Anomalous Ward Identities and ω-Meson Decays

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The strong decay $\omega \to \pi^+\pi^-\pi^0$ and the photopion reaction $\gamma+\pi \to \pi+\pi$ are discussed using the hard-pion current-algebra technique. We use the anomalous Ward identities for three- and four-point functions to determine the structure of the contact term, but we are not able to determine it completely. Both the $\omega \to 3\pi$ and the $\gamma - 3\pi$ amplitude are then expressed in terms of three parameters $g_{\omega\rho\pi}$, ξ (which characterizes the contact term), and δ (the parameter which appears in the $A_1\rho\pi$ system). We determine $g_{\omega\rho\pi}$ from a fit to the experimental $\omega \to \pi^0\gamma$ decay width and use this value of $g_{\omega\rho\pi}$ to determine the $\omega \to 3\pi$ width and the ratio $\Gamma(\omega \to \pi^0\gamma)/\Gamma(\omega \to 3\pi)$ for the various values of δ, ignoring the contact term. Next we estimate ξ by using the experimental decay width for $\omega \to 3\pi$ and use the resulting $\omega \to 3\pi$ amplitude to estimate $f_{\gamma-3\pi}$. We also estimate the effect of anomalous Ward identities on the $\pi^0 \to \gamma\gamma$ form factor and obtain the value of $F(0,0,0)$ for $b_1=0$ and $2b_1=3y$.

I. INTRODUCTION

SEVERAL authors have successfully used the hard-pion technique[1] to calculate the decay widths of various vector and axial-vector mesons and the K_{l3} and K_{l4} form factors.[2] We further investigate this method and apply it to the process $\omega \to 3\pi$. In this process we have to consider all the Ward identities obtained from the four-point function $\langle AAAV\rangle_0$, where A and V are the axial-vector and the vector currents, respectively. Brown *et al.*[3] have enumerated all the anomalies which arise in this abnormal-parity series of Ward identities. They have shown that it is impossible to avoid anomalies in the abnormal-parity series. We try to make use of these anomalies to determine the primitive contact term appearing in the four-point function $\langle AAAV\rangle_0$. We find that whatever choice for the b_i we make, we are not able to completely determine the contact term. Thus we are able to express the $\omega \to 3\pi$ amplitude in terms of three parameters $g_{\omega\rho\pi}$, ξ, and δ.

We fix $g_{\omega\rho\pi}$ from the $\omega \to \pi^0\gamma$ decay and use this $g_{\omega\rho\pi}$ to evaluate the decay width $\Gamma(\omega \to 3\pi)$ for the various values of δ, keeping $\xi=0$. We find that $\Gamma(\omega \to 3\pi)$ varies from 9.4 MeV for $\delta=-1$ to 7.5 MeV for $\delta=0$. We also estimate the ratio $\Gamma(\omega \to \pi^0\gamma)/\Gamma(\omega \to \pi^+\pi^-\pi^0)$ for the various values of δ. Next we determine ξ by fitting the $\omega \to 3\pi$ amplitude with the $\omega \to 3\pi$ decay width and use the resulting expression (off mass shell) to find $f_{\gamma-3\pi}$.

We also discuss the $\pi^0 \to 2\gamma$ decay in the context of anomalous Ward identities and obtain estimates for $F(0,0,0)$ for the various choice of b_i's.

In Sec. II, we write all the Ward identities necessary for the problem and write the general expression for the $\omega \to 3\pi$ amplitude. In Sec. III, we use the anomalous vector Ward identity to determine the structure of the contact term. Section IV contains results and discussion. We also present a brief derivation of the $\pi^0 \to \gamma\gamma$ form factor $F(0,0,0)$.

II. WARD IDENTITIES FOR $\langle AAAV\rangle$ SYSTEM

The Ward identities which we use are not the naive Ward identities, but are those which include the anomalies after regularization and addition of counter terms. This problem has been discussed in detail by Brown, Shih, and Young[3] in a recent paper (see Appendix B). The four-point function for our currents is defined as

$$\langle j_1{}^a(q_1) j_2{}^b(q_2) j_3{}^c(q_3) j_4{}^d(q_4)\rangle \equiv \int d^4x d^4y d^4z e^{-iq_1\cdot x - iq_2\cdot y - iq_3\cdot z} \times \langle T\{ j_1{}^a(x) j_2{}^b(y) j_3{}^c(z) j_4{}^d(0)\}\rangle_0\,,$$

with similar definitions holding for three- and two-point functions. Here $j_1{}^a$, $j_2{}^b$, etc., can be either vector, axial-vector, or the divergence of axial-vector currents. In particular, for the purposes of this paper we shall consider all the Ward identities arising from $\langle AAAV\rangle$. This gives rise to a so-called abnormal-parity series of Ward identities. The explicit pole structures of the four-point functions are displayed in Appendix A. The relevant Ward identities then are

$$\begin{aligned} q_{3\lambda}\langle \partial_\mu A_\mu{}^a(q_1)\partial_\nu A_\nu{}^b(q_2) A_\lambda{}^c(q_3) V_\sigma{}^d(q_4)\rangle &= -i\langle \partial_\mu A_\mu{}^a(q_1)\partial_\nu A_\nu{}^b(q_2)\partial_\lambda A_\lambda{}^c(q_3) V_\sigma{}^d(q_4)\rangle \\ &\quad -i\langle \sigma^{ac}(q_1+q_3)\partial_\nu A_\nu{}^b(q_2) V_\sigma{}^d(q_4)\rangle \\ &\quad -i\langle \sigma^{bc}(q_2+q_3)\partial_\mu A_\mu{}^a(q_1) V_\sigma{}^d(q_4)\rangle\,, \end{aligned} \quad (2.1)$$

$$\begin{aligned} q_{3\lambda}\langle \partial_\mu A_\mu{}^a(q_1) A_\nu{}^b(q_2) A_\lambda{}^c(q_3) V_\sigma{}^d(q_4)\rangle &= -i\langle \partial_\mu A_\mu{}^a(q_1)\partial_\lambda A_\lambda{}^c(q_3) A_\nu{}^b(q_2) V_\sigma{}^d(q_4)\rangle \\ &\quad +2f^{cbe}\langle \partial_\mu A_\mu{}^a(q_1) V_\nu{}^e(q_2+q_3) V_\sigma{}^d(q_4)\rangle \\ &\quad -i\langle \sigma^{ac}(q_1+q_3) A_\nu{}^b(q_2) V_\sigma{}^d(q_4)\rangle\,, \end{aligned} \quad (2.2)$$

[1] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).
[2] S. Fenster and F. Hussain, Phys. Rev. **169**, 1314 (1968); Riazuddin and A. Q. Sarker, *ibid.* **173**, 1752 (1968); C. S. Lai and Bing-Lin Young, *ibid.* **169**, 1241 (1968).
[3] R. W. Brown, Chia-Chang Shih, and Bing-Lin Young, Phys. Rev. **186**, 1491 (1969).

