Behavior of Form Factors and Scattering Amplitudes from Analyticity and Other Constraints*

D. S. NARAYAN†
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

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From the singularity structure of a form factor $G(t)$ and that of the imaginary part $F(s,t)$ of a scattering amplitude, in the t plane, and from certain assumptions on their growth outside the physical region, we obtain representations of $G(t)$ and $F(s,t)$ from which we deduce an asymptotic form of these functions. It is shown that the imaginary part of the scattering amplitude for identical particles has the same analytical structure in the P_t^2 plane (square of the transverse momentum) as $G(t)$ in the t plane, and they have the same asymptotic behavior in the variables P_t^2 and t, respectively. For the former case, we also obtain an exact representation of $\ln F(s,t)$ as a power series in trigonometric functions with coefficients which are functions of s. These coefficients can, in principle, be determined from experiment by a procedure similar to that employed in partial-wave analysis. It is argued that a few terms in this series would suffice for a parametrization of the scattering amplitude at all angles and energies. The representations of $G(t)$ and $F(s,t)$ are also used to obtain asymptotic lower bounds on the form factors and the scattering amplitudes similar to those previously obtained by Martin. For the scattering amplitude, we have obtained bounds separately for identical and nonidentical particles, and in the latter case our bound is well behaved at $\theta=90°$, unlike that of Martin.

I. INTRODUCTION

At the present stage of our knowledge of the theory of hadron scattering, one does not know whether any of the existing theoretical models will prove, ultimately, to be the correct description of hadron dynamics. It is therefore of considerable interest to obtain results of a general nature which do not lean heavily on a model and which are derivable from general principles like analyticity and invariance under various transformations, etc. In this category one may mention the lower bounds on the large-momentum-transfer dependence of form factors and scattering amplitudes obtained by Martin[1] from analyticity and certain conditions on the growth of these functions outside the physical region. These bounds could be taken according to Martin, as illustrations of "minimal interactions" in the terminology of Kinoshita.[2] The result obtained by Martin for the scattering amplitude is, however, not well behaved at $\theta=90°$, and the predicted large-angle distribution, which appears to be symmetric around 90°, cannot be true for the general case of the scattering of nonidentical particles. Moreover, he handles the two cases of the form factor and the scattering amplitude in slightly different ways—relying on a theorem in entire functions and imposing conditions to suit the theorem for one case, and carrying out a rather involved calculation for the other.

In this paper we present a unified approach in handling both the form factor and the scattering amplitude, and obtain certain lower bounds, including those of Martin, under various assumptions on their growth outside the physical region. Under a further condition, which is likely to be satisfied, it is shown that the structure of the inequality obtained by Martin for the form factor could actually be valid as an equality in which the parameters would, in general, differ from those in the inequality. This could possibly explain why the principle of minimal interaction, which has no sound theoretical motivation, seems to be satisfied. The behavior of the form factor as $e^{-c\sqrt{|t|}}$ was suggested before by Wu and Yang,[3] and the hypothesis of minimal interaction leads to the same asymptotic dependence (up to polynomials). In the present approach, the presence of complex zeros in the form factor (which are assumed to be finite in number) is to multiply the function $e^{-c\sqrt{|t|}}$ by a polynomial in t. Since the zeros occur in conjugate pairs, the polynomial, which is normalized to unity at $t=0$, remains non-negative on the real axis and increases unboundedly as $t\to-\infty$. Experimentally it has been found that the function $e^{-c\sqrt{|t|}}$ gives too fast a decrease for large $|t|$. This deviation can be taken as an indication of the presence of complex zeros. We have found that the function

$$G(t)=(1+at^2+bt)\exp\{-c[(4\mu^2-t)^{1/2}-2\mu]\},\quad a>0,\quad b>0 \qquad (1)$$

which incorporates a conjugate pair of zeros and the correct analytical structure with a cut from $t=4\mu^2$ to ∞, where μ is the pion mass, and which is normalized to unity at $t=0$, gives a good fit to the existing data on the proton form factor. The parametrization of the proton form factor according to (1) and the parametrization of p-p scattering, according to a scheme discussed below, will be reported separately.

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† Present address: Tata Institute of Fundamental Research, Bombay, India.
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[3] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

We also show that the function $F(s,t)$ which is, apart from a factor depending on s, the imaginary part of the scattering amplitude for identical particles, such that $F(s,0)=1$, has the same structure in the transverse momentum variable P_t^2 ($P_t^2=-t-t^2/4q^2$, where q is the c.m. momentum) as the form factor has in the t variable. The parameters which are constants in the form factor can now depend on the energy variables. In analogy to (1), we have

$$F(s,t)=(\text{polynomial in } p_t^2 \text{ with real coefficients}) \times\exp\{-c'[(4\mu^2+P_t^2)^{1/2}-2\mu]\}. \quad (2)$$

We also obtain inequalities for the scattering amplitude of identical particles, analogous to those of the form factor, and finally some inequalities for the scattering of nonidentical particles which are well behaved in the whole angular interval $0\leq\theta\leq\pi$.

From considerations of analyticity and the knowledge of the experimentally observed distribution of transverse momenta, the function $\exp\{-c[(d+p_t^2)^{1/2}-d]\}$ occurring in (2) was actually introduced by Chavda and the author[5] to describe p-p scattering. It was found that the experimental data then available could be fitted to this form within a factor of 2 or 3 over several orders of magnitude. From the presently available enlarged experimental data it appears that, if the parameters in (2) are taken to be energy independent, one would not be able to get a reasonable fit over the whole range of angles and energies. We find that one can, however, get a reasonably good fit to the data by combining, in a suitable matter, two expressions of the same form as in (2) with constant coefficients. The details of this parametrization and a discussion of the significance of such a sum in relation to a structure in the proton will be reported, as mentioned earlier, along with this parametrization of the proton form factor.

