by

$$\nu=2n+l. \tag{3.4}$$

Instead of using expansion (3.3), one can express Ψ' in terms of two-quasiparticle states classified by permutational symmetry quantum numbers (i.e., the Young diagrams [3], [21], or [111] for a three-particle system) and by the total orbital angular momentum Λ. This classification scheme is useful when W' is a symmetrical operator with respect to the exchange of particles and invariant with respect to rotations in orbital space. Since we shall assume here these invariance properties for W', the Young diagram $[f]$ and the angular momentum Λ are good quantum numbers. Three-particle translationally invariant states may be written in terms of the Jacobi "internal coordinates"

$$\mathbf{x}_a{}^{(12)} \equiv (1/\sqrt{2})(\mathbf{x}_1-\mathbf{x}_2),$$
$$\mathbf{x}_b{}^{(12)} \equiv (1/\sqrt{6})(\mathbf{x}_1+\mathbf{x}_2-2\mathbf{x}_3),$$

with

$$\begin{aligned}
\mathbf{x}_a{}^{(31)} &\equiv (1/\sqrt{2})(\mathbf{x}_3-\mathbf{x}_1) = -\tfrac{1}{2}\mathbf{x}_a{}^{(12)}-\tfrac{1}{2}\sqrt{3}\mathbf{x}_b{}^{(12)}, \\
\mathbf{x}_b{}^{(31)} &\equiv (1/\sqrt{6})(\mathbf{x}_3+\mathbf{x}_1-2\mathbf{x}_2) = \tfrac{1}{2}\sqrt{3}\mathbf{x}_a{}^{(12)}-\tfrac{1}{2}\mathbf{x}_b{}^{(12)}, \\
\mathbf{x}_a{}^{(23)} &\equiv (1/\sqrt{2})(\mathbf{x}_2-\mathbf{x}_3) = -\tfrac{1}{2}\mathbf{x}_a{}^{(12)}+\tfrac{1}{2}\sqrt{3}\mathbf{x}_b{}^{(12)}, \\
\mathbf{x}_b{}^{(23)} &\equiv (1/\sqrt{6})(\mathbf{x}_2+\mathbf{x}_3-2\mathbf{x}_1) \\
&= -\tfrac{1}{2}\sqrt{3}\mathbf{x}_a{}^{(12)}-\tfrac{1}{2}\mathbf{x}_b{}^{(12)}.
\end{aligned} \tag{3.5}$$

Using Eqs. (3.5) one may express the symmetrical operator W' in terms of $\mathbf{x}_a{}^{(12)}$ and $\mathbf{x}_b{}^{(12)}$ only. Taking into account the transformations of the end of Sec. II, one may write the corresponding W as a function of $\mathcal{C}_{\mu\nu}{}^{(t)}$, $t=1, 2$.

We now extract form factors from the wave function Ψ. Using the equivalence with Ψ', valid in the rest frame, we use the same formula as Moshinsky,[5] except that we replace the single-quasiparticle function ψ_{nl} by its relativistic generalization $\psi_{Nnl}(P_\mu)$. In this way, the correct Lorentz transformation properties are taken into account. We have obtained the detailed expressions in a previous paper,[4] and we repeat them here for convenience.

The scalar form factor $F(q^2)$, where q^2 is the invariant momentum transfer, corresponding to the wave function (2.2) classified by the partition [3] and $\Lambda=0$, is given by

$$\begin{aligned}
F(q^2) = \sum_{n_1l_1, n_2l_2} \sum_{\bar{n}_1\bar{l}_1, \bar{n}_2\bar{l}_2} \sum_{n_1'l_1', n_2'l_2'} \sum_{\bar{n}_1'\bar{l}_1', \bar{n}_2'\bar{l}_2'} & a^*(n_i'l_i')a(n_il_i) \\
\times \langle \bar{n}_1\bar{l}_1\bar{n}_2\bar{l}_2\Lambda | n_1l_1n_2l_2\Lambda\rangle & \langle n_1'l_1'n_2'l_2'\Lambda | \bar{n}_1'\bar{l}_1'\bar{n}_2'\bar{l}_2'\Lambda\rangle \\
\times \langle \psi_{N\bar{n}_2'\bar{l}_2}(p_\mu)\psi_{N\bar{n}_2l_2}(p_\mu+q_\mu)\rangle & \delta_{\bar{n}_1'\bar{n}_1}\delta_{\bar{l}_1'l_1}.
\end{aligned} \tag{3.6}$$