$$
\begin{aligned}
&q_{3\lambda}\langle A_\mu{}^a(q_1)A_\nu{}^b(q_2)A_\lambda{}^c(q_3)V_\sigma{}^d(q_4)\rangle \\
&\quad=-i\langle\partial_\lambda A_\lambda{}^c(q_3)A_\mu{}^a(q_1)A_\nu{}^b(q_2)V_\sigma{}^d(q_4)\rangle \\
&\qquad+2f^{cae}\langle V_\mu{}^e(q_1+q_3)A_\nu{}^b(q_2)V_\sigma{}^d(q_4)\rangle \\
&\qquad+2f^{cbe}\langle A_\mu{}^a(q_1)V_\nu{}^e(q_2+q_3)V_\sigma{}^d(q_4)\rangle \\
&\qquad+\Delta_{\mu\nu\sigma}(AA\mathbf{A}V)\,,
\end{aligned}
\tag{2.3}
$$

where $a,b,c=1,2,3$, and $d=0$ or 8. Here we have used the usual $SU(3)\times SU(3)$ commutation relations[4] among currents along with the following additional commutator:

$$
\delta(x_0-y_0)[A_0{}^a(x),\partial_\mu A_\mu{}^b(y)]=\sigma^{ba}(x)\delta(x-y)\,,
\tag{2.4}
$$

where $\sigma^{ba}(x)=\sigma^{ab}(x)$ because of conservation of the isospin current. $\Delta_{\mu\nu\sigma}(AA\mathbf{A}V)$ is the axial-vector anomaly as discussed in Ref. 3, whose authors have shown that if the general counter terms are nonzero, then we must also have anomalies in the abnormal-parity vector Ward identities. So, instead of the naive vector constraint condition, we have

$$
q_{4\sigma}\langle A_\mu{}^a(q_1)A_\nu{}^b(q_2)A_\lambda{}^c(q_3)V_\sigma{}^d(q_4)\rangle
=\Delta_{\mu\nu\lambda}{}^V(AAA\mathbf{V})\,,
\tag{2.5}
$$

where $\Delta_{\mu\nu\lambda}{}^V(AAA\mathbf{V})$ is the vector anomaly (Appendix B).

In the process $\omega\to\pi^0(\pi^+\pi^-)$, the two pions must necessarily be emitted in an $I=1$ state if the ω meson is to be an isosinglet. Hence the σ terms, which are symmetric in isospin indices, will not contribute in this process. Using (2.1)–(2.3), after some algebra we can write

$$
\begin{aligned}
&\langle\partial_\mu A_\mu{}^a(q_1),\partial_\nu A_\nu{}^b(q_2),\partial_\lambda A_\lambda{}^c(q_3),V_\sigma{}^d(q_4)\rangle=-iq_{1\mu}q_{2\nu}q_{3\lambda}\langle A_\mu{}^a(q_1)A_\nu{}^b(q_2)A_\lambda{}^c(q_3)V_\sigma{}^d(q_4)\rangle \\
&\qquad+f^{cae}(q_1-q_3)_\mu\langle V_\mu{}^e(q_1+q_3)\partial_\nu A_\nu{}^b(q_2)V_\sigma{}^d(q_4)\rangle+f^{cbe}(q_2-q_3)_\nu\langle V_\nu{}^e(q_2+q_3)\partial_\mu A_\mu{}^a(q_1)V_\sigma{}^d(q_4)\rangle \\
&\qquad+f^{bae}(q_1-q_2)_\mu\langle V_\mu{}^e(q_1+q_2)\partial_\lambda A_\lambda{}^c(q_3)V_\sigma{}^d(q_4)\rangle+i(q_1-q_3)_\mu f^{cae}\Delta_{\mu\sigma}(\mathbf{A}^b(q_2)V^e(q_1+q_3)V^d(q_4)) \\
&\qquad+i(q_2-q_3)_\nu f^{cbe}\Delta_{\nu\sigma}(\mathbf{A}^a(q_1)V^e(q_2+q_3)V^d(q_4))+i(q_1-q_2)_\mu f^{bae}\Delta_{\mu\sigma}(\mathbf{A}^c(q_3)V^e(q_1+q_2)V^d(q_4)) \\
&\qquad\qquad+\tfrac{1}{3}i[q_{1\mu}q_{2\nu}\Delta_{\mu\nu\sigma}(AA\mathbf{A}V)+q_{1\mu}q_{3\lambda}\Delta_{\lambda\mu\sigma}(A\mathbf{A}AV)+q_{2\nu}q_{3\lambda}\Delta_{\nu\lambda\sigma}(\mathbf{A}AAV)]\,,
\end{aligned}
\tag{2.6}
$$

where $\Delta_{\mu\sigma}(\mathbf{A}VV)$ is the three-point axial-vector anomaly. Equations (2.1)–(2.3) and (2.6) can be rewritten in terms of the vertex functions M and Γ defined in Appendix A. These are all standard techniques,[1,5] and we shall not give the details here. There will be two sets of equations depending on the choice of the vector current; i.e., d is either 0 or 8. Then from these two sets of equations we eliminate the φ vertex functions and write $M^\omega(q_1,q_2,q_3)_\sigma{}^{abc}$ in terms of $M^\omega(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}$:

$$
\begin{aligned}
&M^\omega(q_1,q_2,q_3)_\sigma{}^{abc}=-C_A{}^3g_{A_1}{}^{-3}F_\pi{}^{-3}q_{1\mu}q_{2\nu}q_{3\lambda}M^\omega(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc} \\
&\qquad+f^{bae}(q_1-q_2)_\mu F_\pi{}^{-2}g_\rho{}^{-1}\Delta_V(q_1+q_2)_{\mu\mu'}\Gamma^{(2)\omega}(q_3,\,q_1+q_2)_{\mu'\sigma}{}^{ce}+(a\leftrightarrow c;\,q_1\leftrightarrow q_3)+(b\leftrightarrow c;\,q_2\leftrightarrow q_3) \\
&\qquad+8i(q_1-q_3)_\mu f^{abc}[(1/\sqrt{6})f_B\cos\theta_Y-\tfrac{1}{3}f_Y\sin\theta_B]\cos^{-1}(\theta_B-\theta_Y)F_\pi{}^{-3}m_\omega{}^{-2} \\
&\qquad\times(2b_1-3y)\epsilon_{\mu\sigma'\alpha\beta}(q_1+q_3)_\alpha q_{4\beta}\Delta_\omega{}^{-1}(q_4)_{\sigma\sigma'}+(b\leftrightarrow a;\,q_1\leftrightarrow q_2)+(c\leftrightarrow b;\,q_3\leftrightarrow q_2) \\
&\qquad+\tfrac{1}{3}q_{1\mu}q_{2\nu}[\tfrac{1}{2}f_B\cos\theta_Y\Delta_{\mu\nu\sigma'}(AA\mathbf{A}V^0)-\tfrac{1}{2}f_Y\sin\theta_B\Delta_{\mu\nu\sigma'}(AA\mathbf{A}V^8)] \\
&\qquad\qquad\times\Delta_\omega{}^{-1}(q_4)_{\sigma\sigma'}F_\pi{}^{-3}m_\omega{}^{-2}\cos^{-1}(\theta_B-\theta_Y)+(q_2\leftrightarrow q_3;\,b\leftrightarrow c)+(q_1\leftrightarrow q_2;\,a\leftrightarrow b)\,.
\end{aligned}
\tag{2.7}
$$