In Sec. II we develop a technique for deriving lower bounds on the form factor $G(t)$ for $t\rightarrow-\infty$ in terms of its growth along the physical cut. In Secs. III and V we do a similar thing for the absorptive parts of scattering amplitudes of identical and nonidentical particles, respectively. In Sec. IV we give an exact representation for the scattering of identical particles which has been derived in Appendix C. The problem presented by the presence of an infinite number of zeros is briefly discussed in Appendix A.

II. FORM FACTOR

The form factor is assumed to be analytic in the cut t plane with a cut from $t=4\mu^2$ to $t=+\infty$. We also impose the following conditions: (i) $|\ln|G(t)||\leq A|t|^\alpha$, where A is a constant; $\alpha<\frac{3}{2}$ for $|t|\rightarrow+\infty$ and $0<\arg t<2\pi$. (ii) On the cut, $|\ln|G(t)||\leq A'|t|^{\alpha'}$, $\alpha'\leq\alpha<\frac{3}{2}$ for $t\rightarrow+\infty$. (iii) $G(t)$ has only a finite number of zeros in the region where $G(t)$ is analytic. This assumption about the finite number of zeros may appear rather restrictive. But it raises difficult problems in the present approach to include an arbitrary distribution of an infinite number of zeros. Even though one can relax the restriction on the number of zeros in a certain manner and carry out, as shown in Appendix A, a modified analysis the final result is less informative. So we proceed with the simplifying condition (iii). Without loss of generality we can assume that $G(0)=1$, as one can always add a constant to a given function or divide it by a constant without changing its analytical structure.

Let the zeros be situated at $t=t_1, t_2, \ldots, t_N$, and let

$$h(t)=\prod_{i=1}^{N}\left(1-\frac{t}{t_i}\right). \quad (3)$$

In (3) all complex zeros must occur in conjugate pairs to make $h(t)$ real for t real and $t<4\mu^2$. We now define a new function $g(t)$ given by

$$G(t)=h(t)g(t), \quad (4)$$

so that $g(t)$ is again analytic in the cut plane but has no zeros in it. Further, $g(t)$ also satisfies conditions (i) and (ii) and $g(0)=1$. The modulus of the function $h(t)$ increases unboundedly for $t\rightarrow-\infty$.

Applying Cauchy's theorem to the function $[\ln g(t)]/(4\mu^2-t)^{1/2}$, which is analytic in the cut plane, and using the conditions on the growth of $g(t)$ and the identity $g(0)=1$, one can easily obtain the integral representation for $g(t)$, given by

$$\frac{\ln g(t)}{(4\mu^2-t)^{1/2}}=\frac{t}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\ln|g(t')|}{t'(t'-t)(t'-4\mu^2)^{1/2}}. \quad (5)$$

One can obtain the asymptotic behavior of $g(t)$ from (5) if one can find the asymptotic behavior of the integral in (5). Though this is, in general, difficult, it can be done in the particular case when

$$\ln|g(t')| \underset{t'\rightarrow\infty}{\sim} t'^{\alpha}, \quad \alpha<\tfrac{1}{2}$$

and $\ln|g(t')|$ is everywhere finite in the range of integration except possibly for a logarithmic singularity at $t'=4\mu^2$. One can then show[6] by uniform convergence that the integral in (5) behaves asymptotically for $t\rightarrow-\infty$ as $-\pi c/|t|$, where

$$c=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\ln|g(t')|}{t'(t'-4\mu^2)^{1/2}}, \quad (6)$$

provided $c\neq0$. Consequently the right-hand side of (5) behaves as a constant for large positive values of $(-t)$ and the form factor $G(t)$ will have the asymptotic be-

[4] D. A. Coward *et al.*, Phys. Rev. Letters **20**, 291 (1968).

[5] L. K. Chavda and D. S. Narayan, Nuovo Cimento **43**, 382 (1965).

[6] We first take out a factor $1/t$ from the integrand in (5) and consider the resulting integral as $t\rightarrow\infty$. By a theorem on uniform convergence [see G. A. Gibson, *Advanced Calculus* (MacMillan, London, 1954), p. 441], this integral tends to the integral given in (6).

havior given by

$$G(t) \sim h(t) \exp[-c(4\mu^2+|t|)^{1/2}]. \tag{7}$$

The sign of c in (7) is not determined by our analysis and it has to be inferred from experiment.

We show now how one can obtain certain inequalities starting from Eq. (5). We have the obvious inequality

$$\frac{|\ln|g(t)||}{(4\mu^2-t)^{1/2}} \leq \frac{|\ln g(t)|}{(4\mu^2-t)^{1/2}} \leq \frac{|t|}{\pi}\int_{4\mu^2}^{\infty} \frac{dt' |\ln|g(t')||}{t'(t'-t)(t'-4\mu^2)^{1/2}} \tag{8}$$

for values of $t<0$. For the case $|\ln|g(t')||\leq t'^{\alpha}, \alpha<\frac{1}{2}$, one can find the asymptotic behavior of the integral in (8) as before and one then obtains the inequality

$$|\ln|g(t)||\leq c'(4\mu^2-t)^{1/2} \tag{9}$$

valid for $t\rightarrow -\infty$, where

$$c'=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'|\ln|g(t')||}{t'(t'-4\mu^2)^{1/2}}. \tag{10}$$

Since $|\ln|A||\leq B$ implies $\ln|A|\geq -B$, inequality (9) together with (5) implies

$$|G(t)|\geq |h(t)|e^{-c'(4\mu^2+|t|)^{1/2}}\geq e^{-c'(4\mu^2+|t|)^{1/2}}. \tag{11}$$

The last inequality in (11) follows from the fact that $|h(t)|$ increases unboundedly for $t\rightarrow -\infty$. Structurally (7) and (11) are similar if we take the equality sign in (11) and assume c in (7) to be positive. This can possibly explain why the hypothesis of minimal interaction seems to be satisfied experimentally. It should be emphasized that while the parameter c' in (11) is never zero except for a trivial case, the parameter c in (7) can be zero. In the latter case we would not be able to obtain the form of the scattering amplitude given by (7). We have therefore to impose the condition that $c\neq 0$.