Here $\langle n_1l_1n_2l_2\Lambda | n_1'l_1'n_2'l_2'\Lambda\rangle$ is a coefficient of a geometrical character expressed in terms of the oscillator transformation brackets of Brody and Moshinsky[12] as indicated explicitly in Ref. 5. We have obtained a closed expression for the matrix element:

$$\langle \psi_{Nn'l}(p_\mu)\psi_{Nnl}(p_\mu+q_\mu)\rangle \tag{3.7}$$

[12] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Gordon and Breach, New York, 1967).

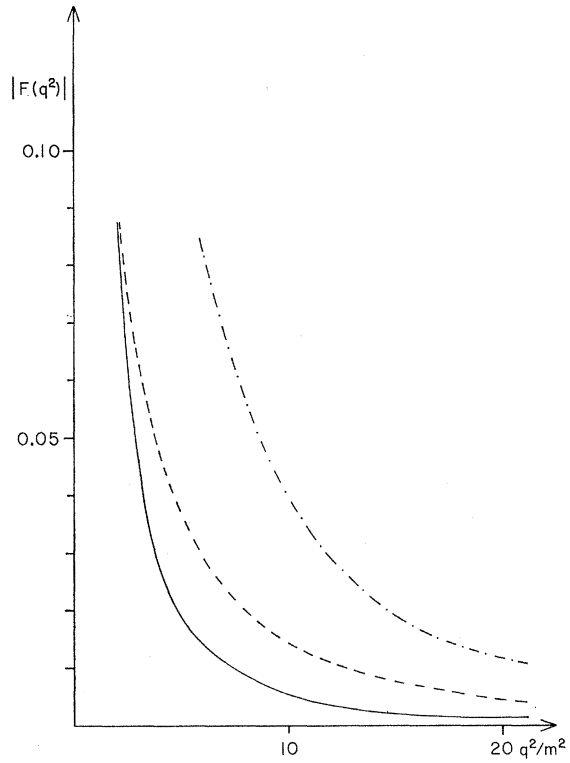

FIG. 1. Comparison of the experimental (solid line) with the calculated proton form factor. The dashed line corresponds to $a_1=0.63$, $a_2=0$, $a_3=0.45$, and $a_4=-0.63$, while the dot-dashed line corresponds to the same values of a_i, except that $a_4=0.63$. Here q^2 is given in units of m^2 (m stands for the mass of the proton).

as a function of q^2 which is given explicitly in the Appendix (see also Ref. 4).

We shall now compare such a scalar form factor with the electromagnetic form factors of the proton. These form factors are rather well known experimentally up to $q^2\simeq 20$ GeV^2. One gets a reasonable fit to both the electric and magnetic form factors using the dipole formula[7]

$$F_p(q^2)=[1+q^2/0.71\ (\mathrm{GeV}/c)^2]^{-2}, \qquad (3.8)$$

which is plotted in Fig. 1 (solid line). Although the comparison of our scalar form factor with the experimental electromagnetic form factors of the proton is questionable and a better analysis of the electromagnetic interaction within the frame of our model is necessary, we shall assume for the time being that the error in such a comparison is not large.

We now use expansion (3.2) and vary the coefficients $a(n_il_i)$ to fit the form factor (3.8). We assume that the basis functions have $\Lambda=0$ and correspond to the Young diagram [3]. This is consistent with the assumption of spatial symmetry of the usual quark model and with the fact that the proton seems to correspond to the symmetrical irreducible representation of $SU(6)$.[10] We further restrict the expansion to states of up to four quanta, although the calculation is feasible for a larger number of quanta and indeed we have performed it in some cases. In Table I we give a list of the states considered.

TABLE I. Quantum numbers for three-particle states. The total number of quanta $2n_1+l_1+2n_2+l_2$ is less than or equal to 4.

