to suppose that $\rho_\gamma(x,p_T,M^{*2})$ in Eq. (3.4) also tends to a limiting value[6,7] (>0) for fixed x and p_T, but M^{*2} going to infinity. In that case, we find from Eq. (3.4) that as $E\to\infty$, $\rho_e(\mathbf{p},E)\sim\ln^2E$ for a fixed $\mathbf{p}$. To the extent that the total inelastic ep cross section $\sigma_e{}^{\text{in}}(E)$ itself goes like $\ln^2E$ for $E\to\infty$, $\rho_e(\mathbf{p},E)$ seems to attain a limiting distribution[7] only relative to $\sigma_e{}^{\text{in}}(E)$.

Secondly, the behavior of the average multiplicity of a hadron as a function of the incident energy in a multihadron production process at very high energies is one of the features that are likely to reveal some basic characteristics when hadrons take part in high-energy collisions, whether in hadron-hadron scattering[8] or in photoproduction and electroproduction processes. These basic characteristics could very well be such that the average multiplicity shows the same kind of behavior at very high energies irrespective of the primary process. In such a case, the conclusion of a logarithmic dependence of average multiplicity on the incident energy is quite consistent with observations from cosmic-ray data[9] and also with some current theoretical works.[6,10]

We also mention that if, instead of assuming that the total photoabsorption cross section tends to a constant at high energies, we assume that it increases logarithmically (or with some powers of the logarithm) with energy, there will be a corresponding modification in the high-energy behavior of $\sigma_e{}^{\text{in}}(E)$; the conclusion about the behavior of the average multiplicity in this situation, however, remains unchanged.

ACKNOWLEDGMENTS

The author would like to thank Dr. R. Rajaraman and Dr. K. V. L. Sarma for discussions.

[6] R. P. Feynman, Phys. Rev. Letters **23**, 1415 (1969).

[7] J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. **188**, 2159 (1969).

[8] A similar point of view has been recently expressed by Feynman (Ref. 6).

[9] See, for example, E. M. Friedlander, review paper, in *Proceedings of the Eleventh International Conference on Cosmic Rays, Budapest, 1969* (Central Research Institute for Physics, Budapest, 1970), and references quoted therein.

[10] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968).

PHYSICAL REVIEW D VOLUME 3, NUMBER 5 1 MARCH 1971

Two-Pion-Exchange Contribution to Nucleon-Nucleon Scattering*

JUDITH BINSTOCK

Texas A & M University, College Station, Texas 77843

(Received 19 August 1970)

The two-pion-exchange contribution to nucleon-nucleon scattering is calculated, following the method of Goldberger, Grisaru, MacDowell, and Wong. The results are shown for two cases. In the first case, the input $\pi\pi\to\bar{N}N$ amplitude is approximated by a nucleon pole term only; in the second case it is approximated by a nucleon pole term plus a $\Delta(1236)$ pole term. It is found that for the physical value of the $\Delta(1236)$ width, $\Gamma_\Delta=120$ MeV, there is very little difference for the two cases in the resulting effect on nucleon-nucleon scattering below 400 MeV. The model is mathematically equivalent to the Amati-Leader-Vitale approach for all but ${}^1S_0$, ${}^3P_0$, and ${}^3P_1$ NN partial waves. The results are found to be sensitive to the value of Γ_Δ.

I. INTRODUCTION

It has long been contended that elastic nucleon-nucleon scattering below 400 MeV lab kinetic energy can be quantitatively described by one-pion exchange (OPE) alone, for the higher partial waves. The suggestion first by Taketani *et al.* in 1951 and later by Chew in 1958, that the long-range part of the interaction is given by OPE,[1] has been supported by successive phase-shift analyses, the latest of which include Seamon *et al.*[2] and MacGregor *et al.*[3] When the long-range part of the interaction is assumed to be OPE and a x^2 search is made for the pion-nucleon coupling constant, the values obtained include $g_{NN\pi}{}^2=15.4\pm0.4$ [2] for fits to $pp+np$ data, and $g_{NN\pi}{}^2=14.4\pm0.4$ [3] and $g_{NN\pi}{}^2=16.0\pm0.6$ [2] for fits to pp data. If we compare these values with the value of $g_{NN\pi}{}^2=14.9\pm0.3$ determined from forward $\pi^\pm p$ elastic scattering,[4] we find some support for the concept of the essentially OPE nature of the long-range part of the nucleon-nucleon interaction.

To OPE we can add shorter-range terms, in order to improve the model. For instance, the amplitude can be considered as due to a sum of one-boson-exchange (OBE) poles or potentials. This has been done by several

* Work supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. 69-1817.

[1] M. Taketani, S. Nakemura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **28**, 991 (1962); **32**, 380 (1964); G. F. Chew, Phys. Rev. **112**, 1380 (1958).

[2] R. E. Seamon, K. A. Friedman, G. Breit, R. D. Haracz, J. M. Holt, and A. Prakash, Phys. Rev. **165**, 1579 (1968).

[3] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **182**, 1714 (1969).

[4] V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965).

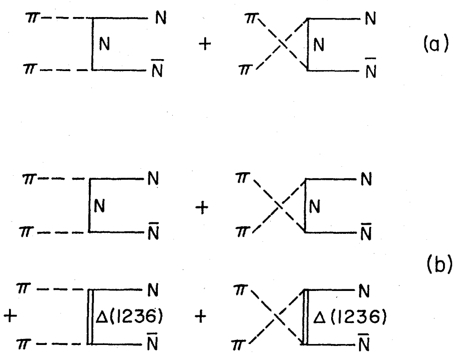


FIG. 1. Approximations for $\pi\pi \rightarrow \bar{N}N$.

authors.[5–20] It has generally been found necessary to add to the π, ρ, and ω exchange terms the exchange of an isoscalar scalar boson (an ϵ) of mass between about 400 and 600 MeV/c^2. The mass depends on whether a pole model or potential model is chosen. If the exchanged ϵ is allowed to be very wide, instead of being of zero width, it may have an even higher mass. For example, Stagat *et al.*[18] use a wide ϵ of mass $\sim$715 MeV, corresponding to the $I=0$ S-wave pion-pion resonance that has been reported.

Pole models have the shortcoming that they fit the $^3D_1$ phase shift badly[19] if the physical value[21,22] of f_ρ/g_ρ is taken. Potential models, however, can fit the $^3D_1$ phase shift. Since an OBE potential model corresponds to summing a set of ladder diagrams, this suggests that perhaps the diagram corresponding to uncrossed two-pion exchange (TPE) may be responsible for this improvement. Note that the crossed TPE diagram is still left out in the OBE potential model.

So we can say that the TPE contribution is of interest because

(a) it is of longer range than the contributions from the exchange of zero-width η, ω, ρ, or ϵ;
(b) it may be that the coupling of the epsilon to the nucleon is too small to give the intermediate range attraction required by the data;
(c) pole models (zero width or otherwise) cannot fit the $^3D_1$ phase shift well, with a physical value of f_ρ/g_ρ. Although OBE potential models can, they include the effect of uncrossed TPE and not the effect of crossed TPE, which is also present.

