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Coherent States and Indefinite Metric—Applications to the Free Electromagnetic and Gravitational Fields*

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The "coherent states" are constructed within the framework of an indefinite-metric linear vector space. It is found that these states have positive definite norm and are eigenstates of the annihilation operator. The theory is then applied to the free electromagnetic field and the linearized free gravitational field in the Gupta formulation. The "coherent states," restricted by the weak subsidiary condition, decompose into the usual coherent states involving only the physical degrees of freedom and an orthogonal vector of vanishing norm.

I. INTRODUCTION

The coherent-state representation[1] has found useful applications in quantum optics and quantum field theory. But these coherent states were constructed by an infinite superposition of state vectors belonging to a Hilbert space with a positive definite metric. In view of the usefulness of the indefinite metric[2] in quantum field theory, the properties of "coherent states" (hereafter called C states)[3] in indefinite-metric vector spaces are of importance. Previous studies[4] in this direction were primarily confined to the so-called pseudo-oscillator in one dimension. We propose here a method of constructing covariant C states with an indefinite metric, with applications to the electromagnetic[5] and Einstein's linearized gravitational[6] fields in the framework of the Gupta formalism. The most important property of C states which has not been emphasized in earlier attempts[4] is the positive definiteness of the norm. The fact that the C-state vectors are amenable to direct physical interpretation, coupled with their positive definiteness despite the basic vector space being of indefinite metric, suggests the possibility that the C-state vectors may point to a way of extracting physical states out of the totality of indefinite-metric states. Imposing the Gupta subsidiary conditions[5,6] on the C-state vectors, we arrive at the remarkable result that every C-state vector decomposes into a physical coherent-state vector containing only the physical degrees of freedom and an orthogonal vector of vanishing norm.

The coherent representation of the electromagnetic field has been extensively investigated[1] in the framework of a positive-definite-metric Hilbert space. Starting with the quantization condition for the physical degrees of freedom of the vector potential, the coherent states were constructed[7,8] using standard techniques. In our investigation based on the indefinite metric, we mete out equal treatment to all four components of the electromagnetic vector potential and show that the Gupta subsidiary condition on the C states leads to only the physical degrees of freedom. The result that the C states are eigenstates of annihilation operator a_μ with eigenvalue z_μ provides a remarkable advantage, viz., Gupta's[5] weak subsidiary condition on the four-vector potential is most easily realized as a restriction on the four-dimensional complex manifold of vectors by stipulating $k^\mu z_\mu=0$, where of course $k^\mu k_\mu=0$. Thus, in our formalism, while the totality of all C states $|\{z_\mu\}\rangle$ already possess an essential ingredient of acceptable physical interpretation, viz., positive definiteness of the norm, the subclass obtained by imposing the constraint $k^\mu z_\mu=0$ satisfies the weak condition automatically. This subclass turns out to be the same (modulo vectors of vanishing norm) as the coherent states constructed by starting with the commutation relations between physical degrees of freedom only.

The insight provided by the indefinite-metric approach to coherent states raises the possibility of a similar approach to Einstein's linear model of the quantized gravitational[9] field. Here again the Gupta[6] method of quantization of the gravitational field provides the basic framework. The degrees of freedom that introduce an indefinite metric are suppressed in the

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[1] The literature on coherent states is vast. A concise report of references especially on finite degrees of freedom is found in J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968) and H. Haken, *Light and Matter* (Springer, New York, 1970). For infinite degrees of freedom leading to the quantum-field-theoretic case, see S. S. Schweber, J. Math. Phys. **3**, 831 (1962) and F. Rohrlich, in *Analytic Methods in Mathematical Physics*, edited by R. P. Gilbert and R. G. Newton (Gordon and Breach, New York, 1970).

[2] K. L. Nagy, *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (Noordhoff, Groningen, The Netherlands, 1966).

[3] The term "coherent states" is used in the literature where the underlying Hilbert space has a positive definite metric, with definite physical connotation. In our case although the algebraic construction resembles the standard techniques, the underlying Hilbert space has an indefinite metric. To avoid confusion, we will call our coherent states the C states.

[4] M. G. Gundzik, J. Math. Phys. **7**, 641 (1966); G. S. Agarwal, Nuovo Cimento **65B**, 266 (1970).

[5] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[6] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161 (1952).

[7] V. Chung, Phys. Rev. **140**, B1110 (1965).

[8] T. W. B. Kibble, J. Math. Phys. **9**, 316 (1968).

[9] The construction of an overcomplete family of states starting with commutation relations belonging to an affine group has been considered by J. R. Klauder, in *Relativity: Proceedings of the Midwest Conference, Cincinnati, Ohio* (Plenum, New York, 1970), where models that are algebraically similar to gravitational theory are discussed. Here the Hilbert space considered is of positive definite metric. Our approach is based on an indefinite metric.

physical state by imposing subsidiary conditions on the C state. The treatment is quite analogous to the electromagnetic case.

The construction of C states with an indefinite metric is outlined in Sec. II. The theory is applied to the free electromagnetic field in Sec. III and to the linearized free gravitational field in Sec. IV. A discussion is presented in Sec. V.