$\Delta_V(q_1+q_2)_{\mu\mu'}$, etc., are the vector current propagators. We will approximate the current propagators by the corresponding meson propagators. We see here that the anomalous terms have a zero at $q_4{}^2=m_\omega{}^2$ and hence will not contribute to the physical process $\omega\to3\pi$.[6] However, they will contribute to $\gamma+\pi\to\pi+\pi$. In order to utilize the tree structure implied by Eq. (2.7), we shall define the contact term $M_C{}^\omega$ by subtracting the pole structure from the function $M(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}$. The advantage of adopting such a procedure is that the Ward identities then contain the functions $M_C{}^\omega$, which are simpler than the functions M in that, at least as far as the current algebra is concerned, they do not have to have any poles. Hence they can be approximated by smooth functions of 4-momenta.[5] We define

$$
\begin{aligned}
&M^\omega(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}\equiv M_C{}^\omega(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc} \\
&\quad+[ig_\rho{}^{-2}\Gamma^{(2)\omega}(q_2,\,q_1+q_3)_{\nu\alpha\sigma}{}^{be}\Delta_V(q_1+q_3)_{\alpha\alpha'} \\
&\quad\times\Gamma^{(1)}(q_1,q_3)_{\mu\lambda\alpha'}{}^{ace}+(q_1\leftrightarrow q_2;\,\mu\leftrightarrow\nu;\,a\leftrightarrow b) \\
&\qquad+(q_2\leftrightarrow q_3;\,\nu\leftrightarrow\lambda;\,b\leftrightarrow c)]\,.
\end{aligned}
\tag{2.8}
$$

[4] Our currents are normalized such that the commutators are $\delta(x_0-y_0)[A_0{}^a(x),V_\mu{}^b(y)]=2if^{abc}A_\mu{}^c(x)\delta^4(x-y)$, etc.

[5] I. S. Gerstein and H. J. Schnitzer, Phys. Rev. **170**, 1638 (1968).

[6] Brown *et al.* have assumed explicitly that there are no vector mesons in their model (free quark model only). If there are vector mesons in their theory, then Ward-identity anomalies will in general contain the vector-meson propagators. See Ref. 14 for a discussion of this point.

Incorporating this into (2.7), we obtain

$$M^{\omega}(q_1,q_2,q_3)_{\sigma}{}^{abc}=-C_{A_1}{}^3 g_{A_1}{}^{-3}F_{\pi}{}^{-3}q_{1\mu}q_{2\nu}q_{3\lambda}M_C{}^{\omega}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}$$
$$-ig_{\rho}{}^{-2}C_A{}^2 g_{A_1}{}^{-2}F_{\pi}{}^{-2}q_{1\mu}q_{2\nu}\Bigg[\Gamma^{(2)\omega}(q_3,q_1+q_2)_{\alpha\sigma}{}^{ce}\Delta_V(q_1+q_2)_{\alpha\alpha'}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}{}^{abe}$$
$$+\frac{8(2b_1-3y)}{F_{\pi}m_{\omega}{}^2\cos(\theta_B-\theta_Y)}\left(\frac{f_B\cos\theta_Y}{\sqrt{6}}-\frac{f_Y\sin\theta_B}{3}\right)\epsilon_{\alpha\sigma'\gamma\delta}(q_1+q_2)_{\gamma}q_{4\delta}\Delta_{\omega}{}^{-1}(q_4)_{\sigma\sigma'}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha}{}^{abc}\Bigg]$$
$$+\frac{f^{bae}}{F_{\pi}{}^2 g_{\rho}}(q_1-q_2)_{\mu}\Delta_V(q_1+q_2)_{\mu\mu'}\Gamma^{(2)\omega}(q_3,q_1+q_2)_{\mu'\sigma}{}^{ce}$$
$$+\frac{if^{acb}8(2b_1-3y)}{m_{\omega}{}^2\cos(\theta_B-\theta_Y)}\left(\frac{f_B\cos\theta_Y}{\sqrt{6}}-\frac{f_Y\sin\theta_B}{3}\right)\epsilon_{\mu\sigma'\alpha\beta}(q_1-q_2)_{\mu}(q_1+q_2)_{\alpha}q_{4\beta}\Delta_{\omega}{}^{-1}(q_4)_{\sigma\sigma'}$$
$$+\frac{q_{1\mu}q_{2\nu}\Delta_{\omega}{}^{-1}(q_4)_{\sigma\sigma'}}{3F_{\pi}{}^3 m_{\omega}{}^2\cos(\theta_B-\theta_Y)}\left[\frac{f_B\cos\theta_Y}{2}\Delta_{\mu\nu\sigma'}(AA\mathbf{A}V^0)-\frac{f_Y\sin\theta_B}{\sqrt{3}}\Delta_{\mu\nu\sigma'}(AA\mathbf{A}V^8)\right]$$
$$+(q_2\leftrightarrow q_3;\,b\leftrightarrow c)+(q_1\leftrightarrow q_2;\,a\leftrightarrow b)\,. \tag{2.9}$$

In order to simplify further, we use the expression for the proper A_1-A_1-ρ vertex discussed by Schnitzer and Weinberg[1] in the context of hard pions. In our normalization it has the following form:

$$\Gamma^{(1)}(q_1,q_2)_{\nu\lambda\sigma}{}^{abc}=2if^{abc}g_{\rho}{}^{-1}m_{\rho}{}^2$$
$$\times[g_{\nu\lambda}(q_2-q_1)_{\sigma}+g_{\lambda\sigma}(k-q_2)_{\nu}+g_{\sigma\nu}(q_1-k)_{\lambda}$$
$$-\delta(g_{\nu\sigma}k_{\lambda}-g_{\lambda\sigma}k_{\nu})]\,, \tag{2.10}$$

where q_1, q_2 are the A_1 momenta and $k=-q_1-q_2$. We write the phenomenological ω-$\rho\pi$ vertex

$$\Gamma^{(2)\omega}(q_1,q_2)_{\nu\lambda}{}^{ab}=\delta^{ab}g_{\omega\rho\pi}\epsilon_{\delta\beta\nu\lambda}q_{1\delta}q_{2\beta}\,. \tag{2.11}$$

Furthermore, it was shown in Ref. 1 that the Ward identities for the three-point functions imply that if the primitive three-point function (in the A_1-A_1-ρ system) is as slowly varying as possible, then the two-point functions have the form of free propagators. In our notation,

$$\Delta_V(q)_{\mu\nu}=\frac{C_V m_{\rho}{}^2}{m_{\rho}{}^2-q^2}\left(g_{\mu\nu}-\frac{q_{\mu}q_{\nu}}{m_{\rho}{}^2}\right),$$
$$\Delta_A(q)_{\mu\nu}=\frac{C_A m_{A_1}{}^2}{m_{A_1}{}^2-q^2}\left(g_{\mu\nu}-\frac{q_{\mu}q_{\nu}}{m_{A_1}{}^2}\right). \tag{2.12}$$

Using (2.10)–(2.12), we can now write the proper matrix element for the decay $\omega\to\pi^+\pi^-\pi^0$ by taking the limit $q_4{}^2\to m_{\omega}{}^2$ in Eq. (2.9). We obtain

$$\lim_{q_4{}^2\to m_{\omega}{}^2}M^{\omega}(q_1,q_2,q_3)_{\sigma}{}^{abc}\epsilon_{\sigma}{}^{\omega}(q_4)=-C_A{}^3 g_{A_1}{}^{-3}F_{\pi}{}^{-3}q_{1\mu}q_{2\nu}q_{3\lambda}M_C{}^{\omega}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}\epsilon_{\sigma}{}^{\omega}(q_4)+2f^{abc}g_{\omega\rho\pi}\epsilon_{\mu\nu\lambda\sigma}q_{1\mu}q_{2\nu}q_{3\lambda}\epsilon_{\sigma}{}^{\omega}(q_4)$$
$$\times\left[-\frac{m_{\rho}{}^2}{g_{\rho}}\frac{g_{A_1}{}^2}{m_{A_1}{}^4}F_{\pi}{}^{-2}(1+\delta)\left(\frac{s}{m_{\rho}{}^2-s}+\frac{u}{m_{\rho}{}^2-u}+\frac{t}{m_{\rho}{}^2-t}\right)+\frac{g_{\rho}}{F_{\pi}{}^2}\left(\frac{1}{m_{\rho}{}^2-s}+\frac{1}{m_{\rho}{}^2-u}+\frac{1}{m_{\rho}{}^2-t}\right)\right], \tag{2.13}$$

where $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, and $u=(q_2+q_3)^2$.