In this paper, we calculate the TPE contribution, using the method of Goldberger, Grisaru, MacDowell, and Wong (GGMW).[23] Unitarity is applied in the $\bar{N}N \rightarrow \bar{N}N$ channel to yield the imaginary part of the $\bar{N}N \rightarrow \bar{N}N$ scattering amplitude. The input is the $\pi N \rightarrow \pi N$ amplitude. Then real analyticity and crossing are used to determine the discontinuity (in the momentum transfer variable) of the ten $NN \rightarrow NN$ amplitudes described as $G_i^I(s,t,u)$ $(i=1,\ldots,5; I=0,1)$ in GGMW. We write for the $G_i^I(s,t,u)$ unsubtracted single dispersion relations in the momentum transfer variable u, with a cutoff u_c in momentum transfer suitably chosen.

We make two calculations:

(i) The input $\pi\pi \rightarrow \bar{N}N$ amplitude is represented as Fig. 1(a), with only a nucleon pole.

(ii) The input $\pi\pi \rightarrow \bar{N}N$ amplitude is represented as Fig. 1(b), a sum of N and $\Delta(1236)$ pole terms.

The $\pi N \rightarrow \pi N$ amplitude in case (ii) corresponds to the

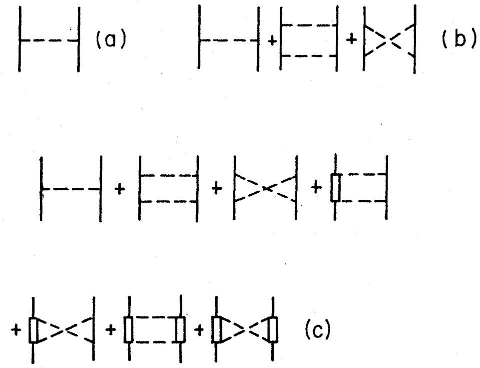


FIG. 2. Models for $NN \rightarrow NN$. In Sec. III A, (a) corresponds to curve OPE; (b) corresponds to curve 3, if $u_c=(3\mu)^2$; 4, if $u_c=(4\mu)^2$; (c) corresponds to curve 3′, if $u_c=(3\mu)^2$; 4′, if $u_c=(4\mu)^2$.

[5] N. Hoshizaki, S. Otsuki, W. Watari, and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) **27**, 1199 (1962).
[6] S. Sawada, T. Ueda, W. Watari, M. Yonezawa, Progr. Theoret. Phys. (Kyoto) **28**, 991 (1962); **32**, 380 (1964); Suppl. **39**, 140 (1967).
[7] R. S. McKean, Jr., Phys. Rev. **125**, 1399 (1962).
[8] R. A. Bryan, C. R. Dismukes, and W. Ramsey, Nucl. Phys. **45**, 353 (1963).
[9] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963); Phys. Rev. **138**, B145 (1965).
[10] J. S. Ball, A. Scotti, and D. Y. Wong, Phys. Rev. **142**, 1000 (1966).
[11] R. A. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964); **164**, 1215 (1967); **177**, 1435 (1969).
[12] G. Köpp and G. Kramer, Phys. Letters **19**, 593 (1965).
[13] G. Köpp, Z. Physik **191**, 273 (1966).
[14] R. A. Arndt, R. A. Bryan, M. H. MacGregor, Phys. Rev. **152**, 1490 (1966); **184**, 1966(E) (1969); Phys. Letters **21**, 314 (1966).
[15] R. A. Bryan and R. A. Arndt, Phys. Rev. **150**, 1299 (1966).
[16] A. E. S. Green and T. Sawada, Rev. Mod. Phys. **39**, 594 (1967); Nucl. Phys. **B2**, 276 (1967).
[17] T. Ueda and A. E. S. Green, Phys. Rev. **174**, 1304 (1968).
[18] R. W. Stagat, F. Riewe, and A. E. S. Green, Phys. Rev. Letters **24**, 631 (1970).
[19] R. A. Bryan, Nucl. Phys. **A146**, 359 (1970).
[20] A. Gersten, R. H. Thompson, and A. E. S. Green, University of Florida Report, 1970 (unpublished).
[21] We define the coupling constants f_ρ and g_ρ as follows:

$$\mathcal{L}_\rho^{\mathrm{int}}=(4\pi)^{1/2}(g_\rho\bar{\Psi}_N\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\varrho}_\mu\Psi_N+(f_\rho/4M)\bar{\Psi}_N\sigma_{\mu\nu}\boldsymbol{\tau}\cdot(\partial_\nu\boldsymbol{\varrho}_\mu-\partial_\mu\boldsymbol{\varrho}_\nu)\Psi_N),$$
$$\sigma_{\mu\nu}=\tfrac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,.$$

The value of f_ρ/g_ρ probably lies within or near the range 3.7–5.2. The lower limit is for a model in which the ρ saturates the isovector part of the electromagnetic nucleon form factor, at zero momentum transfer. The upper limit is for a one-pole fit to the isovector electromagnetic nucleon form factor, at zero momentum transfer, taken from Eq. (18) of Ref. 22.
[22] E. B. Hughes, R. A. Griffy, M. R. Yearian, and R. Hofstadter, Phys. Rev. **139**, B458 (1965).

[23] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

CGLN[24] model. The choice of cutoff u_c, as well as the assumption that the pion-nucleon amplitude is a sum of N and $\Delta(1236)$ pole terms, is discussed in the Appendix. The results for case (ii) are mathematically equivalent to the model Durso[25] refers to as CGLN for all NN phase shifts other than ${}^3P_0$, ${}^3P_1$, and ${}^1S_0$. The difference in these three phase shifts is due to the different set of NN amplitudes[26,27] for which Durso writes dispersion relations. The results shown in this paper for case (ii) differ from the results in Durso's paper in that we take the physical value of the $\Delta(1236)$ width $\Gamma_\Delta=120$ MeV.[28] Durso took $\Gamma_\Delta \sim 140$ MeV, in order to compare with previous calculations of Amati, Leader, and Vitale,[27] whose calculational approach he followed. Also, in this paper we include the results for the NN phases ${}^3D_1$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$, ${}^1P_1$, because we feel these are also of interest. The model in this paper differs from the one by Furuichi and Watari[29] in that we do not extract the left-hand-cut contribution to the partial-wave amplitude, but rather we retain the entire amplitude.

In Sec. II we derive the TPE contribution, which is given by Eqs. (2.10)–(2.22). In Sec. III A, we plot the results. The curves labeled 3 and 4 are for OPE+TPE [case (i)], with cutoffs u_c of $(3\mu)^2$ and $(4\mu)^2$, respectively. The curves labeled 3′ and 4′ are similar, except they are for case (ii) instead of case (i). Schematically, in Sec. III A we are plotting the phase shifts (after geometric unitarization[30]) for the models corresponding to (a)–(c) of Fig. 2. The TPE diagrams here do not represent Feynman diagrams. In Sec. III B we give the numerical results for the TPE contribution [cutoff $u_c=(3.5\mu)^2$] in tabular form. In Sec. IV we discuss the

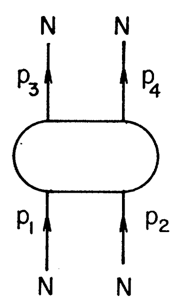


FIG. 3. $NN \to NN$ kinematics.

[24] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[25] J. W. Durso, Phys. Rev. **149**, 1234 (1966).