[f]	Λ	π	n_1	l_1	n_2	l_2	i
3	0	+	0	0	0	0	1
			0	1	0	1	2
			0	2	0	2	3
			1	0	1	0	4
	2	+	0	1	0	1	1
			0	2	0	2	2
			0	2	1	0	3
21	0	+	0	0	1	0	1
			0	1	1	1	2
			0	0	2	0	3
	1	−	0	0	0	1	1
			0	1	0	2	2
			0	1	1	0	3

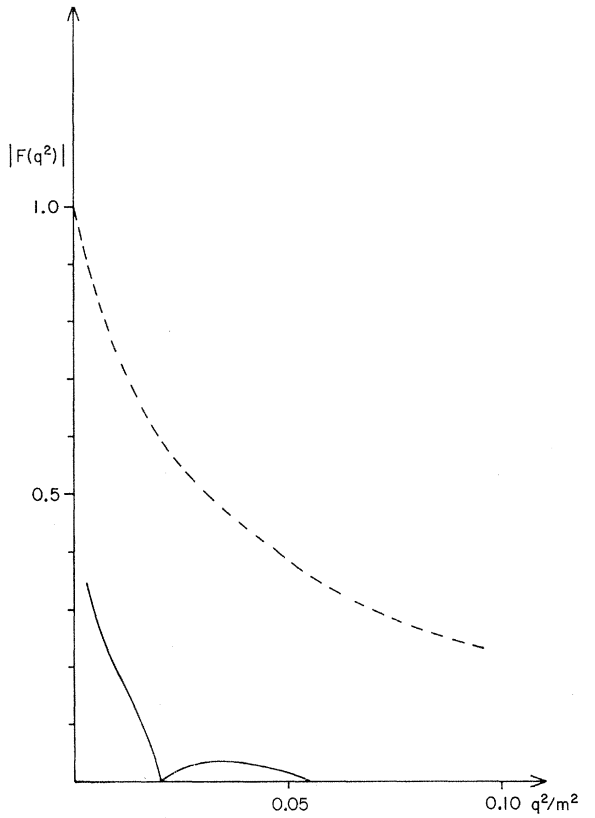

FIG. 2. Comparison of the calculated form factors with $|N|=2$ (dashed line) and with $|N|=20$ (solid line). The set of coefficients a_i is the same as in Fig. 1, with $a_1/a_4>0$.

As can be seen from Eq. (3.6) and the explicit formula for the scalar product (3.7), the scalar form factor $F(q^2)$, arising from a truncated $U(3,1)$ basis, behaves for large transfers as $(q^2)^{-|N|}$. A comparison with the dipole formula (3.8) then fixes $|N|=2$. Using

this value, the scalar form factor for $q^2 \simeq 0$ becomes

$$F(q^2) \simeq 1 - f(a_1,a_2,a_3,a_4)q^2 \tag{3.9}$$

and for large values of q^2

$$F(q^2) \simeq g(a_1,a_2,a_3,a_4)q^{-4}, \tag{3.10}$$

where f and g are definite quadratic, homogeneous functions of the expansion parameters a_i, $i=1, 2, 3, 4$, referring to the wave functions of Table I, with partition [3] and $\Lambda=0$. Comparing (3.9) and (3.10) with the dipole formula, one can restrict the possible values of these coefficients. In particular, using the explicit forms of f and g, one can see that the limits (3.9) and (3.10) do not coincide with the experimental limits if all a_i are taken real and with the same sign.

Having this in mind and with the value $|N|=2$, we have varied a_i to obtain a good fit to Eq. (3.8). The

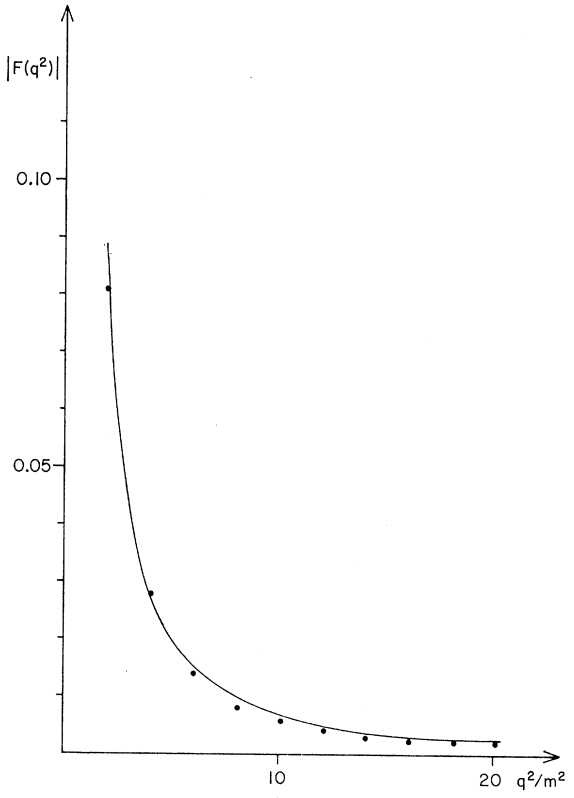

FIG. 3. Comparison of the experimental points with the calculated proton form factor (solid line) computed with $a_1=-0.761$, $a_2=0.165$, $a_3=0.229$, and $a_4=0.583$.