If we relax the condition that $\alpha<\frac{1}{2}$ and take $\alpha=\frac{1}{2}$, the integral in (10) will not exist. To handle this case we go back to Eq. (8), and split the range of integration in (8) into $4\mu^2$ to τ and τ to $+\infty$, such that in the latter range $|\ln|g(t')||\leq Kt'^{\alpha}$, where K is a constant and $\alpha=\frac{1}{2}$. The first integral being an integral in a finite range behaves as $1/|t|$ while the second integral can be shown to behave as $\ln(1+|t|/\tau)/|t|$. So one gets, instead of (11), the inequality

$$|G(t)|\geq |h(t)|\exp[-c''(4\mu^2+|t|)^{1/2}\ln(1+|t|/\tau)]. \tag{12}$$

Finally, if $\frac{1}{2}<\alpha<\frac{3}{2}$, one can show that

$$|G(t)|\geq |h(t)|\exp[-c'''(4\mu^2+|t|)^{\alpha}]. \tag{13}$$

In (13) we have relaxed the condition on the growth outside the physical region, but we get a weaker bound.

III. SCATTERING OF IDENTICAL PARTICLES

Let $F(s,t)$ represent, apart from the factor depending on s, the imaginary part of the amplitude for the scattering of identical particles, such that $F(s,0)=1$. The function $F(s,t)$ is analytic in the cut plane with cuts from $t=4\mu^2$ to $t=+\infty$ and $t=-4q^2-4\mu^2$ to $t=-\infty$, and it is real for $-4q^2-4\mu^2<t<4\mu^2$. In analogy to (4), we write

$$F(s,t)=H(s,t)f(s,t), \tag{14}$$

where $H(s,t)$ is a product of factors as in (3) corresponding to the zeros of $F(s,t)$ in the t plane where the numbers t_i could, in general, depend on s. The function $f(s,t)$ is analytic in the cut plane but has no zeros in it. We now apply Cauchy's theorem to the function $[\ln f(s,t)]/(4\mu^2-t)^{1/2}(t+4q^2+4\mu^2)^{1/2}$, making the contour to run suitably round the cuts and in two semicircles above and below the real axis. Suitably choosing the branches of $(4\mu^2-t)^{1/2}(t+4q^2+4\mu^2)^{1/2}$, making some changes of variables, and noting that

$$(4\mu^2-t)^{1/2}(t+4q^2+4\mu^2)^{1/2}=2q(d+P_t^2)^{1/2}, \tag{15}$$

where $d=4\mu^2(1+\mu^2/q^2)\simeq 4\mu^2$, one gets

$$\frac{\ln f(s,t)}{(d+P_t^2)^{1/2}}=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{(t'-t)}\frac{\ln|f(s,t')|}{(t'^2/4q^2+t'-d)^{1/2}} +\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{(t'-u)}\frac{\ln|f(s,-t'-4q^2)|}{(t'^2/4q^2+t'-d)^{1/2}}. \tag{16}$$

Noting that for the scattering of identical particles, $f(s,t')\equiv f(s,-t'-4q^2)$, Eq. (16) can be rewritten, after some algebra, as

$$\frac{\ln f(s,t)}{(d+P_t^2)^{1/2}} =\frac{2}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'(1+t'/2q^2)\ln|f(s,t')|}{(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}}. \tag{17}$$

Since $f(s,0)=1$, we also have

$$0=\frac{2}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'(1+t'/2q^2)\ln|f(s,t')|}{t'(1+t'/4q^2)(t'^2/4q^2+t'-d)^{1/2}}. \tag{18}$$

Subtracting (18) from (17), we finally get

$$\frac{\ln f(s,t)}{(d+P_t^2)^{1/2}}=\frac{2P_t^2}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'(1+t'/2q^2)\ln|f(s,t')|}{t'(1+t'/4q^2)(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}} \equiv\frac{2P_t^2}{\pi}I(s,P_t^2)=\frac{2P_t^2}{\pi}\left[I_1(s,P_t^2)+\frac{1}{2q^2}I_2(s,P_t^2)\right], \tag{19}$$

where

$$I_1(s,P_t^2)=\int_{4\mu^2}^{\infty}\frac{dt'\ln|f(s,t')|}{t'(1+t'/4q^2)(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}}$$

and

$$I_2(s,P_t^2)=\int_{4\mu^2}^{\infty}\frac{dt'\ln|f(s,t')|}{(1+t'/4q^2)(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}}.$$

In the above equation $P_t^2=q^2\sin^2\theta$, where θ is the c.m. angle of scattering. We want to find the asymptotic behavior of the integral in (19) when $q^2\to\infty$, keeping θ fixed and $\theta\neq 0$.