II. CONSTRUCTION OF C STATES WITH INDEFINITE METRIC

The construction[4] of C states in an indefinite-metric linear vector space V is straightforward. However, we adopt a normalization different from the one used in the literature[4] with a view to exhibiting covariance explicitly in the four-dimensional case. We start with the so-called pseudo-oscillator quantization condition

$$[a,a^\dagger]=-1. \tag{2.1}$$

A realization of (2.1) in V is the analog of the usual Fock representation, however, with indefinite norm for the vectors $|n\rangle\in V$. The vacuum states are defined in the usual way:

$$a|0\rangle=0, \tag{2.2a}$$

$$\langle 0|a^\dagger=0. \tag{2.2b}$$

Note that $a^\dagger$ is the pseudoadjoint of a. We define the vectors $|n\rangle$ by

$$|n\rangle\equiv\frac{1}{(n!)^{1/2}}(a^\dagger)^n|0\rangle, \tag{2.3}$$

where n is a positive integer. Now the definition (2.3) together with (2.1) and (2.2) implies

$$a|n\rangle=-(n)^{1/2}|n-1\rangle, \tag{2.4a}$$

$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle. \tag{2.4b}$$

The operator $N\equiv -a^\dagger a$ plays the role of the usual number operator:

$$(-a^\dagger a)|n\rangle=n|n\rangle. \tag{2.5}$$

Thus the Hamiltonian $H=EN$ has positive definite eigenvalues nE. The orthonormality and the completeness conditions are, respectively,

$$\langle n|m\rangle=(-1)^n\delta_{nm}, \tag{2.6}$$

$$\sum_{n=0}^{\infty}|n\rangle(-1)^n\langle n|=1, \tag{2.7}$$

where the quantity $\langle n|m\rangle$ is the inner product in V. Since

$$\||n\rangle\|^2=(-1)^n, \tag{2.8}$$

the metric in V is indefinite.

We define the C state $|z\rangle$ through

$$|z\rangle\equiv e^{|z|^2/2}\sum_{n=0}^{\infty}\frac{(-z)^n}{(n!)^{1/2}}|n\rangle. \tag{2.9}$$

Note the positive sign in front of $|z|^2$ and the negative sign in front of z. The adjoint state is

$$\langle z|\equiv e^{|z|^2/2}\sum_{n=0}^{\infty}\frac{(-z^*)^n}{(n!)^{1/2}}\langle n|. \tag{2.10}$$

The vectors $|z\rangle$ constitute a linear vector space $\mathcal{C}$ with the inner product

$$\langle z'|z\rangle=\exp(\tfrac{1}{2}|z|^2+\tfrac{1}{2}|z'|^2-z'^*z) \tag{2.11a}$$

$$=\exp(\tfrac{1}{2}|z'-z|^2-i\,\mathrm{Im}z'^*z). \tag{2.11b}$$

For our purposes, the most significant property of the inner product (2.11) is that it leads to a positive definite norm for $|z\rangle$:

$$\||z\rangle\|^2\equiv\langle z|z\rangle=1. \tag{2.12}$$

Thus, the C states $|z\rangle$ constructed by an infinite superposition of indefinite-metric states $|n\rangle$ possess a positive definite norm. This is understandable in view of the exponential-type summation involved in the definition of C states. The previous term dominates the subsequent term, thus sequentially making the vacuum state the most important term in the summation leading to a positive norm. This property plays an important role in the construction of C states for the electromagnetic field in the Gupta formalism.

The "completeness" condition needs careful handling when the metric is indefinite. The relation reads as follows:

$$\int d\mu(z)|-z\rangle\langle z|=1, \tag{2.13}$$

where

$$d\mu(z)\equiv(1/\pi)e^{-2|z|^2}dxdy, \tag{2.14}$$

with $z=x+iy$. Note that the factor $e^{-2|z|^2}$ is introduced to secure convergence of the measure. Though this might seem artificial at this point, we have to bear in mind that the degrees of freedom leading to the indefinite metric occur in addition to those that possess positive-definite-metric vector spaces. Then the finiteness of the measure will be related to the physical dominance of the positive definiteness over negative definiteness in the exponent. The subsidiary condition will provide a precise statement of this dominance. Thus the completeness relation will find expression only in terms of physical degrees of freedom.

The states $|z\rangle$ are eigenstates of the annihilation operator with eigenvalue z,

$$a|z\rangle=z|z\rangle. \tag{2.15}$$

The C states $|z\rangle$ can be obtained from the vacuum by a pseudounitary transformation

$$|z\rangle=U(z,z^*)|0\rangle, \tag{2.16}$$

where

$$U(z,z^*)\equiv\exp[-(a^\dagger z-az^*)]. \tag{2.17}$$

That U is pseudounitary can be inferred from the fact that

$$(a^\dagger z-az^*)^\dagger=-(a^\dagger z-az^*).$$

III. FREE ELECTROMAGNETIC FIELD

A. Monochromatic Model

We will generalize the results of Sec. II to four degrees of freedom, one of which leads to an indefinite metric. The C states for the Minkowski oscillator

$$[a_\mu, a_\nu^\dagger] = g_{\mu\nu} \tag{3.1}$$

(where the signature of $g_{\mu\nu}$ is $+2$) will be first constructed. We call such states covariant C states. These results will later be generalized to the electromagnetic field. The spatial components a_1, a_2, a_3 satisfy the normal oscillator commutation relations, while a_0 obeys the pseudo-oscillator commutation condition of Sec. II. A realization of (3.1) in $V_4^{(4)}$ is governed by the following:

$$a_\mu|0,0,0,0\rangle \equiv a_\mu|0\rangle = 0, \tag{3.2a}$$

$$\langle 0|a_\mu^\dagger = 0, \tag{3.2b}$$

for all μ.

$$|n_1n_2n_3n_0\rangle \equiv |\{n_\mu\}\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}(a_0^\dagger)^{n_2}}{(n_1!n_2!n_3!n_0!)^{1/2}}|0\rangle, \tag{3.3}$$

$$a_1a_2a_3a_0|\{n_\mu\}\rangle = -(n_1n_2n_3n_0)^{1/2}|n_1-1,\, n_2-1,\, n_3-1,\, n_0-1\rangle, \tag{3.4}$$

$$a_1^\dagger a_2^\dagger a_3^\dagger a_0^\dagger|\{n_\mu\}\rangle = [(n_1+1)(n_2+1)(n_3+1)(n_0+1)]^{1/2} \times|n_1+1,\, n_2+1,\, n_3+1,\, n_0+1\rangle, \tag{3.5}$$

$$\langle\{n_\mu'\}|\{n_\mu\}\rangle = (-1)^{n_0}\delta_{n_0'n_0}\delta_{n_1'n_1}\delta_{n_2'n_2}\delta_{n_3'n_3}. \tag{3.6}$$