[26] For a definition of these amplitudes see (2.18)–(2.21) of the first paper of Ref. 27.

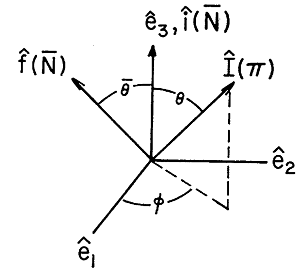
[27] D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960); **18**, 409 (1960); Phys. Rev. **130**, 750 (1963).

[28] A. Barbaro-Galtieri, S. E. Derenzo, L. Price, A. Rittenberg, A. H. Rosenfeld, N. Barash-Schmidt, C. Bricman, M. Roos, P. Söding, and C. G. Wohl, Rev. Mod. Phys. **42**, 87 (1970).

[29] S. Furuichi and W. Watari, Progr. Theoret. Phys. (Kyoto) **34**, 594 (1965); **36**, 348 (1966); see also S. Furuichi, *ibid.* Suppl. **39**, 190 (1967).

[30] What we plot as the nuclear-bar phase shift δ_{JLS} (degrees) is our result for $\langle JLS|(S-1)/2i|JLS\rangle\times 180/\pi$. What we plot as the nuclear-bar coupling parameter ϵ_J (degrees) is our result for $\langle J, L=J\pm 1, S|(S-1)/2i|J, L=J\mp 1, S\rangle\times 180/\pi$. That is what is meant by geometric unitarization in this paper.

FIG. 4. u-channel angles for $\bar{N}N \to \bar{N}N$ with $\pi\pi$ intermediate state.



model, analyzing the graphs in Sec. III and comparing the model to other models for the TPE contribution.

In a subsequent paper we plan to derive the NN phase shifts resulting from the combination of the TPE contribution and single-boson exchanges.

II. OUTLINE OF PROCEDURE

We define the Mandelstam variables s, t, u in the usual way. If p_1–p_4 are the 4-momenta of the four nucleons in $NN \to NN$ elastic scattering (Fig. 3), then

$$NN \to NN \quad s\text{ channel}, \quad s=(p_1+p_2)^2=(p_3+p_4)^2, \tag{2.1a}$$

$$\bar{N}N \to \bar{N}N \quad t\text{ channel}, \quad t=(p_1-p_3)^2=(p_4-p_2)^2, \tag{2.1b}$$

$$\bar{N}N \to \bar{N}N \quad u\text{ channel}, \quad u=(p_1-p_4)^2=(p_3-p_2)^2, \tag{2.1c}$$

$$s+t+u=4M^2, \tag{2.1d}$$

where $p_1^2=p_2^2=p_3^2=p_4^2=M^2$, and M is the mass of the N or $\bar{N}$.

The u-channel c.m. variables are

$$\bar{E}=\text{energy of } N \text{ or } \bar{N}=(u/4)^{1/2}, \tag{2.2a}$$

$$\mu=\text{mass of pion}, \tag{2.2b}$$

$$q=\text{momentum of pion in intermediate } \pi\pi \text{ state} =(\tfrac{1}{4}u-\mu^2)^{1/2}, \tag{2.2c}$$

$$\bar{z}=\text{cosine of scattering angle for } \bar{N}N \to \bar{N}N =2s/(4M^2-u)-1, \tag{2.2d}$$

$$\bar{p}=\text{momentum of } N \text{ or } \bar{N} \text{ in initial or final } \bar{N}N \text{ state} =(\tfrac{1}{4}u-M^2)^{1/2}. \tag{2.2e}$$

Let $\mathfrak{F}_{ab}(\hat{\imath},\hat{I})$ be the $\pi\pi \to \bar{N}N$ amplitude for scattering from π in the intermediate direction $\hat{I}$ to $\bar{N}$ in the initial direction $\hat{\imath}$ (a, b are the helicities of $\bar{N}$ and N, respectively) (see Fig. 4).

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi\pi\to\bar{N}N}=\frac{\bar{p}}{q}|\mathfrak{F}_{ab}|^2 \tag{2.3}$$

defines the normalization of $\mathfrak{F}_{ab}$. Then $\mathfrak{F}_{ab}^*(\hat{f},\hat{I})$ is the complex conjugate of the $\pi\pi \to \bar{N}N$ amplitude for scattering from π in the intermediate direction $\hat{I}$ to $\bar{N}$ in the final direction $\hat{f}$. The unit directional vectors can be expressed as

$$\hat{\imath}=(0,0,1)\,, \tag{2.4a}$$

$$\hat{f}=(\sin\bar{\theta},\,0,\,\cos\bar{\theta})\,, \tag{2.4b}$$

$$\hat{I}=(\sin\theta\cos\varphi,\,\sin\theta\sin\varphi,\,\cos\theta)\,. \tag{2.4c}$$

We can write

$$\bar{z}=\hat{\imath}\cdot\hat{f}=\cos\bar{\theta}\,, \tag{2.5a}$$

$$Z_1=\hat{\imath}\cdot\hat{I}=\cos\theta\,, \tag{2.5b}$$

$$Z_2=\hat{f}\cdot\hat{I}=\sin\bar{\theta}\sin\theta\cos\varphi+\cos\bar{\theta}\cos\theta\,. \tag{2.5c}$$

We shall make the approximation that the $\pi\pi\to\bar{N}N$ amplitude is as shown in Fig. 1(b), a sum of N and $\Delta(1236)$ pole terms. This corresponds to the CGLN model.[24]

We define

$$M_\Delta=\text{mass of }\Delta(1236)\,, \tag{2.6a}$$

$$\lambda_N=(u-2\mu^2)/[(u-4M^2)^{1/2}(u-4\mu^2)^{1/2}]\,, \tag{2.6b}$$

$$\lambda_\Delta=(u-2\mu^2+2M_\Delta^2-2M^2)/[(u-4M^2)^{1/2}(u-4\mu^2)^{1/2}]\,, \tag{2.6c}$$

$$\Gamma_\Delta=\text{width of }\Delta(1236)\,, \tag{2.6d}$$

$$E_N=(M_\Delta^2+M^2-\mu^2)/(2M_\Delta)\,, \tag{2.6e}$$

$$p_{33}^2=E_N^2-M^2\,, \tag{2.6f}$$

$$A_1^{(I)}=-\begin{pmatrix}\sqrt{6}\\ 2\end{pmatrix}g_{NN\pi}^2\Big/[2(u-4M^2)^{1/2}]\,, \tag{2.7a}$$

$$A_2^{(I)}=\begin{pmatrix}2\sqrt{6}\\ -2\end{pmatrix}\frac{\Gamma_\Delta M_\Delta}{3p_{33}^3(\sqrt{u})(u-4\mu^2)^{1/2}}\times\left[3(E_N-M)(M_\Delta+M)\left(1+\frac{u}{2p_{33}^2}\right)+(M_\Delta-M)(E_N+M)\right], \tag{2.7b}$$

$$F^{(I)}=\begin{pmatrix}2\\ -1\end{pmatrix}\frac{\Gamma_\Delta M_\Delta}{3p_{33}^3 g_{NN\pi}^2}\times\left[3(E_N-M)\left(1+\frac{u}{2p_{33}^2}\right)-(E_N+M)\right], \tag{2.7c}$$

$$A_3^{(I)}=A_2^{(I)}/\lambda_\Delta+A_1^{(I)}F^{(I)}M/\bar{E}\,. \tag{2.7d}$$

The upper component corresponds to $I=0$, the lower to $I=1$, I being the total isospin in the $\bar{N}N$ channel. [Setting $\Gamma_\Delta=0$ would give us the approximation that the $\pi\pi\to\bar{N}N$ amplitude is as shown in Fig. 1(a), with only the N pole.]