The expression (2.13) can be further simplified if we use (a) Weinberg's first sum rule[6]

$$g_{A_1}{}^2/m_{A_1}{}^2+F_{\pi}{}^2=g_{\rho}{}^2/m_{\rho}{}^2\,, \tag{2.14}$$

(b) the second spectral function sum rule[7]

$$g_{\rho}\simeq g_{A_1}\,, \tag{2.15}$$

and (c) the KSRF relation[8]

$$g_{\rho}{}^2=2F_{\pi}{}^2m_{\rho}{}^2\,, \tag{2.16}$$

which together yield $m_{A_1}{}^2=2m_{\rho}{}^2$.

[7] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[8] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

Using these expressions, (2.13) can be written as

$$M^{\omega}(q_1,q_2,q_3)_\sigma{}^{abc}\epsilon_\sigma{}^{\omega}(q_4)=-\frac{1}{2\sqrt{2}m_\rho{}^3}q_{1\mu}q_{2\nu}q_{3\lambda}M_C{}^{\omega}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}\epsilon_\sigma{}^{\omega}(q_4)+2f^{abc}\frac{g^{\omega\rho\pi}}{F_\pi}\epsilon_{\mu\nu\lambda\sigma}q_{1\mu}q_{2\nu}q_{3\lambda}\epsilon_\sigma{}^{\omega}(q_4)$$
$$\times\left[\sqrt{2}m_\rho\left(\frac{1}{m_\rho{}^2-s}+\frac{1}{m_\rho{}^2-u}+\frac{1}{m_\rho{}^2-t}\right)-\frac{1+\delta}{2\sqrt{2}m_\rho}\left(\frac{s}{m_\rho{}^2-s}+\frac{u}{m_\rho{}^2-u}+\frac{t}{m_\rho{}^2-t}\right)\right]. \quad (2.17)$$

III. VECTOR-CONSTRAINT CONDITION

In this section we try to evaluate the structure of the contact term appearing in the physical $\omega\to 3\pi$ amplitude. We make use of the work done recently by Brown *et al.*[3] on Ward-identity anomalies. They show, and we have already incorporated this in the work up to now, that after universal regularization and addition of counter terms, the only first-order anomalies in the system under consideration are the axial-vector anomalies $\Delta(\mathbf{A}VV)$ and $\Delta(\mathbf{A}AAV)$ and the vector anomalies $\Delta(A\mathbf{V}V)$ and $\Delta(AAA\mathbf{V})$. We just cannot avoid anomalies in this problem.

To investigate the contact term, we use the vector constraint equation (2.5). Introducing the contact term, the vector constraint condition (2.5) can be written as

$$q_{4\sigma}M_C{}^{\omega}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}+8b_1g_Ag_\rho{}^{-1}C_{A_1}{}^{-1}m_{A_1}{}^{-2}\cos^{-1}(\theta_B-\theta_Y)\left(\frac{f_B\cos\theta_Y}{\sqrt{6}}-\frac{f_Y\sin\theta_B}{3}\right)f^{abc}(m_{A_1}{}^2-q_2{}^2)$$
$$\times\{2\epsilon_{\mu'\nu\lambda'\sigma}q_{1\mu'}q_{3\lambda'}q_{4\sigma}g_{\mu\lambda}-\epsilon_{\nu\lambda\mu'\sigma}(q_1+q_3)_{\mu'}q_{4\sigma}(q_1+2q_3)_\mu+\epsilon_{\nu\mu\lambda'\sigma}(q_1+q_3)_{\lambda'}q_{4\sigma}(2q_1+q_3)_\lambda$$
$$+\delta[\epsilon_{\nu\mu\lambda'\sigma}(q_1+q_3)_{\lambda'}q_{4\sigma}(q_1+q_3)_\lambda-\epsilon_{\nu\lambda\mu'\sigma}(q_1+q_3)_{\mu'}q_{4\sigma}(q_1+q_3)_\mu]\}$$
$$+(q_1\leftrightarrow q_2;\, a\leftrightarrow b;\, \mu\leftrightarrow\nu)+(q_2\leftrightarrow q_3;\, \nu\leftrightarrow\lambda;\, b\leftrightarrow c)$$
$$=192ib_2\left(\frac{f_B\cos\theta_Y}{\sqrt{6}}-\frac{f_Y\sin\theta_B}{3}\right)\cos^{-1}(\theta_B-\theta_Y)f^{abc}(g_{A_1}C_A{}^{-1}m_{A_1}{}^{-2})^3[(m_{A_1}{}^2-q_1{}^2)(m_{A_1}{}^2-q_2{}^2)(m_{A_1}{}^2-q_3{}^2)q_{4\sigma}\epsilon_{\mu\nu\lambda\sigma}$$
$$+(m_{A_1}{}^2-q_2{}^2)(m_{A_1}{}^2-q_3{}^2)q_{1\mu}q_{1\mu'}\epsilon_{\mu'\nu\lambda\sigma}q_{4\sigma}+(m_{A_1}{}^2-q_1{}^2)(m_{A_1}{}^2-q_2{}^2)q_{3\lambda}q_{3\lambda'}\epsilon_{\mu\nu\lambda'\sigma}q_{4\sigma}$$
$$+(m_{A_1}{}^2-q_3{}^2)(m_{A_1}{}^2-q_1{}^2)q_{2\nu}q_{2\nu'}\epsilon_{\mu\nu'\lambda\sigma}q_{4\sigma}+(m_{A_1}{}^2-q_1{}^2)q_{2\nu}q_{3\lambda}\epsilon_{\mu\nu'\lambda'\sigma}q_{2\nu'}q_{3\lambda'}q_{4\sigma}$$
$$+(m_{A_1}{}^2-q_2{}^2)q_{3\lambda}q_{1\mu}\epsilon_{\mu'\nu\lambda'\sigma}q_{1\mu'}q_{3\lambda'}q_{4\sigma}+(m_{A_1}{}^2-q_3{}^2)q_{1\mu}q_{2\nu}\epsilon_{\mu'\nu'\lambda\sigma}q_{1\mu'}q_{2\nu'}q_{4\sigma}]. \quad (3.1)$$