We recall that $\langle\{n_\mu'\}|\{n_\mu\}\rangle$ is the inner product in the indefinite-metric linear vector space $V_4^{(4)}$. The completeness condition is

$$\sum_{\{n_\mu\}=0}^{\infty} |\{n_\mu\}\rangle(-1)^{n_0}\langle\{n_\mu\}| = 1. \tag{3.7}$$

The covariant C states belonging to $\mathcal{C}_4^{(4)}$ are defined by

$$|\{z_\mu\}\rangle \equiv \exp(-\tfrac{1}{2}z^{\mu *}z_\mu) \times \sum_{\{n_\mu\}=0}^{\infty} \frac{(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}(-z_0)^{n_0}}{(n_1!n_2!n_3!n_0!)^{1/2}}|\{n_\mu\}\rangle, \tag{3.8}$$

where $z^{\mu *}z_\mu = z^{k*}z_k - |z^0|^2 \equiv |z_k|^2 - |z^0|^2 \equiv |z_\mu|^2$. The adjoint states are introduced in an obvious manner. It is easily checked, using (3.4), that

$$a_\mu|\{z_\mu\}\rangle = z_\mu|\{z_\mu\}\rangle. \tag{3.9}$$

Of course, there is no summation over μ in (3.9). The inner product in $\mathcal{C}_4^{(4)}$ is

$$\langle\{z_\mu'\}|\{z_\mu\}\rangle = \exp(-\tfrac{1}{2}|z_\mu'|^2 - \tfrac{1}{2}|z_\mu|^2 + z_\mu'^* z^\mu). \tag{3.10}$$

Note the important result that

$$\||\{z_\mu\}\rangle\|^2 = 1. \tag{3.11}$$

Therefore, the covariant C states possess positive norm.

The completeness condition on $|\{z_\mu\}|$ merits a careful study. When all z_μ are independent, we define the weight function

$$d\mu(\{z_\mu\}) \equiv (1/\pi^4)e^{-2|z^0|^2}dx_1dx_2dx_3dx_0dy_1dy_2dy_3dy_0 \tag{3.12}$$

and the completeness condition reads

$$\int d\mu(\{z_\mu\})|z_1,\, z_2,\, z_3,\, -z_0\rangle\langle z_1,\, z_2,\, z_3,\, +z_0| = 1. \tag{3.13}$$

However, in physical applications, not all z_μ are independent because of the subsidiary condition, as we will see presently. We stipulate that the states (3.8) satisfy the subsidiary condition in the Gupta[5] form,

$$k^\mu a_\mu|\{z_\mu\}\rangle = 0, \tag{3.14}$$

where k^μ is the null vector $k^\mu k_\mu = 0$. Since the C states are eigenstates of a_μ with eigenvalue z_μ, we can replace the left-hand side of (3.14) by $k^\mu z_\mu|\{z_\mu\}\rangle$. Thus the subsidiary condition will be satisfied if

$$k^\mu z_\mu|\{z_\mu\}\rangle = 0. \tag{3.15}$$

This implies that

$$k^\mu z_\mu = 0 \tag{3.16}$$

is necessary and sufficient to guarantee (3.14). We will denote the class of C states restricted by (3.16) by ${}_P\mathcal{C}_4^{(4)}$. Obviously, ${}_P\mathcal{C}_4^{(4)} \subset \mathcal{C}_4^{(4)}$.

There exists a pseudounitary transformation

$$U[\{z_\mu\},\{z_\mu^*\}] \equiv \exp(z^\mu a_\mu^\dagger - z^{\mu *}a_\mu) \tag{3.17}$$

such that

$$|\{z_\mu\}\rangle = U[\{z_\mu\},\{z_\mu^*\}]|0\rangle. \tag{3.18}$$

To see this we rewrite (3.8) as follows:

$$|\{z_\mu\}\rangle = \exp(-\tfrac{1}{2}|z_\mu|^2)\exp(z^\mu a_\mu^\dagger)|0\rangle \tag{3.19}$$

$$= \exp(-\tfrac{1}{2}|z_\mu|^2)\exp(z^\mu a_\mu^\dagger)\exp(-z^{\mu *}a_\mu)|0\rangle \tag{3.20}$$

and use the identity $e^{\frac{1}{2}[A,B]}e^{(A+B)} \equiv e^A e^B$.

An interesting aspect of $|\{z_\mu\}\rangle$ is revealed by relation (3.20). With due regard to the commutation properties of a_μ, we rewrite (3.20) as

$$\begin{aligned} |\{z_\mu\}\rangle = {} & \exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\exp[-\tfrac{1}{2}(|z_3|^2-|z_0|^2)] \\ & \times\exp(z_1a_1^\dagger + z_2a_2^\dagger)\exp[-(z_1^*a_1+z_2^*a_2)] \\ & \times\exp(z_3a_3^\dagger - z_0a_0^\dagger)\exp[-(z_3^*a_3 - z_0^*a_0)]|0\rangle \end{aligned} \tag{3.21a}$$

$$\begin{aligned} = {} & \exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\exp(z_1a_1^\dagger+z_2a_2^\dagger) \\ & \times\exp[-(z_1^*a_1+z_2^*a_2)]\exp[(z_3a_3^\dagger - z_0a_0^\dagger) \\ & -(z_3^*a_3 - z_0^*a_0)]|0\rangle, \end{aligned} \tag{3.21b}$$

where we have used

$$[(z_3a_3^\dagger - z_0a_0^\dagger)\ -(z_3^*a_3 - z_0^*a_0)] = |z_3|^2 - |z_0|^2.$$