Then we have

$$\mathfrak{F}_{++}^{(I)}(\hat{\imath},\hat{I})=\frac{M}{\bar{E}}A_1^{(I)}\left[\frac{Z_1}{\lambda_N-Z_1}+\begin{pmatrix}-1\\ +1\end{pmatrix}\frac{Z_1}{\lambda_N+Z_1}+\frac{F^{(I)}Z_1}{\lambda_\Delta-Z_1}+\begin{pmatrix}-1\\ +1\end{pmatrix}\frac{F^{(I)}Z_1}{\lambda_\Delta+Z_1}\right]+A_2^{(I)}\left[\frac{1}{\lambda_\Delta-Z_1}+\begin{pmatrix}+1\\ -1\end{pmatrix}\frac{1}{\lambda_\Delta+Z_1}\right], \tag{2.8a}$$

$$\mathfrak{F}_{++}^{(I)*}(\hat{f},\hat{I})=\mathfrak{F}_{++}^{(I)}(\hat{\imath},\hat{I})\,|_{Z_1\to Z_2}\,, \tag{2.8b}$$

$$\mathfrak{F}_{+-}^{(I)}(\hat{\imath},\hat{I})=-e^{-i\varphi}A_1^{(I)}\sin\theta\left[\frac{1}{\lambda_N-Z_1}+\begin{pmatrix}-1\\ +1\end{pmatrix}\frac{1}{\lambda_N+Z_1}+\frac{F^{(I)}}{\lambda_\Delta-Z_1}+\begin{pmatrix}-1\\ +1\end{pmatrix}\frac{F^{(I)}}{\lambda_\Delta+Z_1}\right], \tag{2.8c}$$

$$\mathfrak{F}_{+-}^{(I)*}(\hat{f},\hat{I})=-[\cos\bar{\theta}\cos\varphi\sin\theta-\sin\bar{\theta}\cos\theta+i\sin\theta\sin\varphi]A_1^{(I)}\left[\frac{1}{\lambda_N-Z_2}+\begin{pmatrix}-1\\ +1\end{pmatrix}\times\frac{1}{\lambda_N+Z_2}+\frac{F^{(I)}}{\lambda_\Delta-Z_2}+\begin{pmatrix}-1\\ +1\end{pmatrix}\frac{F^{(I)}}{\lambda_\Delta+Z_2}\right], \tag{2.8d}$$

$$\mathfrak{F}_{-+}^{(I)*}(\hat{f},\hat{I})=-\text{complex conjugate of }\mathfrak{F}_{+-}^{(I)*}(\hat{f},\hat{I})\,. \tag{2.8e}$$

Using the notation of GGMW, we apply the unitarity relation in the u channel ($\bar{N}N\to\bar{N}N$). Taking only the $\pi\pi$ intermediate state, we derive for the u-channel helicity amplitudes $\bar{f}_i$ ($i=1$, 5)

$$\operatorname{Im}\bar{f}_1=0\,, \tag{2.9a}$$

$$\operatorname{Im}\bar{f}_2=\frac{q\bar{E}}{8\pi}\int d\Omega(\hat{I})\;\mathfrak{F}_{++}(\hat{\imath},\hat{I})\mathfrak{F}_{++}^*(\hat{f},\hat{I})\,, \tag{2.9b}$$

$$\operatorname{Im}(\bar{f}_3+\bar{f}_4)=\frac{q\bar{E}}{8\pi}\frac{1}{(1+\bar{z})}\int d\Omega(\hat{I})\times\mathfrak{F}_{+-}(\hat{\imath},\hat{I})\mathfrak{F}_{+-}^*(\hat{f},\hat{I})\,, \tag{2.9c}$$

$$\operatorname{Im}(\bar{f}_3-\bar{f}_4)=\frac{q\bar{E}}{8\pi}\frac{1}{(\bar{z}-1)}\int d\Omega(\hat{I})\times\mathfrak{F}_{+-}(\hat{\imath},\hat{I})\mathfrak{F}_{-+}^*(\hat{f},\hat{I})\,, \tag{2.9d}$$

$$\operatorname{Im}\bar{f}_5=\frac{Mq}{8\pi}\frac{1}{(1-\bar{z}^2)^{1/2}}\int d\Omega(\hat{I})\times\mathfrak{F}_{++}(\hat{\imath},\hat{I})\mathfrak{F}_{+-}^*(\hat{f},\hat{I})\,. \tag{2.9e}$$

In writing the $\bar{f}$'s and $\mathfrak{F}$'s, we have suppressed the u channel isospin indices. With our approximation for

$\mathfrak{F}$, we get

$$\operatorname{Im}\bar{f}_1{}^{(I)}=0\,, \tag{2.10a}$$

$$\begin{aligned}\operatorname{Im}\bar{f}_2{}^{(I)}=&\frac{q}{\bar{E}}M^2A_1{}^2(1+F)\begin{pmatrix}2\\0\end{pmatrix}[1+F-2T_1(\lambda_N)]+qMA_1A_3\begin{pmatrix}-4\\0\end{pmatrix}(1+F)T_1(\lambda_\Delta)\\&+\frac{q}{\bar{E}}M^2A_1{}^2\left[T_2(\lambda_N,\lambda_N)+\begin{pmatrix}1\\-1\end{pmatrix}T_3(\lambda_N,\lambda_N)\right]+2qMA_1A_3\left[T_2(\lambda_N,\lambda_\Delta)+\begin{pmatrix}1\\-1\end{pmatrix}T_3(\lambda_N,\lambda_\Delta)\right]\\&+q\bar{E}A_3{}^2\left[T_2(\lambda_\Delta,\lambda_\Delta)+\begin{pmatrix}1\\-1\end{pmatrix}T_3(\lambda_\Delta,\lambda_\Delta)\right],\end{aligned}\tag{2.10b}$$