results are shown in Fig. 1 in which two sets of a_i are used. The two sets differ from each other only in the sign of the ratio a_1/a_4; as can be seen from the comparison, using $a_1/a_4<0$ gives a better result.

As we have shown before,[6,9] the nonrelativistic limit for $F(q^2)$ is obtained for $|N|\to\infty$. In the case of the proton we are therefore in the relativistic limit. It is interesting to analyze, however, what is the effect on the form factor when, using fixed values of the parameters a_i, one varies the value of $|N|$. In particular, we show in Fig. 2 the comparison of $F(q^2)$ calculated with $|N|=2$ and with $|N|=20$. We see that for $|N|=20$, F shows a diffraction minimum which disappears for low values of $|N|$, i.e., in the relativistic limit. This shows that nonrelativistic calculations[13] could be seriously questioned when dealing with the quark model of baryons.

Having fitted the form factor reasonably well, we have obtained the linear coefficients a_i and, therefore, the proton effective wave function. Using it, we shall show how to obtain an effective interaction and from it a wave operator W'. Solving the corresponding wave equation will then provide us with the low-energy baryonic spectrum.

B. Effective Interaction

In the previous section we have used a truncated relativistic oscillator basis to calculate the proton scalar form factor. Here we obtain a wave operator W' consistent with this form factor. In other words, we shall find W' such that the ground-state eigenvector of the corresponding wave equation reproduces the scalar form factor.

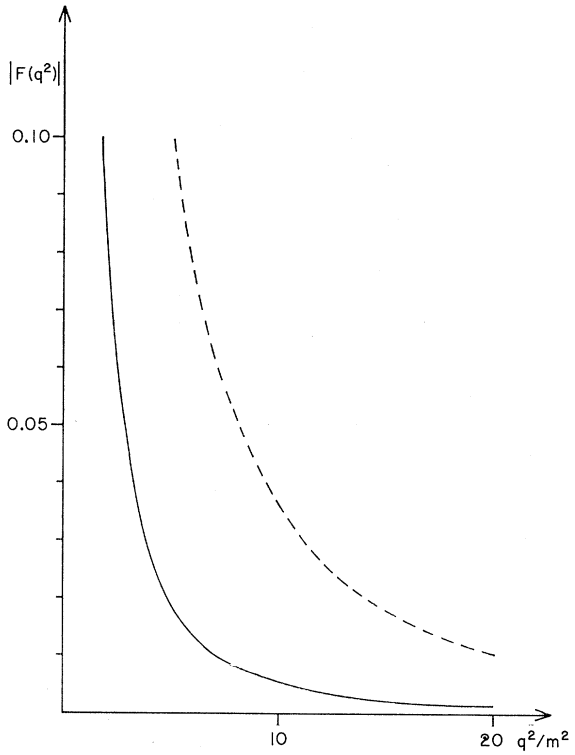

FIG. 4. Comparison of the experimental proton form factor (solid line) with the one obtained using a Coulomb potential as an effective interaction.

[13] C. A. Heusch and F. Ravndal, Phys. Rev. Letters 25, 253 (1970).