Note that in the calculation from (3.19) to (3.22) only the commutation properties of a_μ were used and the subsidiary condition was not used. The appearance of the factors $\exp[-\frac{1}{2}(|z_1|^2+|z_2|^2)]\ \exp(z_1a_1^\dagger+z_2a_2^\dagger) \times\exp[-(z_1^*a_1+z_2^*a_2)]$ as a unit is significant. Now we will realize the subsidiary condition (3.16) in a special frame of reference F,

$$k_1=k_2=0\,,\quad k_3=k_0\,,\quad z_3=z_0\,. \tag{3.22}$$

The frame of reference F is not a covariant frame. But since every observer can choose the frame F, the treatment remains invariant. With this choice, we have

$$\begin{aligned}|\{z_\mu\}\rangle_F=&\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\\ &\times\exp(z_1a_1^\dagger+z_2a_2^\dagger)\exp[-(z_1^*a_1+z_2^*a_2)]\\ &\times\exp[z_3(a_3-a_0)^\dagger-z_3^*(a_3-a_0)]|0\rangle\,.\end{aligned} \tag{3.23}$$

With $\mathcal{A}=a_3-a_0$, we note that

$$[\mathcal{A},\mathcal{A}^\dagger]=0\,. \tag{3.24}$$

Thus

$$\begin{aligned}|\{z_\mu\}\rangle_F=&\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\\ &\times\exp(z_1a_1^\dagger+z_2a_2^\dagger)\exp[-(z_1^*a_1+z_2^*a_2)]\\ &\times\exp(z_3\mathcal{A}^\dagger)\exp(-z_3^*\mathcal{A})|0\rangle\end{aligned} \tag{3.25a}$$

$$\begin{aligned}=&\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\exp(z_1a_1^\dagger+z_2a_2^\dagger)\\ &\times\exp[-(z_1^*a_1+z_2^*a_2)]\exp(z_3\mathcal{A}^\dagger)|0\rangle\,.\end{aligned} \tag{3.25b}$$

The operator $\exp(z_3\mathcal{A}^\dagger)$ generates exactly those vectors that satisfy the subsidiary condition in F because

$$\mathcal{A}\exp(z_3\mathcal{A}^\dagger)|0\rangle=0\,. \tag{3.26}$$

We rewrite (3.25b) as

$$|\{z_\mu\}\rangle_F=|z_1,z_2\rangle_F+|z_1,z_2,\chi(z_3)\rangle_F\,, \tag{3.27}$$

where

$$\begin{aligned}|z_1,z_2\rangle_F\equiv&\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\exp(z_1a_1^\dagger+z_2a_2^\dagger)\\ &\times\exp[-(z_1^*a_1+z_2^*a_2)]|0\rangle\,,\end{aligned} \tag{3.28}$$

$$|z_1,z_2,\chi(z_3)\rangle_F\equiv|z_1,z_2\rangle_F\otimes\sum_{n=1}^{\infty}\frac{(z_3\mathcal{A}^\dagger)^n}{n!}|0\rangle\,. \tag{3.29}$$

We see easily that

$$\||z_1,z_2\rangle_F\|^2=1\,, \tag{3.30}$$

$$\||z_1,z_2,\chi(z_3)\rangle_F\|^2=0\,, \tag{3.31}$$

$${}_F\langle z_1,z_2|z_1,z_2,\chi(z_3)\rangle_F=0\,. \tag{3.32}$$

Thus $|\{z_\mu\}\rangle_F$ decomposes into a vector of unit norm and an orthogonal vector of zero norm. Since

$$\||\{z_\mu\}\rangle_F\|^2=1\,, \tag{3.33}$$

the states $|z_1,z_2,\chi(z_3)\rangle_F$ do not make a physically relevant contribution to the C states and we can set

$$|\{z_\mu\}\rangle_F=|z_1,z_2\rangle_F\,, \tag{3.34}$$

where $|z_1,z_2)_F$ are just the coherent states for the degrees of freedom a_1 and a_2 in the frame F. Then the completeness condition reads

$$\int d\mu_F(z_1,z_2)|z_1,z_2\rangle_F\ {}_F\langle z_1,z_2|=1\,, \tag{3.35}$$

where

$$d\mu_F(z_1,z_2)\equiv(1/\pi^2)dx_1dx_2dy_1dy_2\,.$$

B. Construction of C States for Free Electromagnetic Field

The states (3.8) can be extended to the field-theoretic case by suitably replacing each degree of freedom represented by a_μ by an infinite degree of freedom, $a_\mu(k)$, where k is the momentum four-vector. The basic equations[10] for the vector potential $A_\mu(x)$ are

$$\Box A_\mu(x)=0\,, \tag{3.36}$$

$$[A_\mu(x),A_\nu(y)]=-ig_{\mu\nu}D(x-y)\,, \tag{3.37}$$

$$\partial^\mu A_\mu^{(-)}(x)|\Psi\rangle=0\,, \tag{3.38}$$

where $A_\mu^{(-)}(x)$ is the negative-frequency (annihilation-operator) component of $A_\mu(x)$ and $|\Psi\rangle$ is an arbitrary physical state in $V_\infty^{(4)}$. We introduce the inner product

$$\langle f,g\rangle=i\int f^{\mu*}(x)\overleftrightarrow{\partial}_\nu g_\mu(x)d\sigma^\nu(x)\,, \tag{3.39}$$

where the integration is to be carried out over a space-like surface and $\overleftrightarrow{\partial}_\mu\equiv\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu$. If $f^\mu(x)$ and $g_\mu(x)$ are solutions of (3.36), then the inner product (3.39) is independent of time, and therefore we can choose a constant-time surface to carry out integration. Then

$$\langle f,g\rangle=i\int f^{\mu*}(x)\overleftrightarrow{\partial}_0 g_\mu(x)d^3x\,. \tag{3.40}$$

Since $f^\mu(x)$ is a solution of (3.36),

$$f^\mu(x)=\frac{1}{(2\pi)^2}\int\tilde{f}^\mu(k)e^{ikx}\delta(k^2)d^4k \tag{3.41}$$

$$=\frac{1}{(2\pi)^2}\int[\tilde{f}^\mu(k)e^{ikx}+\tilde{f}^\mu(-k)e^{-ikx}]\frac{d^3k}{2|\mathbf{k}|}\,. \tag{3.42}$$

Since $f^{\mu*}(x)=f^\mu(x)$, we obtain

$$f^\mu(x)=\frac{1}{(2\pi)^{3/2}}\times\int[\tilde{f}^\mu(\mathbf{k})e^{ikx}+\tilde{f}^{\mu*}(\mathbf{k})e^{-ikx}]\frac{d^3k}{(2|\mathbf{k}|)^{1/2}}\,, \tag{3.43}$$

where

$$\tilde{f}^\mu(\mathbf{k})\equiv(4\pi|\mathbf{k}|)^{-1/2}\tilde{f}^\mu(k)$$

and

$$\tilde{f}^{\mu*}(\mathbf{k})\equiv(4\pi|\mathbf{k}|)^{-1/2}\tilde{f}^\mu(-k)\,.$$

[10] J. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1959).