$$\begin{aligned}\operatorname{Im}(\bar{f}_3{}^{(I)}+\bar{f}_4{}^{(I)})=&\frac{q\bar{E}A_1{}^2}{(1+\bar{z})^2}\left\{\begin{pmatrix}-2\\0\end{pmatrix}(1+\bar{z})(1+F)^2+4\left[1+F+\begin{pmatrix}0\\1\end{pmatrix}F\left(\frac{\lambda_\Delta}{\lambda_N}-1\right)\right]T_1(\lambda_N)\right.\\&+4F\left[1+F+\begin{pmatrix}0\\1\end{pmatrix}\left(\frac{\lambda_N}{\lambda_\Delta}-1\right)\right]T_1(\lambda_\Delta)+\left[\frac{\bar{z}+1}{\lambda_N{}^2}+\bar{z}-3\right]T_2(\lambda_N,\lambda_N)+\left[\frac{\bar{z}+1}{\lambda_N\lambda_\Delta}-\frac{\lambda_N}{\lambda_\Delta}-\frac{\lambda_\Delta}{\lambda_N}+\bar{z}-1\right]\\&\times 2FT_2(\lambda_N,\lambda_\Delta)+\left[\frac{\bar{z}+1}{\lambda_\Delta{}^2}+\bar{z}-3\right]F^2T_2(\lambda_\Delta,\lambda_\Delta)+\begin{pmatrix}+1\\-1\end{pmatrix}(\bar{z}+1)\left(1-\frac{1}{\lambda_N{}^2}\right)T_3(\lambda_N,\lambda_N)\\&\left.+\begin{pmatrix}+1\\-1\end{pmatrix}\left[\bar{z}-1+\frac{\lambda_N}{\lambda_\Delta}+\frac{\lambda_\Delta}{\lambda_N}-\frac{\bar{z}+1}{\lambda_N\lambda_\Delta}\right]2FT_3(\lambda_N,\lambda_\Delta)+\begin{pmatrix}+1\\-1\end{pmatrix}(\bar{z}+1)\left(1-\frac{1}{\lambda_\Delta{}^2}\right)F^2T_3(\lambda_\Delta,\lambda_\Delta)\right\},\end{aligned}\tag{2.10c}$$

$$\operatorname{Im}(\bar{f}_3{}^{(I)}-\bar{f}_4{}^{(I)})=\begin{pmatrix}1\\-1\end{pmatrix}\operatorname{Im}(\bar{f}_3{}^{(I)}+\bar{f}_4{}^{(I)})\Big|_{\bar{z}\to-\bar{z}}\,, \tag{2.10d}$$

$$\begin{aligned}\operatorname{Im}\bar{f}_5{}^{(I)}=&\frac{qMA_1}{2(1-\bar{z}^2)}\left\{\left[\begin{pmatrix}4\\0\end{pmatrix}\bar{z}\frac{M}{\bar{E}}A_1(1+F)+\begin{pmatrix}0\\-4\end{pmatrix}\left(\frac{M}{\bar{E}}A_1+\frac{\lambda_\Delta}{\lambda_N}A_3\right)\right]T_1(\lambda_N)\right.\\&+\left[\begin{pmatrix}4\\0\end{pmatrix}\bar{z}A_3(1+F)+\begin{pmatrix}0\\-4\end{pmatrix}F\left(\frac{M}{\bar{E}}A_1\frac{\lambda_N}{\lambda_\Delta}+A_3\right)\right]T_1(\lambda_\Delta)+2\frac{M}{\bar{E}}A_1(1-\bar{z})T_2(\lambda_N,\lambda_N)\\&+\begin{pmatrix}-1\\1\end{pmatrix}2\frac{M}{\bar{E}}A_1(\bar{z}+1)T_3(\lambda_N,\lambda_N)-2\left[\frac{M}{\bar{E}}A_1F\left(\bar{z}-\frac{\lambda_\Delta}{\lambda_N}-\frac{\lambda_N}{\lambda_\Delta}\right)+A_3\bar{z}-\frac{A_2}{\lambda_N}\right]T_2(\lambda_N,\lambda_\Delta)\\&+\begin{pmatrix}-1\\1\end{pmatrix}2\left[\frac{M}{\bar{E}}A_1F\left(\bar{z}+\frac{\lambda_\Delta}{\lambda_N}+\frac{\lambda_N}{\lambda_\Delta}\right)+\bar{z}A_3+\frac{A_2}{\lambda_N}\right]T_3(\lambda_N,\lambda_\Delta)\\&\left.+2FA_3(1-\bar{z})T_2(\lambda_\Delta,\lambda_\Delta)+\begin{pmatrix}-1\\1\end{pmatrix}2FA_3(\bar{z}+1)T_3(\lambda_\Delta,\lambda_\Delta)\right\}.\end{aligned}\tag{2.10e}$$

We have suppressed the isospin index on F, A_1, A_2, and A_3. After analytic continuation to the region $u>4\mu^2$, $s>4M^2$ (where λ_N and λ_Δ are imaginary if $u<4M^2$), the T's are given in terms of standard functions [i.e., $-\frac{1}{2}\pi\leq\tan^{-1}(x)\leq\frac{1}{2}\pi$, x any real argument] by the following:

$$T_1(\lambda)=A\,\frac{1}{4\pi}\int\frac{d\Omega(\hat{l})}{1-\hat{\imath}\cdot\hat{l}/\lambda}\to\begin{cases}i\lambda\tan^{-1}(1/i\lambda)\,, & \text{if } u<4M^2\\ \lambda\tanh^{-1}(1/\lambda)\,, & \text{if } u>4M^2,\end{cases}\tag{2.11a}$$

$$T_2(\lambda_1,\lambda_2)=\frac{1}{4\pi}\int\frac{d\Omega(\hat{l})}{(1-\hat{\imath}\cdot\hat{l}/\lambda_1)(1-\hat{\jmath}\cdot\hat{l}/\lambda_2)}=T_2(\lambda_2,\lambda_1)\to g(x)\,,\tag{2.11b}$$

$$T_3(\lambda_1,\lambda_2)=\frac{1}{4\pi}\int\frac{d\Omega(\hat{l})}{(1-\hat{\imath}\cdot\hat{l}/\lambda_1)(1+\jmath\cdot\hat{l}/\lambda_2)}=T_3(\lambda_2,\lambda_1)\to g(y)\,,\tag{2.11c}$$

where

$$x=\frac{(1-\hat{t}\cdot\hat{f}/\lambda_1\lambda_2)}{(1-1/\lambda_1^2)^{1/2}(1-1/\lambda_2^2)^{1/2}}, \tag{2.12a}$$

$$y=\frac{(1+\hat{t}\cdot\hat{f}/\lambda_1\lambda_2)}{(1-1/\lambda_1^2)^{1/2}(1-1/\lambda_2^2)^{1/2}}, \tag{2.12b}$$

$$\begin{aligned} g(x)&=\frac{\ln[x+|x^2-1|^{1/2}]}{|x^2-1|^{1/2}(1-1/\lambda_1^2)^{1/2}(1-1/\lambda_2^2)^{1/2}} \quad \text{if} \quad |x|>1\\ &=\frac{\tan^{-1}[(1-x^2)^{1/2}/x]}{(1-x^2)^{1/2}(1-1/\lambda_1^2)^{1/2}(1-1/\lambda_2^2)^{1/2}} \quad \text{if} \quad 0<x<1\\ &=\frac{\pi+\tan^{-1}[(1-x^2)^{1/2}/x]}{(1-x^2)^{1/2}(1-1/\lambda_1^2)^{1/2}(1-1/\lambda_2^2)^{1/2}} \quad \text{if} \quad -1<x<0. \end{aligned} \tag{2.13}$$