As before we consider the case $\ln|f(s,t')|\sim t'^{\alpha}$ as $t'\to\infty$, where $\alpha<\frac{1}{2}$. Here there is an extra complication due to the fact that $\ln|f(s,t')|$ is both a function of t' and a function of q^2. In general $\ln|f(s,t')|$ need not be bounded as $s\to\infty$. We shall, however, assume that there is a function $\phi(s)$ such that $\psi(s,t')\equiv\ln|f(s,t')|/\phi(s)$ is uniformly bounded by Kt'^{α}, where K is a constant. This assumption is much weaker than the usual assumption of polynomial boundedness as we are making the assumption for $\ln|f(s,t')|$ rather than $|f(s,t')|$. We can then show (Appendix B) that $I(s,P_t^2)$ behaves asymptotically for large P_t^2 as

$$I(s,P_t^2)\simeq\tfrac{1}{2}\pi\bar{c}/P_t^2,\tag{20}$$

where

$$\bar{c}=\frac{2}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'\ln|f(s,t')|}{t'(t'-4\mu^2)^{1/2}},\tag{21}$$

provided $\bar{c}\neq 0$. Finally from (19), (20), and (14) one finds that

$$F(s,t)\simeq H(s,t)\exp[-\bar{c}(d+P_t^2)^{1/2}].\tag{22}$$

The function $H(s,t)$ is a product of factors as in (3) and $H(s,0)=1$. Since the scattering amplitude $F(s,t)$ is invariant for the interchange $t\leftrightarrow u$ $(s+t+u=4M^2$, M being the common mass of the particles in scattering), the function $H(s,t)$ is also invariant for the interchange $t\leftrightarrow u$. Hence, if there is a zero at $t=t_i$, there would also be another zero at $u=t_i$. A pair of such zeros will contribute a factor to $H(s,t)$ which is given, after suitable normalization, by

$$\frac{(1-t/t_i)(1-u/t_i)}{(1+4q^2/t_i)}=\left(1+\frac{1}{1+t_i/4q^2}\frac{P_t^2}{t_i}\right).\tag{23}$$

Further, if t_i is complex, $H(s,t)$ will also contain the complex conjugate of (23) as a factor. One finds, therefore, that $H(s,t)$ is a polynomial in P_t^2 with real coefficients and remains non-negative for physical values of P_t^2. In the future we shall write $H(s,P_t^2)$ instead of $H(s,t)$. If the t_i are initially independent of s, the coefficients in the polynomial would become asymptotically independent of s. It should be pointed out that the constant $\bar{c}$ given by (22) would, in general, depend on s. It may happen that $\ln|f(s,t)|$ factorizes on the cut into a function of s and a function of t. For a Regge type of behavior $f(s,t)\sim s^{\alpha(t)}$, we have $\ln|f(s,t)|=|\alpha(t)|\ln s$ and the parameter $\bar{c}=c_1\ln s$, where c_1 is independent of s. This corresponds to a shrinking diffraction pattern. There is, however, no compelling reason to assume $\bar{c}$ has this logarithmic dependence on s. For a nonshrinking diffraction, $\bar{c}$ will have no dependence on s. The interesting point to be noticed from (7) and (22) is that the imaginary part of the scattering amplitude has the same structure in the variable P_t^2 as the form factor has in the in variable t.

IV. EXACT REPRESENTATION FOR IDENTICAL PARTICLES

Note that the behavior of the scattering amplitude according to (22) is an asymptotic behavior. One has no way of telling at what energies the predicted asymptotic behavior is applicable without detailed information on the form of the function $f(s,t)$ on the t-plane cut. However, we have obtained, in Appendix C, an exact representation for $\ln|f(s,t)|$ as a power series in trigonometric functions of θ with coefficients which could, in general, depend on s. According to this result, the scattering amplitude, which satisfies earlier-stated conditions on the growth outside the physical region, will have an exact representation given by

$$F(s,t)=H(s,P_t^2)\times\exp\left\{-[(d+P_t^2)^{1/2}]\left[\sum_{n=0}^{\infty}C_n\cos^{2n}\theta+d^{1/2}\times\left(\sin^2\theta+\frac{d}{q^2}\right)^{1/2}\sum_{n=0}^{\infty}D_n\cos^{2n}\theta\right]\right\},\tag{24}$$

where C_0 is approximately equal to $\bar{c}$ defined in (22) and $D_0=(1/q)\bar{c}$. The coefficients C_n and D_n are expected to decrease rapidly with n so that the series will converge rapidly. In principle it should be possible to determine the energy dependence of the coefficients C_n and D_n from experiment in the same way that one determines the energy dependence of the partial-wave amplitudes from experiments at various angles and energies. One already knows that the first term of the series in the exponent in (24) is enough to reproduce approximately the experimental data over several orders and that the coefficient C_0 has at most a weak dependence on s like $\ln s$. Consequently two or three terms of the series in (24) may be enough for a complete parametrization of a scattering process.