Thus, in terms of momentum variables,

$$\langle f,g\rangle=\frac{1}{(2\pi)}\int d^3k[\tilde{f}^{\mu *}(\mathbf{k})\tilde{g}_\mu(\mathbf{k})-\tilde{f}^{\mu}(\mathbf{k})\tilde{g}_\mu{}^*(\mathbf{k})]. \qquad (3.44)$$

Note that

$$\langle f,g\rangle^*=\langle g,f\rangle=-\langle f,g\rangle. \qquad (3.45)$$

Similarly, we can introduce

$$A_\mu(x)=\frac{1}{(2\pi)^{3/2}}\times\int[a_\mu(\mathbf{k})e^{ikx}+a_\mu{}^\dagger(\mathbf{k})e^{-ikx}]\frac{d^3k}{(2|\mathbf{k}|)^{1/2}}, \qquad (3.46)$$

with corresponding inner product $\langle a,f\rangle$. The commutation relations (3.37) can be written down in terms of $a_\mu(\mathbf{k})$:

$$[a_\mu(\mathbf{k}),a_\nu{}^\dagger(\mathbf{k}')]=g_{\mu\nu}\delta(\mathbf{k}-\mathbf{k}'). \qquad (3.47)$$

The subsidiary condition (3.38) reads

$$k^\mu a_\mu(\mathbf{k})|\Psi\rangle=0. \qquad (3.48)$$

We are now equipped to write down the C states by replacing various inner products in the previous section by their suitably generalized version. We define the product

$$(f,g)=\frac{1}{(2\pi)}\int\tilde{f}^{\mu *}(\mathbf{k})\tilde{g}_\mu(\mathbf{k})d^3k. \qquad (3.49)$$

This product is related to (3.44) through

$$\langle f,g\rangle=(f,g)-(g,f) \qquad (3.50a)$$

$$=(f,g)-(f,g)^*. \qquad (3.50b)$$

The commutation relations (3.47) can be expressed by

$$[(f,a),(a,g)]=(f,g). \qquad (3.51)$$

Note that the product (f,f) is indefinite.

We define the states

$$|f\rangle\equiv e^{-\frac{1}{2}(f,f)}e^{(a,f)}e^{-(f,a)}|0\rangle, \qquad (3.52)$$

or, equivalently,

$$|f\rangle\equiv e^{\langle a,f\rangle}|0\rangle. \qquad (3.53)$$

It is easy to check, using $(f,a)|0\rangle=0$, that

$$(f,a)|g\rangle=(f,g)|g\rangle. \qquad (3.54)$$

The inner product in $\mathcal{C}_\infty{}^{(4)}$ is

$$\langle g|f\rangle=e^{-\frac{1}{2}(f-g,f-g)+\frac{1}{2}\langle f,g\rangle} \qquad (3.55)$$

$$=e^{-\frac{1}{2}(f,f)-\frac{1}{2}(g,g)+(g,f)}. \qquad (3.56)$$

We note the important result,

$$\langle f|f\rangle=1>0. \qquad (3.57)$$

The C states $|f\rangle$ thus form a submanifold $\mathcal{C}_\infty{}^{(4)}$ with positive definite norm. The physical states are restricted by (3.38). The C states satisfying (3.38) are given by

$$k^\mu a_\mu(\mathbf{k})|g\rangle=0$$

or

$$\int k^\mu a_\mu(\mathbf{k})|g\rangle d^3k=0. \qquad (3.58)$$

Now, using (3.54), we rewrite the left-hand side of (3.58) to obtain

$$\frac{1}{(2\pi)}\int k^\mu g_\mu(\mathbf{k})d^3k|g\rangle=0. \qquad (3.59)$$

Thus the C states $|g\rangle$, constructed using $g_\mu(\mathbf{k})$ such that

$$k^\mu g_\mu(\mathbf{k})=0, \qquad (3.60)$$

satisfy the subsidiary condition (3.58).

As before, we can rewrite (3.52) with the help of the inner products

$$(f_1,g_1)\equiv(f,g)_1=\int\tilde{f}^{1*}(\mathbf{k})\tilde{g}_1(k)d^3k, \qquad (3.61)$$

etc., to obtain

$$|f\rangle=e^{-\frac{1}{2}[(f,f)_1+(f,f)_2]}e^{(a,f)_1+(a,f)_2}e^{-[(f,a)_1+(f,a)_2]}\times e^{(a,f)_3-(a,f)_0-[(f,a)_3-(f,a)_0]}|0\rangle. \qquad (3.62)$$

We realize the subsidiary condition (3.60) in the special frame F introduced in the earlier section. Then

$$|f\rangle_F=e^{-\frac{1}{2}[(f,f)_1+(f,f)_2]}e^{(a,f)_1+(a,f)_2}\times e^{-[(f,a)_1+(f,a)_2]}\exp(\mathcal{a},f)_3|0\rangle,$$

where, in the frame F,

$$\tilde{f}_3(\mathbf{k})=\tilde{f}_0(\mathbf{k}) \quad\text{and}\quad (\mathcal{a},f)_3=\int(a_3-a_0)^\dagger\tilde{f}_3(\mathbf{k})dk^3. \qquad (3.63)$$