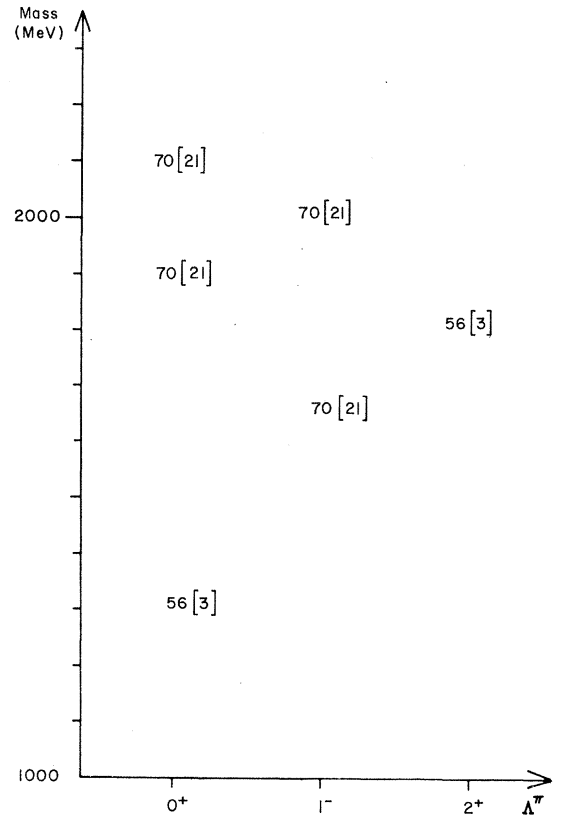

FIG. 5. Low-energy baryon spectrum using the effective interaction $V_{\text{eff}}=5\delta'(r-0.4\text{ F})$.

In order to do this, we use the equivalence of the wave equation (2.1) with the equation

$$W'(m,x_s,p_s)\Psi'=0, \quad s=a,b. \tag{3.11}$$

Here Ψ' is a linear combination of translationally invariant harmonic-oscillator states; m is the rest mass of state Ψ'. The subindices a and b refer to the internal Jacobi coordinates as defined in Eq. (3.5), a similar transformation holding for the momentum operators p. From (3.1), we obtain

$$[p_a{}^2+p_b{}^2+(W'-p_a{}^2-p_b{}^2)]\Psi'=0 \tag{3.12}$$

and define as effective interaction V_{eff}

$$V_{\text{eff}}(x_s,p_s,m)\equiv W'-p_a{}^2-p_b{}^2, \quad s=a,b. \tag{3.13}$$

We shall now restrict ourselves to the simplest possible form for V_{eff} consistent with the form factor. We assume that V_{eff} is not a function of x_b, p_a, and p_b. We also assume that V_{eff} is independent of the spin and unitary spin. With this assumption, we can write

$$V_{\text{eff}}(x_a,m)=(p_a{}^2+p_b{}^2)\Psi'/\Psi'. \tag{3.14}$$

If we use the state Ψ' consistent with the form factor, we see that V_{eff} goes to infinity at certain values of x_a, changing sign at the discontinuities. This is, in fact, intimately related to the ratio a_1/a_4 being negative, as discussed in Sec. III A. Using the set of a_i obtained there, the singularity points lie in the short-range region.

Regarding the long-range part of V_{eff}, we can readily show from (3.14) that it diverges as $x_a{}^2$. This is a consequence of using truncated oscillator bases. This feature of V_{eff} is, however, consistent with the assumption that no ionization is possible for the system and, consequently, no free quarks could be observed.[14]

In order to evaluate the spectrum energies and wave functions, we have used a simple parametrization for V_{eff} that has the above-mentioned properties: a linear combination of a derivative of a δ function and a harmonic potential. With this potential, we have solved the resulting Schrödinger-like equation and have obtained a set of linear coefficients a_i which, although they are not identical to the ones used at the end of Sec. II, reproduce very well the proton form factor (in fact, better than before) as shown in Fig. 3. We can conclude, therefore, that this V_{eff} has the main features required for our purpose. In fact, if one chooses a long-range potential, with no short-range structure, for example the Coulomb potential η/r, one does not reproduce the correct form factor, since the ratio a_1/a_4 is then positive (see Fig. 4).

We can further mention here that if the quark-quark interaction arises from the exchange of a tower, one can show from Regge-pole theory and duality[15] that this interaction contains not only a long-range part, but also a short-range structure. This reinforces, from a completely different point of view, our findings regarding the structure of V_{eff}.

We now proceed to calculate the low-energy part of the spectrum using V_{eff}. (See Fig. 5.) Since different values of m will now enter in the discussion, we shall have to specify an explicit m dependence in W'. We make the simplest choice: W' is a linear function of m. In Sec. III C we discuss the results obtained with this assumption.