V. INEQUALITIES FOR IDENTICAL PARTICLES

Starting from Eq. (19) one can proceed to obtain inequalities analogous to the case of the form factor under various assumptions about the asymptotic behavior of $|\ln|f(s,t')||$ for $t'\to\infty$. For $|\ln|f(s,t')||\leq t'^{\alpha}$, $\alpha<\frac{1}{2}$, it is easy to show, as in the case for the form factor, that

$$|F(s,t)|\geq|H(s,P_t^2)|e^{-\bar{c}'(d+P_t^2)^{1/2}}\geq e^{\bar{c}'(d+P_t^2)^{1/2}}, \quad (25)$$

where

$$\bar{c}'=\frac{2}{\pi}\int_{2\mu}^{\infty}\frac{dx|\ln|F(s,x^2)||}{x(x^2-4\mu^2)^{1/2}}>0. \quad (26)$$

For the case $\alpha=\frac{1}{2}$, one cannot use the argument of uniform convergence as the integral in (26) then does not exist. To deal with this case, we split the range of integration in (19) into $4\mu^2$ to τ and τ to $+\infty$ such that in the latter range $|\ln|F(s,t')||\leq Kt'^{\alpha}$. Converting Eq. (19) into an inequality, we have

$$\frac{|\ln|f(s,t)||}{(d+P_t^2)^{1/2}}<\frac{2P_t^2}{\pi}\times\int_{4\mu^2}^{\tau}\frac{(1+t'/2q^2)|\ln|f(s,t')||dt'}{t'(1+t'/4q^2)(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}}+\frac{K2P_t^2}{\pi}\int_{(\tau+\tau^2/4q^2)^{1/2}}^{\infty}\frac{dx}{x(x^2+P_t^2)}\times\frac{4}{[1+(1+x^2/q^2)^{1/2}]^{1/2}}, \quad (27)$$

where we made some change of variables in the second integral in (27). The first integral in (27) behaves as $1/P_t^2$ for large values of P_t^2. The second integral in (27), by another change of variables, can be written as

$$\int_{(\tau+\tau^2/4q^2)^{1/2}}^{\infty}\frac{dx}{x(x^2+P_t^2)[1+(1+x^2/q^2)^{1/2}]^{1/2}}=\frac{\sqrt{q}}{P_t^2}\int_{q+\tau/2q}^{\infty}\frac{ydy}{(y+q)^{1/2}}\left(\frac{1}{y^2-q^2}-\frac{1}{y^2-q^2+P_t^2}\right). \quad (28)$$

Evaluating the integrals in (28), the second integral in (27) is seen to behave as $(1/P_t^2)\ln(q^2/\tau)$. Consequently the right-hand side of (27) behaves as $\sim$ constant $+\ln(q^2/\tau)$. Finally from (14) and (27) we obtain

$$|F(s,t)|\geq|H(s,P_t^2)|\times\exp[-\bar{c}''(4\mu^2+P_t^2)^{1/2}\ln(q^2/\tau)]. \quad (29)$$

The case $\frac{1}{2}<\alpha<\frac{3}{2}$ is more complicated but one can proceed in the same way as in the case $\alpha=\frac{1}{2}$. One will have an inequality as in (27), where the second term on the right-hand side of (27) gets replaced by

$$\frac{K2P_t^2}{\pi}\int_{\tau}^{\infty}\frac{(1+t'/2q^2)t'^{\alpha-3/2}}{(1+t'/4q^2)^{3/2}(t'^2/4q^2+t'+q^2\sin^2\theta)}\simeq\frac{2K\sin^2\theta}{\pi}(4q^2)^{\alpha-1/2}\int_{\to 0}^{\infty}\frac{(1+2x)x^{\alpha-3/2}dx}{(1+x)^{3/2}(4x^2+4x+\sin^2\theta)}. \quad (30)$$

For a rough idea of the behavior of the integral in (30), we replace it by the integral[7]

$$\int_0^{\infty}\frac{x^{\alpha-3/2}dx}{(1+x)^{1/2}(x+\sin^2\theta)}\simeq(\sin^2\theta)^{\alpha-3/2}\,{}_2F_1(\tfrac{1}{2},\alpha-\tfrac{1}{2},\tfrac{3}{2},\cos^2\theta).$$

Consequently the term in (30) behaves as $(P_t^2)^{\alpha-1/2}$, and finally

$$|F(s,t)|\geq|H(s,P_t^2)|\exp[-\bar{c}'''(4\mu^2+P_t^2)^{\alpha}], \quad (31)$$

which is very similar to (13). In getting the content of (29) and (31) it should be remembered that we have kept $\theta\neq 0$ fixed and allowed q^2 to become large so that we always have $P_t^2=O(q^2)$.

VI. INEQUALITIES FOR NONIDENTICAL PARTICLES

We assume for simplicity that the particles have the same mass M. The right-hand cut extends, as before, from $t=4\mu^2$ to $t=+\infty$, and the left-hand cut from $t=4q^2-4m^2$, where $2m$ is the smallest mass of the multiparticle system which can be exchanged in the u channel. We now apply Cauchy's theorem to the function $[\ln f(s,t)]/(4\mu^2-t)^{1/2}(t+4q^2+4m^2)^{1/2}$. Noting that

$$(2q)^{-1}(4\mu^2-t)^{1/2}(t+4q^2+4m^2)^{1/2}=-\frac{t^2}{4q^2}+t\left[1+\frac{(m^2-\mu^2)}{q^2}\right]+4\mu^2\left(1+\frac{m^2}{q^2}\right)\simeq(P_t^2+4\mu^2),$$

one gets as before

$$\frac{\ln f(s,t)}{(4\mu^2+P_t^2)^{1/2}}=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{(t'-t)}\frac{\ln|f(s,t')|}{(t'^2/4q^2+t'-4\mu^2)^{1/2}}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt'}{(t'-u)}\frac{\ln|f(s,-4q^2-t')|}{(t'^2/4q^2+t'-4\mu^2)^{1/2}}. \quad (32)$$

We put $t=0$ in (32) and subtract the resulting equation from the original equation (32). Noting that $\ln f(s,0)\equiv 0$,

[7] See, Bateman Manuscript Project, *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill, New York, 1954), Vol. II, formula 14.2.9.