Then the states $|f\rangle_F$ decompose into

$$|f\rangle_F=|f_1,f_2\rangle_F+|f_1,f_2,\chi(f_3)\rangle_F \qquad (3.64)$$

in a way analogous to (3.27), with

$$|f_1,f_2\rangle_F\equiv e^{-\frac{1}{2}[(f,f)_1+(f,f)_2]}e^{(a,f)_1+(a,f)_2}\times e^{-[(f,a)_1+(f,a)_2]}|0\rangle, \qquad (3.65)$$

$$|f_1,f_2,\chi(f_3)\rangle_F\equiv e^{-\frac{1}{2}[(f,f)_1+(f,f)_2]}e^{(a,f)_1+(a,f)_2}\times e^{-[(f,a)_1+(f,a)_2]}\sum_{n=1}^{\infty}\frac{(\mathcal{a},f)_3{}^n}{n!}|0\rangle, \qquad (3.66)$$

$${}_F\langle f_1,f_2|f_1,f_2,\chi(f_3)\rangle_F=0, \qquad (3.67)$$

$$\||f_1,f_2,\chi(f_3)\rangle_F\|^2=0. \qquad (3.68)$$

Thus the physically relevant *coherent states*[11] are given by (3.65).

[11] *Note added in proof.* Result (3.65) is valid also in three dimensions. To see this, we introduce the coordinate system defined by the unit polarization vectors such that $e_\mu{}^{(\alpha)}e^{(\alpha')\mu}=g^{\alpha\alpha'}$. Any vector valued function $V_\mu(\mathbf{k})$ can be expanded in terms of the set $\{e_\mu{}^{(\alpha)}\}$. We have

$$V_\mu(\mathbf{k})=\sum_{\alpha=0}^{3}V^{(\alpha)}(\mathbf{k})e_\mu{}^{(\alpha)}(\mathbf{k}).$$

The commutation relations (3.67) are imposed on $a^{(\alpha)}(\mathbf{k})$. We then realize the weak subsidiary condition in the frame G defined by $k^\mu e_\mu{}^{(1)}(\mathbf{k})=0=k^\mu e_\mu{}^{(2)}(\mathbf{k})$ and $k^\mu e_\mu{}^{(3)}(\mathbf{k})=\mp k^\mu e_\mu{}^{(0)}(\mathbf{k})$. Then we have $k^\mu\tilde{g}_\mu(\mathbf{k})=k^\mu e_\mu{}^{(3)}(\mathbf{k})\tilde{g}^{(3)}(\mathbf{k})+k^\mu e_\mu{}^{(0)}(\mathbf{k})\tilde{g}^{(0)}(\mathbf{k})=0$, i.e., $k^\mu e_\mu{}^{(3)}(\mathbf{k})[\tilde{g}^{(3)}(\mathbf{k})\mp\tilde{g}^{(0)}(\mathbf{k})]=|\mathbf{k}|[\tilde{g}^{(3)}(\mathbf{k})\mp\tilde{g}^{(0)}(\mathbf{k})]=0$. Thus we choose states $|g\rangle$ such that $\tilde{g}^{(3)}(\mathbf{k})=\pm\tilde{g}^{(0)}(\mathbf{k})$.

Now we can introduce a complete set of orthonormal Maxwell wave packets $\{h_i^{(n)}\}_{n=1,\ldots,\infty}$ with $i=1$, 2 with respect to the inner product (3.61). Then

$$(f,f)_i=\sum_{n=1}^{\infty} |f_i^{(n)}|^2, \tag{3.69}$$

where

$$f_i^{(n)}\equiv(h_i^{(n)},f_i). \tag{3.70}$$

Thus

$$|f_1,f_2\rangle_F=\exp\Big[-\tfrac{1}{2}\sum_{n=1}^{\infty}(|f_1^{(n)}|^2+|f_2^{(n)}|^2)\Big]$$
$$\times\exp\Big(\sum_{n=1}^{\infty} a_1^{(n)\dagger}f_1^{(n)}\Big)\exp\Big(\sum_{n=1}^{\infty} a_2^{(n)\dagger}f_2^{(n)}\Big)|0\rangle. \tag{3.71}$$

The completeness condition now reads

$$\int |f_1,f_2\rangle_F\ {}_F\langle f_1,f_2|d\mu(f_1,f_2)=1, \tag{3.72}$$

where

$$d\mu(f_1,f_2)=\prod_{n=1}^{\infty}\frac{|f_1^{(n)}|d|f_1^{(n)}|d\theta_1^{(n)}|f_2^{(n)}|d|f_2^{(n)}|d\theta_2^{(n)}}{\pi^2}, \tag{3.73}$$

where

$$\theta_i^{(n)}=\arg f_i^{(n)}. \tag{3.74}$$

IV. LINEAR MODEL OF GRAVITATIONAL FIELD

A. Construction of C States

The coherent states with one of the degrees of freedom leading to an indefinite metric has been already considered in connection with the electromagnetic field. The construction for the linear gravitational field is completely similar. In this section, we will outline the procedure very briefly with a model where the infinite degrees of freedom arising from the momentum variable have been suppressed. The basic equations are adopted from the Gupta formalism[6] with a few minor changes in notation.