C. Low-Energy Baryon Spectrum

We are now prepared to calculate the low-energy spectrum of the three-quark system resulting from the effective interaction of the previous subsection and the truncated basis given in Table I. Using the techniques of Ref. 5, we obtain the matrix elements of W' in the truncated basis and the problem reduces to a conventional matrix eigenvalue problem. A typical spectrum is shown in Fig. 5. We see that the main features of the calculated spectrum are the following. First, the ground state belongs to the Young partition [3] [or the **56** representation in the $SU(6)$ classification] and $\Lambda^\pi=0^+$. Second, the first two excited states with $\Lambda^\pi=0^+$ belong to the partition [21] [or **70** in the $SU(6)$ language], while the first excited [3] $\Lambda^\pi=0^+$ state lies very high in energy. Finally, we see that two $\Lambda^\pi=1^-$

[14] L. I. Schiff, Phys. Letters **31B**, 74 (1970).
[15] G. Cocho, University of Mexico Report, 1970 (unpublished).

states both belonging to [21] lie rather close in energy.

One should mention here that the usual assignment of quantum numbers is different from what we have obtained.[16] In particular, the first $\Lambda^\pi=0^+$ excited state is always assumed to belong to a **56** multiplet and we obtain this state very high in energy. The usual assignments are not beyond all doubts, however, since several predicted resonances are missing. Further experimental and theoretical work is needed to clear up this point.

IV. FINAL REMARKS

We have shown how to construct relativistic wave equations for composite systems and how to obtain their approximate solutions. We have applied our technique to calculate relativistic form factors and the mass spectrum of baryons using the quark model. It is worth emphasizing that our model is a relativistic one and that rather large differences with respect to a nonrelativistic treatment might arise. We have shown this, in particular, in connection with the proton form factor.

We have obtained a simple possible effective interaction consistent with the proton form factor. The quantum number assignments of the mass spectrum obtained is different from the usual ones. However, a more complicated interaction could also be used in principle (containing, for example, three-body forces, spin and unitary-spin dependence, etc.). A different approach to obtain V_{eff} could also be used, calculating it from particle or multiparticle (Regge poles) exchange and comparing it with the potential obtained in our approach.

ACKNOWLEDGMENTS

One of the authors (G.C.) would like to thank Professor Abdus Salam and Professor P. Budini and also the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

[16] H. J. Lipkin, invited paper, European Physical Society Meeting, Florence, 1969 (unpublished).

APPENDIX

In this appendix we write explicitly the matrix element of Eq. (3.7). Such a formula was obtained in Ref. 4.

$$\langle\psi_{Nn'l}(p_\mu)|\psi_{Nnl}(p_\mu+q_\mu)\rangle$$
$$=\mathfrak{N}_{n,l}^{-1/2}\mathfrak{N}_{n',l}^{-1/2}\left(1+\frac{q^2}{2m^2}\right)^{-|N|-2(n+n'+l)}$$
$$\times\sum_{jk}(-1)^{k+l}A_n(n,n',l,k)A_{n'}(n,n',l,k)$$
$$\times\frac{(2n+l-j)!(2n'+l-j)!}{(k-j)!(2n+l-k)!(2n'+l-k)!}D(n,l,n',j)$$
$$\times\left(\frac{|N|q^2}{m^2}\right)^{n+n'+l-k}\left(1+\frac{q^2}{2m^2}\right)^{2k-j}\left(1+\frac{q^2}{4m^2}\right)^{n+n'+l-k},$$

with

$$A_r^2(n,n',l,k)$$
$$=\left(1+\frac{2n+2n'+2l-k-1}{|N|}\right)\cdots\left(1+\frac{2r+2l+1}{|N|}\right)$$
$$\times\left(1+\frac{2r+2l}{|N|}\right).$$

The geometrical coefficients $D(n,l,n',j)$ are given by

$$D(n,l,n',j)=\sum_{s,t}(-1)^{t+s}\frac{(n+t)!(n'+t)!}{(n+t-s)!(n'+t-s)!}$$
$$\times\frac{(j+1)!(2l-2t-1)!}{t!(2s+1)!(j-2s)!(l-t-1)!(l-2t+2s-j)!}.$$

The normalization coefficients $\mathfrak{N}_{n,l}$ are defined as

$$\mathfrak{N}_{n,l}=A^2(n,n,l,2n+1)\sum_i(2n+l-i)!D(n,l,n,i)$$

and m stands for the mass of the system.