Spin and isospin crossing[23] plus real analyticity then give the u discontinuity of the $NN\rightarrow NN$ helicity amplitudes:

$$f_i{}^{(I)}{}_u(s,u)=\sum_{I'=1}^{2}\sum_{j=1}^{5}B_{II'}X_{ij}\,\mathrm{Im}\bar{f}_j{}^{(I')}(s,u) \quad (i=1,\ldots,5), \tag{2.14}$$

where

$$\mathbf{B}=\frac{1}{2}\begin{bmatrix}-1 & 3\\ 1 & 1\end{bmatrix}, \tag{2.15a}$$

$$\mathbf{X}=\frac{1}{2}\begin{bmatrix} 1 & -1 & -\dfrac{s}{2\bar{p}^2} & -\dfrac{s}{2\bar{p}^2} & 0\\ -1 & -1+\dfrac{su}{8p^2\bar{p}^2} & \dfrac{1}{2\bar{p}^2}\left(t+\dfrac{su}{4p^2}\right) & -\dfrac{1}{2\bar{p}^2}\left[t+\dfrac{su}{4p^2}\left(1+\dfrac{s}{2\bar{p}^2}\right)\right] & \dfrac{stu}{8p^2\bar{p}^4}\\ -\dfrac{2p^2}{u} & -\dfrac{p^2}{2\bar{p}^2} & \dfrac{p^2}{\bar{p}^2}\left(1-\dfrac{2M^2}{u}\right) & -\dfrac{p^2}{\bar{p}^2}\left[1-\dfrac{2M^2}{u}\left(1+\dfrac{s}{2\bar{p}^2}\right)\right] & \dfrac{sp^2}{2\bar{p}^4}\\ -\dfrac{2p^2}{u} & -\dfrac{s+4M^2}{8\bar{p}^2} & -\dfrac{p^2}{\bar{p}^2}\left[1+\dfrac{2M^2}{u}\left(1+\dfrac{u}{2p^2}\right)\right] & \dfrac{p^2}{\bar{p}^2}\left[1+\dfrac{2M^2}{u}\left(1+\dfrac{s}{2\bar{p}^2}\right)\left(1+\dfrac{u}{2p^2}\right)\right] & -\dfrac{2sp^2+ts}{4\bar{p}^4}\\ 0 & \dfrac{M^2}{\bar{p}^2} & \dfrac{M^2}{\bar{p}^2} & -\dfrac{M^2}{\bar{p}^2}\left(1+\dfrac{s}{2\bar{p}^2}\right) & \dfrac{2p^2}{\bar{p}^2}\left(1-\dfrac{su}{8p^2\bar{p}^2}\right) \end{bmatrix} \tag{2.15b}$$

and

$$p^2=\tfrac{1}{4}s-M^2. \tag{2.15c}$$

We write dispersion relations not for the helicity amplitudes, but for simple linear combinations of the helicity amplitudes which are free of kinematic singularities (the G's of GGMW)[31]:

$$\mathbf{G}^{(I)}(s,u)=\mathbf{M}(s,u)\mathbf{f}^{(I)}(s,u), \tag{2.16}$$

where

$$\mathbf{M}=\begin{bmatrix} \dfrac{4}{s} & 0 & \dfrac{4M^2}{sp^2} & \dfrac{4(1+u/2p^2)}{s} & \dfrac{1+u/2p^2}{M^2}\\ 0 & 0 & 0 & -\dfrac{1}{p^2} & -\dfrac{s}{4M^2p^2}\\ 0 & 0 & -1/p^2 & 0 & 0\\ 0 & 0 & 0 & 1/p^2 & 1/p^2\\ 0 & -\dfrac{1}{p^2} & 0 & \dfrac{1+u/2p^2}{p^2} & \dfrac{(1+u/2p^2)(s+4M^2)}{4p^2M^2} \end{bmatrix}, \tag{2.17}$$

[31] The contribution from the second integral (the t cut) equals the contribution from the first integral (the u cut) for allowed partial waves. The two terms cancel for nonallowed partial waves. This follows from the Pauli principle.

$$\mathbf{G}^{(I)}(s,u)=\frac{1}{\pi}\int_{(2\mu)^2}^{u_c}\frac{du'}{u'-u}\mathbf{G}_u^{(I)}(s,u')+\frac{1}{\pi}\int_{(2\mu)^2}^{u_c}\frac{dt'}{t'-t}\mathbf{G}_t^{(I)}(s,t'),\tag{2.18}$$

with a cutoff in the momentum transfer variable at u_c. The choice of cutoff u_c, as well as the assumption that the pion-nucleon amplitude $\mathfrak{F}$ is a sum of N and $\Delta(1236)$ pole terms, is discussed in the Appendix.

Then we project out the partial waves:

$$E=(\tfrac{1}{4}s-M^2)^{1/2}=(p^2+M^2)^{1/2},\tag{2.19a}$$

$$z=-(1+u/2p^2),\tag{2.19b}$$

$$f_0{}^J=\frac{p}{2E}\int_{-1}^{1}f_1(s,z)P_J(z)dz,\tag{2.20a}$$

$$f_{11}{}^J=\frac{p}{2E}\int_{-1}^{1}f_2(s,z)P_J(z)dz,\tag{2.20b}$$

$$f_{12}{}^J=\frac{p}{2M}\int_{-1}^{1}f_5(s,z)[P_{J+1}(z)-P_{J-1}(z)]\times dz[J(J+1)]^{1/2}/(2J+1),\tag{2.20c}$$

$$f_{22}{}^J=\frac{p}{2E}\int_{-1}^{1}\left[f_3(s,z)P_J(z)+f_4(s,z)\frac{JP_{J+1}(z)+(J+1)P_{J-1}(z)}{2J+1}\right]dz,\tag{2.20d}$$

$$f_1{}^J=\frac{p}{2E}\int_{-1}^{1}\left[f_4(s,z)P_J(z)+f_3(s,z)\frac{JP_{J+1}(z)+(J+1)P_{J-1}(z)}{2J+1}\right]dz,\tag{2.20e}$$

where the P's are the standard Legendre polynomials. The spin-singlet partial-wave amplitude is $f_0{}^J$, and the spin-triplet uncoupled ($L=J$) partial-wave amplitude is $f_1{}^J$. The three coupled spin-triplet partial-wave amplitudes are

$$f_{L=J-1}=[1/(2J+1)]\{Jf_{11}{}^J+(J+1)f_{22}{}^J+2[J(J+1)]^{1/2}f_{12}{}^J\},\tag{2.21a}$$

$$f_{L=J+1}=[1/(2J+1)]\{(J+1)f_{11}{}^J+Jf_{22}{}^J-2[J(J+1)]^{1/2}f_{12}{}^J\},\tag{2.21b}$$

$$f_{J-1,J+1}=[1/(2J+1)]\{[J(J+1)]^{1/2}\times(f_{11}{}^J-f_{22}{}^J)+f_{12}{}^J\}.\tag{2.21c}$$

The partial-wave amplitudes correspond to partial-wave projections of $(\mathbf{S}-1)/2i$, where $\mathbf{S}$ is the scattering matrix.

We pick a simple unitarization procedure[30] to express the results:

$$\delta_{J,L=J,S=0}=(180/\pi)\ \mathrm{Re}f_0{}^J,\tag{2.22a}$$

$$\delta_{J,L=J,S=1}=(180/\pi)\ \mathrm{Re}f_1{}^J,\tag{2.22b}$$

$$\delta_{J,L=J-1,S=1}=(180/\pi)\ \mathrm{Re}f_{L=J-1},\tag{2.22c}$$

$$\delta_{J,L=J+1,S=1}=(180/\pi)\ \mathrm{Re}f_{L=J+1},\tag{2.22d}$$

$$\epsilon_J=(180/\pi)\ \mathrm{Re}f_{J-1,J+1},\tag{2.22e}$$

where the phase parameters δ and ϵ are in degrees, and Re indicates the real part.