The field equations are given by

$$\Box\gamma_{\mu\nu}(x)=0, \tag{4.1}$$

$$\Box\gamma(x)=0. \tag{4.2}$$

The commutation relations for the γ's are

$$[\gamma_{\mu\nu}(x),\gamma_{\lambda\rho}(y)]=-i(g_{\mu\lambda}g_{\nu\rho}+g_{\mu\rho}g_{\nu\lambda})D(x-y), \tag{4.3}$$

$$[\gamma(x),\gamma(y)]=4iD(x-y), \tag{4.4}$$

where $D(x)$ is the usual zero-mass Green's function.[10] We express the γ's in terms of their Fourier transforms:

$$\gamma_{\mu\nu}(x)=\frac{1}{(2\pi)^{3/2}}\int\Big[a_{\mu\nu}(\mathbf{k})e^{ikx}+a_{\mu\nu}^\dagger(\mathbf{k})e^{-ikx}\Big]\frac{d^3k}{(2|\mathbf{k}|)^{1/2}}, \tag{4.5}$$

$$\gamma(x)=\frac{2}{(2\pi)^{3/2}}\int\Big[a(\mathbf{k})e^{ikx}+a_a(\mathbf{k})e^{-ikx}\Big]\frac{d^3k}{(2|\mathbf{k}|)^{1/2}}. \tag{4.6}$$

The quantization conditions between the γ's lead to the following commutation relations between $a_{\mu\nu}$ and a:

$$[a_{\mu\nu}(\mathbf{k}),a_{\lambda\rho}^\dagger(\mathbf{k}')]=(g_{\mu\lambda}g_{\nu\rho}+g_{\mu\rho}g_{\nu\lambda})\delta(\mathbf{k}-\mathbf{k}'), \tag{4.7}$$

$$[a(\mathbf{k}),a^\dagger(\mathbf{k}')]=-\delta(\mathbf{k}-\mathbf{k}'). \tag{4.8}$$

We note that $a_{l0}(\mathbf{k})$ and $a(\mathbf{k})$ satisfy the pseudocommutation relations. Suppressing the momentum dependence, we introduce the following new operators:

$$\begin{aligned} &b_{11}\equiv\tfrac{1}{2}(a_{11}-a_{22}), && b_{lk}\equiv a_{lk},\\ &b_{22}\equiv\tfrac{1}{2}(a_{11}+a_{22}), && b_{l0}\equiv a_{l0},\\ &b_{33}\equiv a_{33}/\sqrt{2}, && b\equiv a,\\ &b_{00}\equiv a_{00}/\sqrt{2}, \end{aligned} \tag{4.9}$$

where the last three definitions are introduced for the sake of consistency of notation. We note that

$$\begin{gathered} [b_{11},b_{11}^\dagger]=1,\quad [b_{22},b_{22}^\dagger]=1,\quad [b_{33},b_{33}^\dagger]=1,\\ [b_{00},b_{00}^\dagger]=1,\quad [b_{12},b_{12}^\dagger]=1,\quad [b_{23},b_{23}^\dagger]=1,\\ [b_{31},b_{31}^\dagger]=1,\quad [b_{10},b_{10}^\dagger]=-1,\\ [b_{20},b_{20}^\dagger]=-1,\quad [b_{30},b_{30}^\dagger]=-1,\\ [b,b^\dagger]=-1. \end{gathered} \tag{4.10}$$

We can introduce the occupation-number space in a way quite analogous to the electromagnetic case. We merely outline the salient features.

$$b_{\mu\nu}|0\rangle=0, \tag{4.11a}$$

$$\langle 0|b_{\mu\nu}^\dagger=0, \tag{4.11b}$$

$$\langle\{n_{\mu\nu}'\}|\{n_{\mu\nu}\}\rangle=(-1)^{n_{10}+n_{20}+n_{30}+n}\delta_{\{n_{\mu\nu}'\}\{n_{\mu\nu}\}}. \tag{4.12}$$

Note that the metric is indefinite.

We introduce the complex numbers $z=x+iy$ and $z_{\mu\nu}=z_{\nu\mu}=x_{\mu\nu}+iy_{\mu\nu}$ to define the C states.

$$|\{z_{\mu\nu};z\}\rangle=\exp[-\tfrac{1}{2}(\mathfrak{z},\mathfrak{z})]\exp(\mathfrak{B},\mathfrak{z})\exp[-(\mathfrak{z},\mathfrak{B})]|0\rangle, \tag{4.13}$$

where the product $(\mathfrak{B},\mathfrak{z})$ is defined by

$$\begin{aligned}(\mathfrak{B},\mathfrak{z})\equiv{}& z_{11}b_{11}^\dagger+z_{22}b_{22}^\dagger+z_{33}b_{33}^\dagger\\ &+z_{00}b_{00}^\dagger+z_{12}b_{12}^\dagger+z_{23}b_{23}^\dagger+z_{31}b_{31}^\dagger\\ &-(z_{10}b_{10}^\dagger+z_{20}b_{20}^\dagger+z_{30}b_{30}^\dagger+zb^\dagger).\end{aligned} \tag{4.14}$$

The following relation is useful in computations:

$$[(\mathfrak{z}',\mathfrak{B}),(\mathfrak{B},\mathfrak{z})]=(\mathfrak{z}',\mathfrak{z}). \tag{4.15}$$

Using (4.15), we can rewrite (4.13) as

$$|\{z_{\mu\nu};z\}\rangle=\exp[(\mathfrak{B},\mathfrak{z})-(\mathfrak{z},\mathfrak{B})]|0\rangle. \tag{4.16}$$

It is easily checked that

$$b_{\mu\nu}|\{z_{\mu\nu};z\}\rangle = z_{\mu\nu}|\{z_{\mu\nu};z\}\rangle, \tag{4.17}$$

$$\langle\{z_{\mu\nu}';z'\}|\{z_{\mu\nu};z\}\rangle = \exp[-\tfrac{1}{2}(\mathfrak{z}',\mathfrak{z}')-\tfrac{1}{2}(\mathfrak{z},\mathfrak{z})+(\mathfrak{z}'\mathfrak{z})]. \tag{4.18}$$

We arrive at the most significant result that

$$\||\{z_{\mu\nu};z\}\rangle\|^2=1. \tag{4.19}$$

B. Subsidiary Conditions

The subsidiary conditions in the Gupta[6] formalism are imposed on the state vectors.