We assume the smoothness postulate of the contact term and write its most general form using Lorentz covariance, Bose statistics, etc., keeping terms up to second order in 4-momenta. The next higher-order (fourth-order) contact term will make a contribution to $\omega\to 3\pi$ of the order of $m_\pi{}^2/m_{A_1}{}^2$ smaller than the second order so we do not need to consider it. Thus

$$M_C{}^{\omega}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abc}=f^{abc}\left(\frac{f_B\cos\theta_Y}{\sqrt{6}}-\frac{f_Y\sin\theta_B}{3}\right)\cos^{-1}(\theta_B-\theta_Y)$$
$$\times\{\chi\epsilon_{\mu\nu\lambda\sigma}+A_1[\epsilon_{\mu\nu\lambda'\sigma}q_{3\lambda'}(q_1+q_2)_\lambda+\epsilon_{\mu'\nu\lambda\sigma}q_{1\mu'}(q_2+q_3)_\mu+\epsilon_{\mu\nu'\lambda\sigma}q_{2\nu'}(q_3+q_1)_\nu]$$
$$+A_2(\epsilon_{\mu\nu\lambda'\sigma}q_{3\lambda'}q_{3\lambda}+\epsilon_{\mu'\nu\lambda\sigma}q_{1\mu'}q_{1\mu}+\epsilon_{\mu\nu'\lambda\sigma}q_{2\nu'}q_{2\nu})$$
$$+A_3(\epsilon_{\mu\nu\lambda'\sigma}q_{2\lambda'}q_{3\lambda}+\epsilon_{\mu\nu\lambda'\sigma}q_{1\lambda'}q_{3\lambda}+\epsilon_{\nu\lambda\mu'\sigma}q_{2\mu'}q_{1\mu}+\epsilon_{\nu\lambda\mu'\sigma}q_{3\mu'}q_{1\mu}+\epsilon_{\lambda\mu\nu'\sigma}q_{3\nu'}q_{2\nu}+\epsilon_{\lambda\mu\nu'\sigma}q_{1\nu'}q_{2\nu})$$
$$+A_4(\epsilon_{\mu'\nu\lambda\sigma}q_{2\mu'}q_{3\mu}+\epsilon_{\mu'\nu\lambda\sigma}q_{3\mu'}q_{2\mu}+\epsilon_{\mu\nu'\lambda\sigma}q_{1\nu'}q_{3\nu}+\epsilon_{\mu\nu'\lambda\sigma}q_{1\nu}q_{3\nu'}+\epsilon_{\mu\nu\lambda'\sigma}q_{2\lambda'}q_{1\lambda}+\epsilon_{\mu\nu\lambda'\sigma}q_{1\lambda'}q_{2\lambda})$$
$$+A_5(\epsilon_{\mu'\nu'\lambda\sigma}q_{1\mu'}q_{2\nu'}g_{\mu\nu}+\epsilon_{\mu'\nu\lambda'\sigma}q_{1\mu'}q_{3\lambda'}g_{\mu\lambda}+\epsilon_{\mu\nu'\lambda'\sigma}q_{2\nu'}q_{3\lambda'}g_{\nu\lambda})$$
$$+A_6(\epsilon_{\mu\lambda\nu'\sigma}q_{3\nu'}q_{3\nu}+\epsilon_{\lambda\nu\mu'\sigma}q_{3\mu'}q_{3\mu}+\epsilon_{\nu\mu\lambda'\sigma}q_{2\lambda'}q_{2\lambda}+\epsilon_{\lambda\nu\mu'\sigma}q_{2\mu'}q_{2\mu}+\epsilon_{\mu\lambda\nu'\sigma}q_{1\nu'}q_{1\nu}+\epsilon_{\nu\mu\lambda'\sigma}q_{1\lambda'}q_{1\lambda})$$
$$+A_7[g_{\lambda\mu}\epsilon_{\nu\tau\sigma\rho}q_{2\rho}(q_3-q_1)_\tau+g_{\lambda\nu}\epsilon_{\tau\sigma\rho\nu}q_{1\rho}(q_3-q_2)_\tau+g_{\mu\nu}\epsilon_{\lambda\sigma\rho\tau}q_{3\rho}(q_1-q_2)_\tau]\}. \quad (3.2)$$

Introducing (3.2) into (3.1), and noting that all the covariants $\epsilon_{\mu\nu\lambda\sigma}q_{3\lambda}q_{4\sigma}q_{1\lambda}$, etc., are not independent, we replace q_4 by $-q_1-q_2-q_3$ in (3.1), and comparing the coefficients of the various covariants, we get

$$\begin{aligned}
\chi&=192ib_2g_{A_1}{}^3C_A{}^{-3},\\
A_2-A_3&=192ib_2g_{A_1}{}^3C_A{}^{-3}m_{A_1}{}^{-2}+16b_1g_{A_1}g_\rho{}^{-2}m_\rho{}^{-2}C_A{}^{-1}(1+\delta),\\
A_1-A_4&=16b_1g_{A_1}g_\rho{}^{-2}m_\rho{}^{+2}C_A{}^{-1}(2+\delta),\\
A_1+A_6&=0,\\
A_5&=-32b_1g_{A_1}g_\rho{}^{-2}m_\rho{}^2C_A{}^{-1};\quad \text{no constraints on } A_7.
\end{aligned} \quad (3.3)$$

We see that we have three independent parameters A_1, A_2, and A_7 which are not determined. However, A_7 does not contribute to the $\omega \to 3\pi$ amplitude. Substituting (3.3) in (3.1) and utilizing (2.13), we find that the contact term contains essentially one parameter, ξ, which is not determined.[9] This result is independent of the choice of b_i's, as is obvious from Eq. (3.3). Thus the amplitude becomes

$$M^{\omega}(q_1,q_2,q_3)_{\sigma}{}^{abc}\epsilon_{\sigma}{}^{\omega}(q_4)=f^{abc}\epsilon_{\mu\nu\lambda\sigma}q_{1\mu}q_{2\nu}q_{3\lambda}\epsilon_{\sigma}{}^{\omega}(q_4)$$

$$\times\left\{i\xi+2g_{\omega\rho\pi}(F_{\pi})^{-1}\left[\sqrt{2}m_{\rho}\left(\frac{1}{m_{\rho}^2-s}+\frac{1}{m_{\rho}^2-u}+\frac{1}{m_{\rho}^2-t}\right)-\frac{1}{2\sqrt{2}}m_{\rho}^{-1}(1+\delta)\left(\frac{s}{m_{\rho}^2-s}+\frac{u}{m_{\rho}^2-u}+\frac{t}{m_{\rho}^2-t}\right)\right]\right\}. \quad (3.4)$$

IV. DISCUSSION AND RESULTS

We see from Eq. (3.4) that we are left with three parameters, $g_{\omega\rho\pi}$, ξ, and δ. We evaluate $g_{\omega\rho\pi}$ by fitting the expression for $\Gamma(\omega\to\pi^0\gamma)$ obtained from the vector-dominance model, to the experimental value for $\Gamma(\omega\to\pi^0\gamma)$ which is now known to be fairly accurately, 1.16 MeV. In this way we obtain a value of 1.70×10^{-2} MeV^{-1} for $g_{\omega\rho\pi}$. We then feed this value of $g_{\omega\rho\pi}$ back into the expression for $\Gamma(\omega\to3\pi)$ and determine the decay width for the various values of δ, by putting $\xi=0$. With this approximation, the $\omega\to3\pi$ system is reduced to the GSW[10] model with only the ρ pole contributing to the process. The values of $\Gamma(\omega\to3\pi)$ for $\delta=0$ to $\delta=-1$ are given in Table I. We observe that for the choice of $\delta=-1$, $\Gamma(\omega\to3\pi)$ is very close to the experimental value, thus indicating that the contact term may not be very important in this process. The values for the ratio $\Gamma(\omega\to\pi\gamma)/\Gamma(\omega\to3\pi)$ are also presented in Table I.