we get the inequality

$$\frac{|\ln|f(s,t)||}{(4\mu^2+P_t^2)^{1/2}} \leq \frac{|t|}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{t'[t'+4q^2\sin^2(\frac{1}{2}\theta)]} \times\frac{|\ln|f(s,t')||}{(t'^2/4q^2+t'-4\mu^2)^{1/2}}+\frac{|t|}{\pi}\int_{4m^2}^{\infty}\frac{dt'}{(t'+4q^2)} \times\frac{|\ln|f(s,-t'-4q^2)||}{[t'+4q^2\cos^2(\frac{1}{2}\theta)](t'^2/4q^2+t'-4\mu^2)^{1/2}}\equiv J_1+J_2, \quad (33)$$

where we used $t=-4q^2\sin^2(\frac{1}{2}\theta)$ and $u=-4q^2\cos^2(\frac{1}{2}\theta)$. We impose the condition that the function $\ln|f(s,t)|$ on the cuts satisfy asymptotically the inequalities

$$\begin{aligned}|\ln|f(s,t')||&\leq B_1(s)t'^{\lambda_1}, \quad t'\to+\infty\\ |\ln|f(s,t')||&\leq B_2(s)|t'|^{\lambda_2}, \quad t'\to-\infty.\end{aligned} \quad (34)$$

For the case $\lambda_1<\frac{1}{2}$ and $\lambda_2<\frac{1}{2}$, we can use similar arguments, as before, to show that

$$J_1\to\frac{|t|}{\pi}\frac{1}{4q^2\sin^2(\frac{1}{2}\theta)}\int_{4\mu^2}^{\infty}\frac{dt'\,|\ln|f(s,t')||}{t'(t'-4\mu^2)^{1/2}} \equiv D_1(s)>0. \quad (35)$$

We further note that

$$J_2<\frac{|t|}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{t'} \times\frac{|\ln|f(s,-4q^2-t')||}{[t'+4q^2\cos^2(\frac{1}{2}\theta)](t'^2/4q^2+t'-4\mu^2)^{1/2}}\equiv J_2'. \quad (36)$$

We can show

$$J_2'\to\frac{|t|}{\pi}\frac{1}{4q^2\cos^2(\frac{1}{2}\theta)}\int_{4\mu^2}^{\infty}\frac{dt'\,|\ln|f(s,t-4q^2-t')||}{t'(t'-4\mu^2)^{1/2}} \equiv D_2(s)\tan^2(\tfrac{1}{2}\theta). \quad (37)$$

Using (33), (35), and (37) in (32) and the definition $F(s,t)=H(s,t)f(s,t)$, we get

$$|F(s,t)|>|H(s,t)| \times\exp\{-(4\mu^2+P_t^2)^{1/2}[D_1(s)+D_2(s)\tan^2(\tfrac{1}{2}\theta)]\}. \quad (38)$$

For a Regge type of behavior of the amplitude on the cuts, one finds that $D_1(s)=\alpha_1\ln s$ and $D_2(s)=\alpha_2\ln s$ and inequality (38) then reads as

$$|F(s,t)|>\exp\{-(\ln s) \times(4\mu^2+P_t^2)^{1/2}[\alpha_1+\alpha_2\tan^2(\tfrac{1}{2}\theta)]\}. \quad (39)$$

Note that inequality (38) has no singular behavior at any angle and the predicted lower bound decreases continuously for $\theta\to\pi$.

The cases $\lambda_1=\lambda_2=\frac{1}{2}$ and $\frac{1}{2}<\lambda_1<\frac{3}{2}$, $\frac{1}{2}<\lambda_2<\frac{3}{2}$, can be treated in a similar manner as before. For the former case one will have an extra factor $\ln(P_t^2/\tau)$ in the exponential in (39), while in the latter case the factor $(4\mu^2+P_t^2)^{1/2}$ in (39) gets replaced by $(4\mu^2+P_t^2)^{\alpha}$.

VII. CONCLUDING REMARKS

An important point brought out in this paper is the parallelism which exists, in the analytical structure, between a form factor and a scattering amplitude for identical particles. While the natural variable for the form factor is t, the corresponding variable for the scattering amplitude, which exhibits this parallelism, is the square of the transverse momentum P_t^2. In view of this, it is perhaps not surprising to find that large values of $|t|$ and large values of P_t^2 are found, experimentally, to be damped in the same manner, namely, as $e^{-c\sqrt{|t|}}$ and $e^{-\bar{c}\sqrt{P_t^2}}$. In the case of scattering, the amplitude is, however, also a function of s and the parameter $\bar{c}$ could be expected to depend on s. It is rather a significant fact that this dependence is either weak or absent. For nonidentical particles, the analogy between a scattering amplitude and a form factor is not so direct. One can, however, establish a parallelism between a form factor and either the even or the odd part (under $\cos\theta\to-\cos\theta$) of a scattering amplitude. One can see from our results on the bounds that there is considerable resemblance in the damping of transverse momenta between identical and nonidentical particles, at least for forward scattering.

One of the important features in the whole domain of strong-interaction phenomena is the nearly universal character of the damping of transverse momenta in both elastic and inelastic events. The average value of the transverse momentum is also roughly the same for both kinds of scattering. That this approximate equality of these values is not accidental and that it is a consequence of unitarity has been recently shown by the author.[8] The analysis based on analyticity in this paper brings out the fact that the transverse momentum is a natural variable to describe the scattering of identical particles and also to a certain extent the scattering of nonidentical particles. But considerations of analyticity in the t plane alone are not enough to show why the damping of transverse momenta is energy independent.