Since the u and t discontinuities of the $NN\to NN$ scattering amplitude are expected to be most accurate for small values of u and t (in fact they are not calculated for u and t greater than u_c), we have a peripheral model. That is, we do not expect correct S-wave predictions from this model, but we do expect the model to improve in accuracy as we go to higher partial waves.

III. RESULTS

A. Graphs

The graphs of Fig. 5 are of the phase parameters in degrees, after geometric unitarization.[30] The input parameters are $\Gamma_\Delta=120$ MeV, $g_{NN\pi}{}^2=15$, $M=939$ MeV/c^2, $\mu=137$ MeV/c^2, and $M_\Delta=1236$ MeV/c^2.

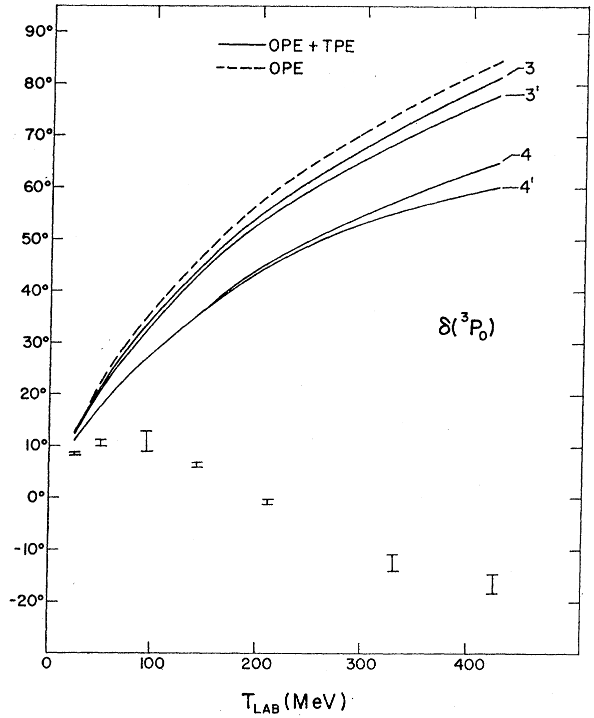
B. Tables

Tables I and II are for the TPE phase parameters in degrees, after geometric unitarization.[30] The input parameters are $\Gamma_\Delta=120$ MeV, $g_{NN\pi}{}^2=15$, $M=939$ MeV/c^2, $\mu=137$ MeV/c^2, $M_\Delta=1236$ MeV/c^2, and cutoff $u_c=(3.5\mu)^2$.

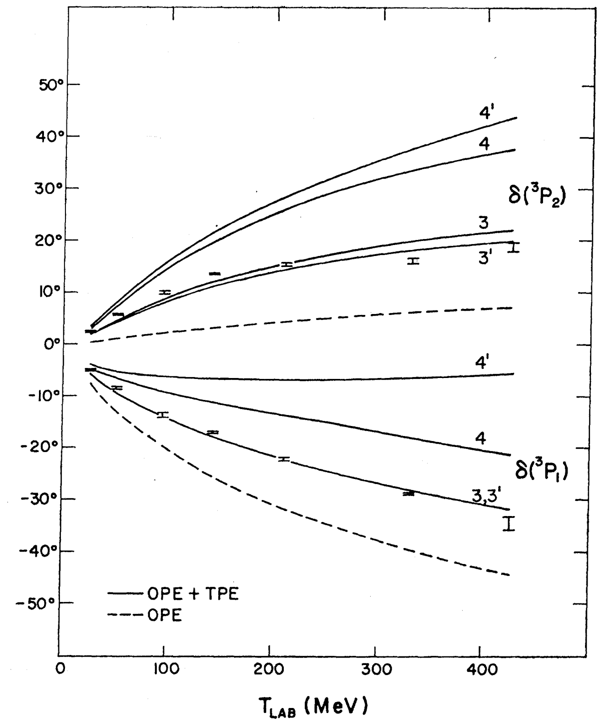
IV. DISCUSSION

In Sec. III A we have shown the graphs for OPE +TPE, with $\Gamma_\Delta=120$ MeV, $g_{NN\pi}{}^2=15$, $M=939$ MeV/c^2, $\mu=137$ MeV/c^2, $u_c=(3\mu)^2$ and $(4\mu)^2$. Note that the curves labeled 3 [N-pole term only in $\pi\pi\to\bar{N}N$, $u_c=(3\mu)^2$] are very similar to the corresponding curves labeled 3′ [both N and $\Delta(1236)$ poles in $\pi\pi\to\bar{N}N$, $u_c=(3\mu)^2$]. The same holds for the graphs labeled 4 and 4′ [$u_c=(4\mu)^2$]. So for practical purposes, we could leave out the $\Delta(1236)$ pole term in the $\pi\pi\to\bar{N}N$ amplitude. Almost all of the TPE contribution to NN scattering (below 425 MeV lab kinetic energy) comes from the N-pole term in the $\pi\pi\to\bar{N}N$ amplitude. This is not true, however, if we take $\Gamma_\Delta=140$ MeV, as do Durso[25] and Furuichi and Watari.[29]

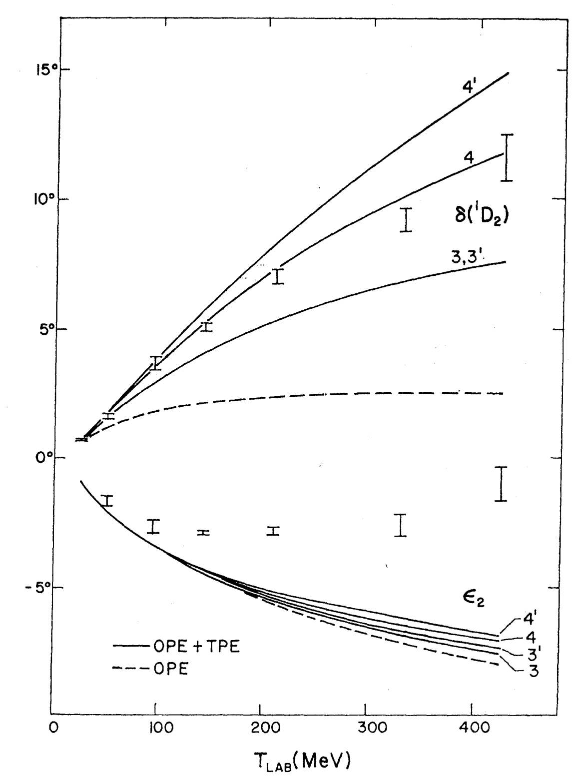
Raising Γ_Δ resembles the addition of an isoscalar scalar boson exchange term to the NN amplitude. There turns out to be considerable cancellation among the terms involving the $\Delta(1236)$ pole, because a small change in Γ_Δ gives rise to a large change in the TPE contribution. While the TPE contribution gives too much attraction for $\Gamma_\Delta=140$ MeV (which corresponds