The photopion reaction $\gamma+\pi\to\pi+\pi$ is related directly to the $\omega\to3\pi$ process through the off-mass-shell amplitude (2.9) and we can evaluate the coupling constant $f_{\gamma-3\pi}$ defined in the following manner:

$$ef_{\gamma-3\pi}/m_{\pi}^3=F(s,t,u=m_{\pi}^2), \quad (4.1)$$

where F is the form factor appearing in the matrix element

$$\langle\gamma|\pi^+\pi^-\pi^0\rangle=(2\pi)^4\delta^4(p^{\gamma}-k^+-k^--k^0)$$
$$\times\epsilon_{\mu\nu\lambda\sigma}\epsilon_{\mu}{}^{\gamma}(p)k_{\nu}{}^+k_{\lambda}{}^-k_{\sigma}{}^0F(s,t,u). \quad (4.2)$$

We observe that the anomalous terms which do not contribute to the physical $\omega\to3\pi$ process will contribute in this reaction, as the ω is off its mass shell. Here we have to make a choice of the two parameters b_1, b_2 appearing in the anomalies. Bardeen's choice[11] $b_1=b_2=0$ gives rise to only axial-vector anomalies. In the choice $2b_1=3y$, $b_2=0$, we eliminate $\Delta(\mathbf{A}VV)$ but retain $(AV\mathbf{V})$ and out of the two four-point anomalies $\Delta(\mathbf{A}AAV)$ and $\Delta(AAA\mathbf{V})$ we eliminate the latter. The question concerning which set should be chosen is a dynamical one. Additional experimental information is needed in order to determine the specific values for the b_i's. We have made the choice $2b_1=3y$, $b_2=0$.

We use the current mixing model, which Oakes and Sakurai[12] have shown to be the only one consistent with Weinberg's first sum rule for $SU(3)$.[7] Assuming an octet symmetry breaking, they have calculated

$$\theta_Y=35.0°, \quad \theta_B=22.5°. \quad (4.3)$$

Knowing θ_Y, we can calculate f_Y by using the first Weinberg sum rule

$$\frac{g_{\rho}^2}{m_{\rho}^2}=\frac{3}{f_Y^2}(m_{\varphi}^2\cos^2\theta_Y+m_{\omega}^2\sin^2\theta_Y)$$

along with the KSRF relation.[8] Thus we obtain

$$\frac{f_Y^2}{4\pi}\frac{1}{\sin^2\theta_Y}=9. \quad (4.4)$$

Using vector dominance, one can show that

$$\frac{\Gamma(\omega\to\pi\gamma)}{\Gamma(\pi\to\gamma\gamma)}=\frac{2}{3}\left(\frac{m_{\omega}^2-m_{\pi}^2}{m_{\omega}m_{\pi}}\right)^3\left(\frac{f_Y}{2}\right)^2\sin^{-2}\theta_Y, \quad (4.5)$$

which on using (4.4) gives a value of 1.3×10^5 for the ratio $\Gamma(\omega\to\pi\gamma)/\Gamma(\pi\to\gamma\gamma)$, consistent with experiment.[13]

For the other mixing parameter f_B, we use the fact that $g_{\varphi NN}$ is experimentally small and by putting $g_{\varphi NN}=0$, we get

$$f_Y\cos\theta_B=-f_B\sin\theta_Y. \quad (4.6)$$

With these choices of the coupling constants and with $\xi=0$, $f_{\gamma-3\pi}$ is listed in Table I. Alternatively, instead of putting $\xi=0$, we can determine ξ by fitting the expression for $\omega\to3\pi$ to the experimental decay width $\Gamma(\omega\to3\pi)=10.9$ MeV. The value of ξ (and consequently $f_{\gamma-3\pi}$) depends on δ. Estimates of $f_{\gamma-3\pi}$ for $\delta=0$ to $\delta=-1$ are listed in Table II. We observe that for $\delta=-1$, $f_{\gamma-3\pi}$ is very close to the upper experimental limit.[14]

[9] ξ lumps together the two parameters A_1 and A_2.

[10] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[11] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

[12] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters **19**, 1266 (1967).

[13] Data have been taken from the tables of Particle Data Group, Rev. Mod. Phys. **42**, 87 (1970).

[14] Aruna Chatterjee, Phys. Rev. **170**, 1578 (1968); K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966).

TABLE I. Decay widths with $\xi=0$.

δ	0	-0.25	-0.5	-0.75	-1.0	Expt
$\Gamma(\rho\to 2\pi)$ MeV	73	92.4	99.5	115	130	90–150
$\Gamma(\omega\to 3\pi)$ MeV	7.5	8.0	8.4	8.9	9.4	10.9 ± 1.1
$\Gamma(\omega\to\pi^0\gamma)/\Gamma(\omega\to 3\pi)$	0.15	0.14	0.13	0.12	0.12	0.13 ± 0.04
$f_{\gamma-3\pi}\times 10^{-2}$	16.1	16.2	16.2	16.2	16.3	4 ± 15

We have estimated the effect of the anomalies in various Ward identities of three-point functions $\langle AVV\rangle$ and $\langle PVV\rangle$ and their consequences on the $\pi^0\gamma\gamma$ form factor. Our approach is similar to that of Young,[15] and our results agree for a particular choice of the b_i's with his quark-model calculations. In our normalization the $\pi^0\gamma\gamma$ form factor is defined as

TABLE II. Form factor for $\gamma+\pi\to\pi+\pi$.

δ	0	-0.25	-0.5	-0.75	-1.0	Expt
$f_{\gamma-3\pi}\times 10^{-2}$	22.8	22.0	21.2	20.4	19.5	4 ± 15

$$i\int d^4x\, e^{ip\cdot x}\langle 0|T(j_\mu^{\text{e.m.}}(x)j_\nu^{\text{e.m.}}(0))|\pi^0(k)\rangle =4\epsilon_{\mu\nu\lambda\sigma}p_\lambda q_\sigma F(p,q)/m_\pi. \quad (4.7)$$

We write all the first-order Ward identities relevant for the three-point function $\langle AVV\rangle$. Using the definitions of the Γ's from (A5) and (A6), the anomalies from Appendix B, and writing the most general form of $\Gamma^{(2)\omega}(k,p)_{\mu\nu\lambda}$ from Lorentz covariance, one can easily obtain

$$\begin{aligned}\Gamma^{(2)\omega}(k,p)_{\nu\lambda}{}^{ab}=\{&-g_{A_1}F_\pi^{-1}m_{A_1}^{-2}(\Gamma_1q^2+\Gamma_2p^2+\Gamma_3k^2)-g_{A_1}F_\pi^{-1}m_{A_1}^{-2}\cos^{-1}(\theta_B-\theta_Y)\\ &\times[\tfrac{1}{2}f_Bd^{ab0}\cos\theta_Y-(1/\sqrt{3})f_Yd^{ab8}\sin\theta_B][8(2b_1-3y)m_{A_1}{}^2m_\omega{}^{-2}(m_\rho{}^2-p^2)(m_\omega{}^2-q^2)]\\ &-8b_1g_{A_1}{}^{-2}(m_{A_1}{}^2-k^2)[(m_\omega{}^2-q^2)m_\rho{}^2m_\omega{}^{-2}-(m_\rho{}^2-p^2)]\}\epsilon_{\mu\nu\lambda\sigma}p_\lambda q_\sigma. \quad (4.8)\end{aligned}$$