An important question which creeps up in any parametrization of the scattering of hadrons is whether the parametrization should be done according to a Gaussian damping $e^{-aP_t^2}$ or to an exponential damping e^{-bP_t} or even to combinations of them. Our derivation of the result in (23) shows that for a whole pattern of behavior of the amplitude on the cut such that $|F(s,t)|<e^{c|t|^{\alpha}}$, where $\alpha<\frac{1}{2}$, we always have a damping according to $e^{-\bar{c}(d+P_t^2)^{1/2}}$, which is a Gaussian for small values of P_t^2 and an exponential for large values of P_t. On the other hand, a purely Gaussian behavior $e^{-aP_t^2}$ can come only as a consequence of a very specific and fast growth of the amplitude on the cut and other unphysical regions as $e^{c|t|}$.

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[8] D. S. Narayan, Phys. Rev. 176, 2154 (1968).

The parametrization of p-p scattering is particularly complicated and various combinations of variables have been suggested from time to time. One source of all the difficulties in the parametrization of p-p scattering could be that one cannot parametrize p-p scattering as a scattering of spinless particles. In this connection one may remark that if one tried to analyze e-p scattering on the basis of a single form factor, one would not have obtained the smooth behavior which the form factor exhibits. We shall return to this question when we report on our parametrization of the proton form factor and p-p scattering.

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APPENDIX A

Instead of the assumption of a finite number of zeros, we assume now that there is no finite limit point of zeros. Since the zeros of an analytic function are isolated points, they form a countable set of points, say, t_1, t_2, ... According to a theorem of Weierstrass,[9] one can always associate an entire function $h(t)$ with zeros at these points and these points only. We assume further that the order of $h(t)$ is finite and that it is actually less than $\frac{1}{2}$. We can then choose $h(t)$ uniquely as the canonical product formed with the zeros of $G(t)$. The function $g(t)$ is then defined as before by writing $g(t)\equiv G(t)/h(t)$. The function $g(t)$ need not satisfy condition (i) which was imposed on the growth of $G(t)$. We can, however, find a sequence $|t_n|=R_n$ such that $g(t_n)$ satisfies condition (i) for $|t_n|\to\infty$. This can be seen from a theorem on the minimum modulus of an entire function of order less than $\frac{1}{2}$.[10] If $m(R)$ is the minimum modulus of such a function on a circle $|t|=R$, then we can find a sequence of circles of increasing radius R_n such that $m(R_n)$ increases monotonically and unboundedly as $n\to\infty$. Hence we have a sequence of circles on which

$$|g(t_n)|=\frac{|G(t_n)|}{|h(t_n)|}<\frac{e^{A|t_n|^\alpha}}{|h(t_n)|}<\frac{e^{A|t_n|^\alpha}}{m(|t_n|)}<e^{A|t_n|^\alpha},$$

which is condition (i). In applying Cauchy's theorem to $g(t)$, we choose the circular part of the contour to be a circle of radius R_n and make $n\to\infty$. The integral on the circle goes to zero as $n\to\infty$ and we get the representation given in (5), provided the integral in (5) converges. The function $h(t)$ is asymptotically unbounded along every ray and, in particular, along $t\to-\infty$. Unlike the case when $h(t)$ is a polynomial, the function $h(t)$ in the present case need not increase monotonically. Consequently one cannot deduce any specific asymptotic behavior for the form factor $G(t)$ as $t\to-\infty$.

APPENDIX B

The condition that there is a function $\phi(s)$ such that $\psi(s,t')\equiv\ln|f(s,t')|/\phi(s)$ is uniformly bounded by Kt'^α implies that the integral, obtained by replacing $\ln|f(s,t')|$ in (19) by $\psi(s,t')$, converges uniformly. Now consider the two integrals given by

$$I_1'(s,P_t^2)=\int_{4\mu^2}^\infty\frac{dt'\psi(s,t')}{t'(1+t'/4q^2)(t'^2/4q^2+t'+P_t^2)(t'^2/4q^2+t'-d)^{1/2}}\tag{B1}$$

and

$$I_1''(s,P_t^2)=\frac{1}{P_t^2}\int_{4\mu^2}^\infty\frac{dt'\psi(s,t')}{t'(t'-d)^{1/2}}.\tag{B2}$$

These two integrals are uniformly convergent and the difference between them can be made arbitrarily small by making q^2 and P_t^2 sufficiently large. Hence the integral $I_1'(s,P_t^2)$ behaves asymptotically for large P_t^2 as $1/P_t^2$. Since the integral $I_1(s,P_t^2)$ defined in (19) and $I_1'(s,P_t^2)$ differ only by the factor $\phi(s)$, it is clear that $I_1(s,P_t^2)$ also behaves as $1/P_t^2$ for large values of P_t^2. One can further show that $(1/2q^2)I_2(s,P_t^2)$ in (19) goes to zero for $q^2\to\infty$. This establishes the asymptotic behavior of $I(s,P_t^2)$ as given in (20).

APPENDIX C

We shall deduce Eq. (24) starting from the representation in (19). We write

$$P_t^2=[(d+P_t^2)^{1/2}-d^{1/2}][(d+P_t^2)^{1/2}+d^{1/2}]$$
$$=\frac{(d+P_t^2)^{1/2}-d^{1/2}}{(d+P_t^2)^{1/2}}\{(d+P_t^2)+[d(d+P_t^2)]^{1/2}\}\tag{C1}$$

and

$$\frac{d+P_t^2+[d(d+P_t^2)]^{1/2}}{x^2+P_t^2}=1-\frac{x^2-d}{x^2+P_t^2}+\frac{[d(d+P_t^2)]^{1/2}}{x^2+P_t^2},\tag{C2}$$

where x^2 stands for $t'^2/4q^2+t'$.