$$\partial^{\mu}\gamma_{\mu\nu}^{(-)}(x)|\Psi\rangle=0, \tag{4.20}$$

$$[\gamma_{\mu}{}^{\mu(-)}(x)-\gamma^{(-)}(x)]|\Psi\rangle=0. \tag{4.21}$$

In terms of Fourier transforms,

$$k^{\mu}a_{\mu\nu}|\Psi\rangle=0, \tag{4.22}$$

$$(a_{\mu}{}^{\mu}-2a)|\Psi\rangle=0. \tag{4.23}$$

Equivalently,

$$(k^1a_{1\nu}+k^2a_{2\nu}+k^3a_{3\nu}+k^0a_{30})|\Psi\rangle=0,$$
$$(a_{11}+a_{22}+a_{33}-a_{00}-2a)|\Psi\rangle=0.$$

In terms of the b's, we have

$$\begin{aligned}
&[k^1(b_{11}+b_{22})+k^2b_{21}+k^3b_{31}+k^0b_{01}]|\Psi\rangle=0,\\
&[k^1b_{12}+k^2(b_{22}-b_{11})+k^3b_{32}+k^0b_{02}]|\Psi\rangle=0,\\
&[k^1b_{13}+k^2b_{23}+\sqrt{2}k^3b_{33}+k^0b_{03}]|\Psi\rangle=0,\\
&[k^1b_{10}+k^2b_{20}+k^3b_{30}+\sqrt{2}k^0b_{00}]|\Psi\rangle=0,\\
&[2b_{22}+\sqrt{2}(b_{33}-b_{00})-2b]|\Psi\rangle=0.
\end{aligned} \tag{4.24}$$

Since we want the C states to satisfy the subsidiary conditions (4.22) and (4.23), the $z_{\mu\nu}$'s and z are constrained as follows:

$$\begin{aligned}
&k^1(z_{11}+z_{22})+k^2z_{21}+k^3z_{31}+k^0z_{01}=0,\\
&k^1z_{12}+k^2(z_{22}-z_{11})+k^3z_{32}+k^0z_{02}=0,\\
&k^1z_{13}+k^2z_{23}+\sqrt{2}k^3z_{33}+k^0z_{03}=0,\\
&k^1z_{01}+k^2z_{02}+k^3z_{03}+\sqrt{2}k^0z_{00}=0,\\
&2z_{22}+\sqrt{2}(z_{33}-z_{00})-2z=0.
\end{aligned} \tag{4.25}$$

Since the above system calls for a tedious algebraic solution, we will realize the subsidiary condition in a particular frame of reference F:

$$k_1=k_2=0,\quad k_3=k_0. \tag{4.26}$$

This implies that

$$z_{31}=z_{01},\quad z_{32}=z_{02},\quad \sqrt{2}z_{33}=z_{03},\quad z_{03}=\sqrt{2}z_{00}. \tag{4.27}$$

The last two constraints in the system (4.27) imply that $z_{33}=z_{00}$ and hence

$$z_{22}=z. \tag{4.28}$$

In the frame F,

$$(\mathfrak{z},\mathfrak{z})_F=|z_{11}|^2+|z_{12}|^2. \tag{4.29}$$

We further notice that in the frame F, the operator

$$Q=z_{22}(b_{22}-b)+z_{33}(b_{33}+b_{00}-\sqrt{2}b_{30})+z_{23}(b_{23}-b_{20})+z_{31}(b_{31}-b_{10}) \tag{4.30}$$

commutes with $Q^{\dagger}$. Thus,

$$\begin{aligned}
|\{z_{\mu\nu};z\}\rangle_F
&=\exp[-\tfrac{1}{2}(|z_{11}|^2+|z_{12}|^2)]\exp(z_{11}b_{11}^{\dagger}+z_{12}b_{12}^{\dagger})\\
&\quad\times\exp[-(z_{11}^*b_{11}+z_{12}^*b_{12})]e^{Q^{\dagger}}e^{-Q}|0\rangle &(4.31)\\
&=\exp[-\tfrac{1}{2}(|z_{11}|^2+|z_{12}|^2)]\exp(z_{11}b_{11}^{\dagger}+z_{12}b_{12}^{\dagger})\\
&\quad\times\exp[-(z_{11}^*b_{11}+z_{12}^*b_{12})]\left[1+\sum_{r=1}^{\infty}\frac{(Q^{\dagger})^r}{r!}\right]|0\rangle &(4.32)\\
&=|z_{11},z_{12}\rangle_F\,|z_{11},z_{12},\chi(z_{22},z_{33},z_{32},z_{31}\rangle_F, &(4.33)
\end{aligned}$$

where

$$|z_{11},z_{12}\rangle_F \equiv \exp[-\tfrac{1}{2}(|z_{11}|^2+|z_{12}|^2)]\exp(z_{11}b_{11}^{\dagger}+z_{12}b_{12}^{\dagger})\times\exp[-(z_{11}^*b_{11}+z_{12}^*b_{12})] \tag{4.34}$$

and

$$|z_{11},z_{12},\chi(z_{22},z_{33},z_{32},z_{31})\rangle_F = |z_{11},z_{12}\rangle_F\otimes\sum_{r=1}^{\infty}\frac{(Q^{\dagger})^r}{r!}|0\rangle. \tag{4.35}$$

Notice that

$$\||z_{11},z_{12}\rangle_F\|^2=1, \tag{4.36}$$

$${}_F\langle z_{11},z_{12}|z_{11},z_{12},\chi(z_{22},z_{33},z_{32},z_{31})\rangle_F=0, \tag{4.37}$$

and

$$\||z_{11},z_{12},\chi(z_{22},z_{33},z_{32},z_{31})\rangle_F\|^2=0. \tag{4.38}$$

Thus only b_{11} and b_{12} gravitons contribute to the coherent states.

V. DISCUSSION

We have dealt with the construction of C states for zero-mass boson systems wherein the basic field variables are further constrained by subsidiary conditions. We have arrived at a remarkable result that the C-state vectors have positive definite norm. We have shown that in a special frame F (chosen for the sake of simplicity of algebraic treatment), the C-state vectors decompose into the usual coherent-state vectors involving only the physical degrees of freedom and an orthogonal null vector. Since the frame F can be chosen by every observer, relativistic invariance is preserved.

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