A similar equation holds for $\Gamma^{(2)\varphi}(k,p)_{\nu\lambda}{}^{ab}$. Using (4.8), the analogous equation for $\Gamma^{(2)\varphi}(k,p)_{\nu\lambda}{}^{ab}$ and Eq. (4.7), one can obtain in a rather straightforward way the following relation for the $\pi^0\gamma\gamma$ form factor:

$$\begin{aligned}F(k^2,p^2,q^2)=&-ie^2\tfrac{4}{3}(2b_1-3y)m_\pi/F_\pi-\left\{\frac{ie^2}{f_Y}\frac{m_\pi}{(p^2-m_\rho{}^2)}\frac{2b_1}{F_\pi m_{A_1}{}^2}(m_{A_1}{}^2-k^2)\cos(\theta_B-\theta_Y)^{-1}\right.\\ &\times\left[\left(\frac{f_B}{\sqrt{6}}\sin\theta_Y\cos\theta_Y+\tfrac{1}{3}f_Y\cos\theta_Y\cos\theta_B\right)\frac{p^2m_\varphi{}^2-q^2m_\rho{}^2}{m_\varphi{}^2-q^2}-\left(\frac{f_B}{\sqrt{6}}\sin\theta_Y\cos\theta_Y-\tfrac{1}{3}f_Y\sin\theta_B\sin\theta_Y\right)\frac{p^2m_\omega{}^2-q^2m_\rho{}^2}{m_\omega{}^2-q^2}\right]\Bigg\}\\ &-\{p\leftrightarrow q\}+\frac{ie^2}{\sqrt{3}}\frac{g_{A_1}m_\pi}{F_\pi m_{A_1}{}^2}\left\{\frac{g_\rho}{p^2-m_\rho{}^2}\left[\frac{g_\varphi}{m_\varphi{}^2-q^2}(\Gamma_1{}^\varphi q^2+\Gamma_2{}^\varphi p^2+\Gamma_3{}^\varphi k^2)+\phi\leftrightarrow\omega\right]\right\}+\{p\leftrightarrow q\}. \quad (4.9)\end{aligned}$$

Therefore,

$$F(0,0,0)=-\frac{4}{3}\frac{m_\pi}{F_\pi}ie^2(2b_1-3y). \quad (4.10)$$

If one uses the value $2b_1=3y$, then

$$F(0,0,0)=0, \quad (4.11)$$

which is the result of Sutherland. If, on the other hand, one uses $b_1=0$, then

$$F(0,0,0)=\frac{2}{3}\frac{\alpha}{\pi}\frac{m_\pi}{F_\pi}\simeq 0.15\alpha, \quad (4.12)$$

which is $\sim\frac{1}{3}$, the experimental value for $F(0,0,0)$.

[15] B. L. Young, Phys. Rev. D 2, 606 (1970).

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APPENDIX A

In our normalization the proper vertices are given by

$$M^\omega(q_1,q_2,q_3)_\sigma{}^{abc}\epsilon_\sigma{}^\omega(q_4)\,,\quad\text{etc.,}$$

where $\epsilon_\sigma{}^\omega$ is the ω polarization vector. With this normalization, we have the following definitions for the various four-point functions:

When $V_\sigma^d = V_\sigma^8$,

$$\langle \partial_\mu A_\mu{}^a(q_1) \partial_\nu A_\nu{}^b(q_2) \partial_\lambda A_\lambda{}^c(q_3) V_\sigma{}^8(q_4)\rangle$$

$$\equiv \frac{i\sqrt{3}}{f_Y} \frac{(F_\pi m_\pi{}^2)^3}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)(m_\pi{}^2-q_3{}^2)} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\sigma'}{}^{abc}], \quad \text{(A1)}$$

$$\langle \partial_\mu A_\mu{}^a(q_1) \partial_\nu A_\nu{}^b(q_2) A_\lambda{}^c(q_3) V_\sigma{}^8(q_4)\rangle$$

$$\equiv \frac{\sqrt{3}}{f_Y} \frac{(F_\pi m_\pi{}^2)^2 g_{A1}{}^{-1}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)} \Delta_{A1}(q_3)_{\lambda\lambda'} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\lambda'\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\lambda'\sigma'}{}^{abc}]$$
$$+ \frac{\sqrt{3}}{f_Y} \frac{F_\pi{}^3 m_\pi{}^4 q_{3\lambda}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)(m_\pi{}^2-q_3{}^2)} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\sigma'}{}^{abc}], \quad \text{(A2)}$$

$$\langle \partial_\mu A_\mu{}^a(q_1) A_\nu{}^b(q_2) A_\lambda{}^c(q_3) V_\sigma{}^8(q_4)\rangle$$

$$\equiv i\frac{\sqrt{3}}{f_Y} \frac{F_\pi m_\pi{}^2 g_{A1}{}^{-2}}{m_\pi{}^2-q_1{}^2} \Delta_{A1}(q_2)_{\nu\nu'} \Delta_{A1}(q_3)_{\lambda\lambda'} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\nu'\lambda'\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\nu'\lambda'\sigma'}{}^{abc}]$$
$$- i\frac{\sqrt{3}}{f_Y} \left\{ \left[\frac{F_\pi{}^2 m_\pi{}^2 q_{2\nu} g_{A1}{}^{-1}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)} \Delta_{A1}(q_3)_{\lambda\lambda'} (\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\lambda'\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\lambda'\sigma'}{}^{abc}) \right] + [q_2 \leftrightarrow q_3;\ \nu \leftrightarrow \lambda;\ b \leftrightarrow c] \right\}$$
$$- i\frac{\sqrt{3}}{f_Y} \frac{F_\pi{}^3 m_\pi{}^2 q_{2\nu} q_{3\lambda}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)(m_\pi{}^2-q_3{}^2)} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\sigma'}{}^{abc}], \quad \text{(A3)}$$

$$\langle A_\mu{}^a(q_1) A_\nu{}^b(q_2) A_\lambda{}^c(q_3) V_\sigma{}^8(q_4)\rangle$$

$$\equiv \frac{\sqrt{3}}{f_Y} g_{A1}{}^{-3} \Delta_{A1}(q_1)_{\mu\mu'} \Delta_{A1}(q_2)_{\nu\nu'} \Delta_{A1}(q_3)_{\lambda\lambda'} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_2,q_3)_{\mu'\nu'\lambda'\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\mu'\nu'\lambda'\sigma'}{}^{abc}]$$
$$+ \frac{\sqrt{3}}{f_Y} F_\pi g_{A1}{}^{-2} \left\{ \frac{q_{2\nu}}{m_\pi{}^2-q_2{}^2} \Delta_{A1}(q_1)_{\mu\mu'} \Delta_{A1}(q_3)_{\lambda\lambda'} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_2,q_1,q_3)_{\mu'\lambda'\sigma'}{}^{bac} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_2,q_1,q_3)_{\mu'\lambda'\sigma'}{}^{bac}] + (q_2 \leftrightarrow q_1;\ \nu \leftrightarrow \mu;\ b \leftrightarrow a) + (q_2 \leftrightarrow q_3;\ \nu \leftrightarrow \lambda;\ b \leftrightarrow c) \right\}$$
$$- \frac{\sqrt{3}}{f_Y} F_\pi{}^2 g_{A1}{}^{-1} \left\{ \frac{q_{1\mu} q_{3\lambda} \Delta_{A1}(q_2)_{\nu\nu'}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_3{}^2)} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma'} M^\varphi(q_1,q_3,q_2)_{\nu'\sigma'}{}^{acb} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_3,q_2)_{\nu'\sigma'}{}^{acb}] + (q_2 \leftrightarrow q_1;\ b \leftrightarrow a;\ \nu \leftrightarrow \mu) + (q_2 \leftrightarrow q_3;\ \nu \leftrightarrow \lambda;\ b \leftrightarrow c) \right\}$$
$$- \frac{\sqrt{3}}{f_Y} \frac{F_\pi{}^3 q_{1\mu} q_{2\nu} q_{3\lambda}}{(m_\pi{}^2-q_1{}^2)(m_\pi{}^2-q_2{}^2)(m_\pi{}^2-q_3{}^2)} [\cos\theta_Y\, m_\varphi{}^2 \Delta_\varphi(q_4)_{\sigma\sigma} M^\varphi(q_1,q_2,q_3)_{\sigma'}{}^{abc} - \sin\theta_Y\, m_\omega{}^2 \Delta_\omega(q_4)_{\sigma\sigma'} M^\omega(q_1,q_2,q_3)_{\sigma'}{}^{abc}], \quad \text{(A4)}$$