Using (C1) and (C2) in (19), we get

$$\ln|f(s,t)|=-\frac{2}{\pi}[(d+P_t^2)^{1/2}-d^{1/2}]\left\{\int\frac{dx}{x}\frac{\ln|f(s,t')|}{(x^2-d)^{1/2}}-\int\frac{dx}{x}\frac{(x^2-d)^{1/2}}{(x^2+P_t^2)}\ln|f(s,t')|+[d(d+P_t^2)]^{1/2}\times\int\frac{dx}{x}\frac{\ln|f(s,t')|}{(x^2+P_t^2)(x^2-d)^{1/2}}\right\}.\tag{C3}$$

[9] E. C. Titchmarsh, *The Theory of Functions* (Oxford U. P., Oxford, 1939), 2nd ed., p. 246.

[10] R. Boas, *Entire Functions* (Academic, New York, 1954), p. 40.

Again we can write

$$\frac{1}{x^2+P_t^2}=\frac{1}{t'^2/4q^2+t'+P_t^2}$$

$$=\frac{1}{q^2(1+t'/2q^2)^2}\sum_{n=0}^{\infty}\frac{(\cos\theta)^{2n}}{(1+t'/2q^2)^{2n}}$$

$$=\frac{1}{q^2(1+x^2/q^2)}\sum_{n=0}^{\infty}\frac{(\cos\theta)^{2n}}{(1+x^2/q^2)^n}. \tag{C4}$$

Using (C4) in (C3), we finally get

$$\ln|f(s,t)|=[(d+P_t^2)^{1/2}-d^{1/2}]\left[\sum_{n=0}^{\infty}C_n(\cos\theta)^{2n}+d^{1/2}\left(\frac{d}{q^2}+\sin^2\theta\right)^{1/2}\sum_{n=0}^{\infty}D_n(\cos\theta)^{2n}\right], \tag{C5}$$

where

$$C_0=\frac{2}{\pi}\left(1+\frac{d}{q^2}\right)\int_d^{\infty}\frac{dx}{x}\frac{\ln|f(s,t')|}{(x^2-d)^{1/2}(1+x^2/q^2)},$$

$$C_n=\frac{-2}{\pi q^2}\int_d^{\infty}\frac{dx}{x}\frac{(x^2-d)^{1/2}\ln|f(s,t')|}{(1+x^2/q^2)^{n+1}},\quad n=1,2,3,\ldots$$

$$D_n=\frac{2}{\pi q}\int_d^{\infty}\frac{dx}{x}\frac{\ln|f(s,t')|}{(x^2-d)^{1/2}(1+x^2/q^2)^{n+1}}.$$

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$SU(3)\times SU(3)$, Symmetry Breaking, and Cabibbo Angle. II*

J. C. Bane†
University of Colorado, Boulder, Colorado 80302

AND

K. T. Mahanthappa
University of Colorado, Boulder, Colorado 80302‡
and
Istituto di Fisica dell'Università, Rome, Italy

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We construct bootstrap equations for symmetry-breaking vectors in $(\bar{3},3)\oplus(3,\bar{3})$ and $(1,8)\oplus(8,1)$ representations of the chiral $SU(3)\times SU(3)$ with both electromagnetic and nonleptonic driving terms which, being obtained from a current-current interaction with currents in $(1,8)\oplus(8,1)$ have parts belonging to both octet and 27-plet representations. Only one solution of these equations is chosen by requiring partial conservation of axial-vector current (PCAC) and the Cabibbo angle θ_C to have a nonzero value. It follows directly from the equations that the existence of the weak 27-plet driving term is a necessary condition to have a nonzero θ_C. The solution has the same form as that suggested by Gell-Mann, Oakes, and Renner for a semistrong symmetry-breaking interaction. Knowing the deviation from $\sqrt{2}$ in their work, we get $\theta_C\approx 0.22$. The solution has both parity-conserving and parity-violating nonleptonic weak interactions in $(1,8)\oplus(8,1)$, and hence the strangeness-changing part cannot be rotated away. In our framework the electromagnetic tadpole cannot be in $(\bar{3},3)\oplus(3,\bar{3})$, but it can be in $(8,1)\oplus(1,8)$.

I. INTRODUCTION

THE most intriguing problems posed by the successful application of $SU(3)$ symmetry to hadron physics[1] are the understanding of the semistrong breaking of $SU(3)$ whose presence is needed to explain the mass spectrum, and the suppression of strangeness-changing semileptonic decays relative to strangeness-conserving ones which is described by the Cabibbo angle. Making use of a self-consistent dynamical framework or bootstrap framework (BF), it has been shown that the spontaneous breaking of $SU(3)$ comes about with the octet symmetry breaking reinforcing itself and other symmetry breakings damping themselves out.[2] But the direction of symmetry breaking in the $SU(3)$ space is arbitrary as long as electromagnetic and weak interactions are ignored. This is obviously true when semistrong symmetry-breaking, electromagnetic, and weak interactions are introduced independent of one another. BF provides a natural framework to allow for dynamical interplay of all these interactions, which in turn can fix the direction of symmetry break-

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† Present address: Research and Development Division, E. I. du Pont de Nemours and Co., Wilmington, Del.

‡ Permanent address.

[1] See, for example, M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (Benjamin, New York, 1964).

[2] R. E. Cutkosky, in *Particle Symmetries and Axiomatic Field Theory*, edited by M. Chrétien and S. Deser (Gordon and Breach, New York, 1966). This article contains an exhaustive list of references to literature on bootstrap and symmetry breaking.