where we have used the field-current identity

$$V_\lambda{}^8 = \frac{\sqrt{3}}{f_Y} (\cos\theta_Y\, m_\varphi{}^2 \varphi_\lambda - \sin\theta_Y\, m_\omega{}^2 \omega_\lambda).$$

The subscripts ω and φ indicate whether the proper vertex functions contain the ω or φ field. We have used the convention of Ref. 5 in that in the M's the divergences of the axial-vector currents stand to the left of the axial-vector currents, which themselves stand to the left of the vector currents.

$q_4=-q_1-q_2-q_3$ and a, b, and c, are the $SU(3)$ indices and vary from 1 to 3. $\Delta_V(q)_{\mu\mu'}$ and $\Delta_{A_1}(q)_{\mu\mu'}$ are the covariant spin-1 parts of the unrenormalized vector and axial-vector propagators as defined in Ref. 5. $\Delta_\varphi(k)_{\mu\mu'}$ is the φ propagator,

$$\Delta_\varphi(k)_{\mu\mu'}=\frac{1}{m_\varphi^2-k^2}\left(g_{\mu\mu'}-\frac{k_\mu k_{\mu'}}{m_\varphi^2}\right),$$

and similarly for $\Delta_\omega(k)_{\mu\mu'}$.

We further assume the ρ and A_1 dominance of the 1^- and 1^+ spectral functions, i.e., $C_V=-g_\rho^2/m_\rho^2$, $C_A=-g_{A_1}^2/m_{A_1}^2$. The definition for the amplitudes $\langle AAAV^0\rangle$, etc., can be written in an analogous manner. The pole structure of the three-point functions $\Gamma^{(1)}$ can be seen in Ref. 5. $\Gamma^{(1)}$ and $\Gamma^{(2)}$ refer to the three-particle vertices with the superscript (1 or 2) indicating the number of vector particles present.

For $\Gamma^{(2)}$ we have the following definitions:

$$\begin{aligned}\langle A_\mu^a(k)V_\nu^b(p)V_\lambda^8(q)\rangle \\ &=\frac{\sqrt{3}}{f_Y}g_{A_1}^{-1}g_\rho^{-1}\Delta_A(k)_{\mu\mu'}\Delta_V(p)_{\nu\nu'}[\cos\theta_Y\, m_\varphi^2\Delta_\varphi(q)_{\lambda\lambda'}\Gamma^{(2)\varphi}(k,p)_{\mu'\nu'\lambda'}{}^{ab}-\sin\theta_Y\, m_\omega^2\Delta_\omega(q)_{\lambda\lambda'}\Gamma^{(2)\omega}(k,p)_{\mu'\nu'\lambda'}{}^{ab}] \\ &\quad+\frac{\sqrt{3}}{f_Y}\frac{F_\pi g_\rho^{-1}k_\mu}{m_\pi^2-k^2}\Delta_V(p)_{\nu\nu'}[\cos\theta_Y\, m_\varphi^2\Delta_\varphi(q)_{\lambda\lambda'}\Gamma^{(2)\varphi}(k,p)_{\nu'\lambda'}{}^{ab}-\sin\theta_Y\, m_\omega^2\Delta_\omega(q)_{\lambda\lambda'}\Gamma^{(2)\omega}(k,p)_{\nu'\lambda'}{}^{ab}],\end{aligned}\tag{A5}$$

$$\begin{aligned}\langle\partial_\mu A_\mu^a(k)V_\nu^b(p)V_\lambda^8(q)\rangle \\ &=\frac{i\sqrt{3}}{f_Y}\frac{F_\pi m_\pi^2}{m_\pi^2-k^2}\Delta_V(p)_{\nu\nu'}[\cos\theta_Y\, m_\varphi^2\Delta_\varphi(q)_{\lambda\lambda'}\Gamma^{(2)\varphi}(k,p)_{\nu'\lambda'}{}^{ab}-\sin\theta_Y\, m_\omega^2\Delta_\omega(q)_{\lambda\lambda'}\Gamma^{(2)\omega}(k,p)_{\nu'\lambda'}{}^{ab}].\end{aligned}\tag{A6}$$

APPENDIX B: ANOMALIES

We follow the notation of Brown *et al.*[3] The only anomalies we are interested in are the two axial-vector anomalies

$$\Delta_{\nu\lambda}(\mathbf{A}^a(k)V^b(p)V^c(q))=8(2b_1-3y)d^{abc}\epsilon_{\nu\lambda\alpha\beta}p_\alpha q_\beta,$$

$$\begin{aligned}\Delta_{\nu\lambda\sigma}(\mathbf{A}^a(k)A^b(p)A^c(q)V^d(t))&=2i\epsilon_{\nu\lambda\sigma\alpha}\\ &\times\{\bar{W}_1[b_1(q+2t)_\alpha+8ib_2k_\alpha+y(t-p)_\alpha]\\ &+\bar{W}_2[b_1(3t-k)_\alpha+8ib_2k_\alpha+y(3t-k)_\alpha]\\ &+\bar{W}_3[-b_1(p+2t)_\alpha-8ib_2k_\alpha+y(q-t)_\alpha]\},\end{aligned}$$

where the above anomaly refers to Ward identities obtained by contracting on the first (leftmost) axial-vector current.

$$\bar{W}_1\equiv\mathrm{Tr}[\lambda_a\lambda_b\lambda_c\lambda_d-\lambda_a^T\lambda_b^T\lambda_c^T\lambda_d^T],$$
$$\bar{W}_2=\bar{W}_1(c\leftrightarrow d),\quad \bar{W}_3=\bar{W}_1(b\leftrightarrow c)$$

and

$$y=1/(24\pi^2 i).$$

We also have the vector anomalies Δ_V as follows:

$$\Delta_{\mu\lambda}{}^V(A^a(k)\mathbf{V}^b(p)V^c(q))=8b_1d^{abc}\epsilon_{\mu\lambda\alpha\beta}p_\alpha q_\beta$$

and

$$\Delta_{\mu\nu\lambda}{}^V(A^a(k)A^b(p)A^c(q)\mathbf{V}^d(t))=16b_2\epsilon_{\mu\nu\lambda\sigma}t_\sigma(\bar{W}_1+\bar{W}_2-\bar{W}_3),$$

where we have expressed the current which is contracted in boldface. Fortunately, there are no anomalies left in the normal-parity series of Ward identities after introduction of counter terms, nor are there any anomalies in the first-order Ward identities involving $\partial_\mu A_\mu$ in the abnormal-parity series.