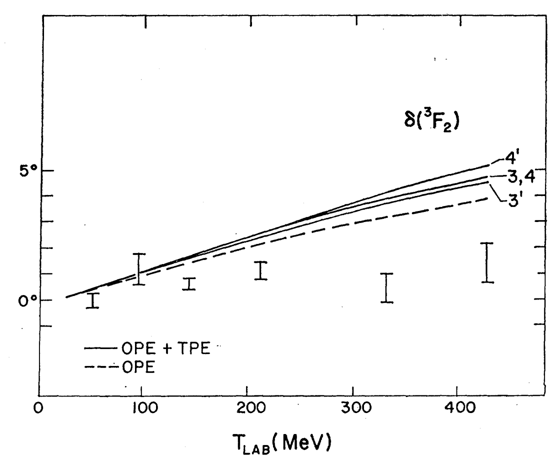
(a)



(b)

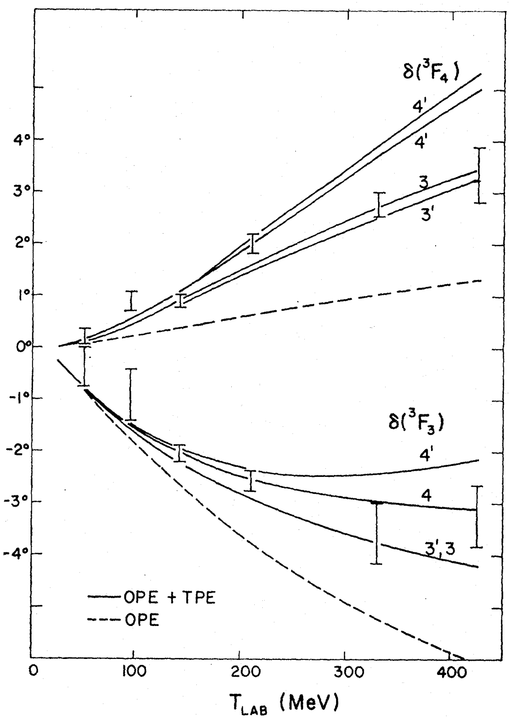


(c)

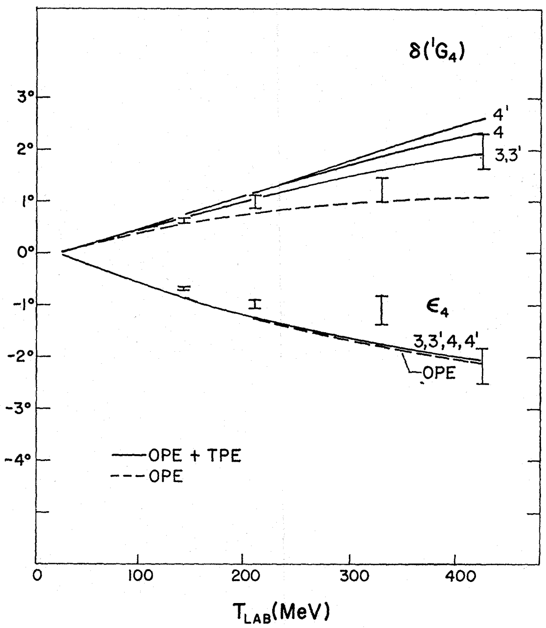


(d)

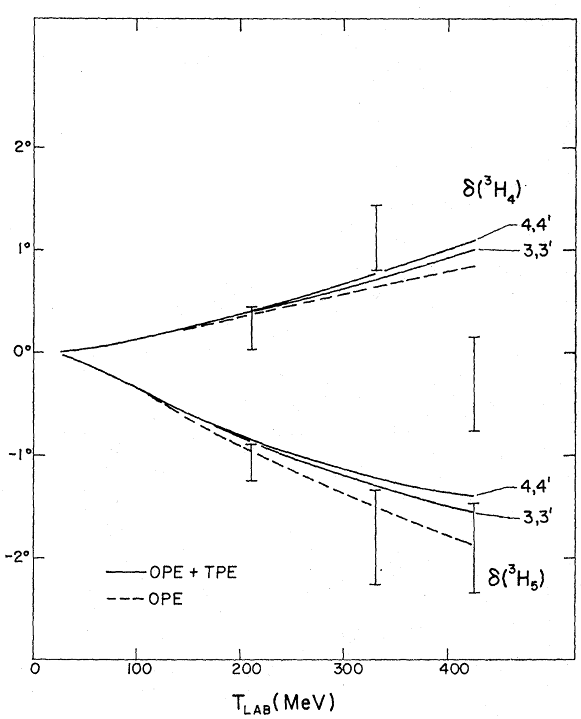
FIG. 5. Graphs of the phase parameters in degrees, after geometric unitarization (see Ref. 30). The labels are identified as follows: 3: N pole only in $\pi\pi \to \bar{N}N$, cutoff $u_c = (3\mu)^2$; 4: N pole only in $\pi\pi \to \bar{N}N$, cutoff $u_c = (4\mu)^2$; 3': N and $\Delta(1236)$ poles in $\pi\pi \to \bar{N}N$, cutoff $u_c = (3\mu)^2$; 4': N and $\Delta(1236)$ poles in $\pi\pi \to \bar{N}N$, cutoff $u_c = (4\mu)^2$. OPE indicates one-pion-exchange. OPE+TPE indicates one-pion-exchange plus two-pion-exchange. I represents error bar for energy-independent solution from (p,p) plus (n,p) data, Table VIII of Ref. 3. Figures 5(a)–5(g), plus $\delta(^3H_6)$ of Fig. 5(h) show the $T=1$ phase parameters; $\delta(^1H_5)$ of Fig. 5(h), plus Figs. 5(i)–5(l) show the $T=0$ phase parameters.



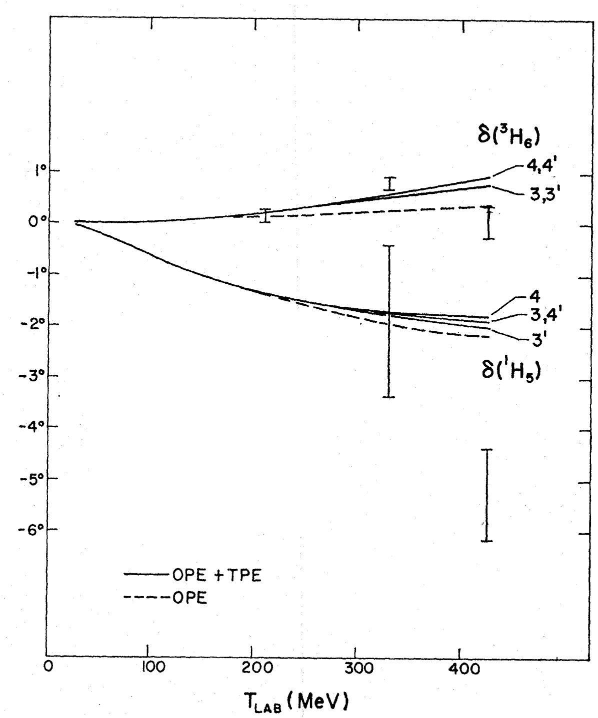
(e)



(f)



(g)



(h)

FIG. 5. (*continued*)

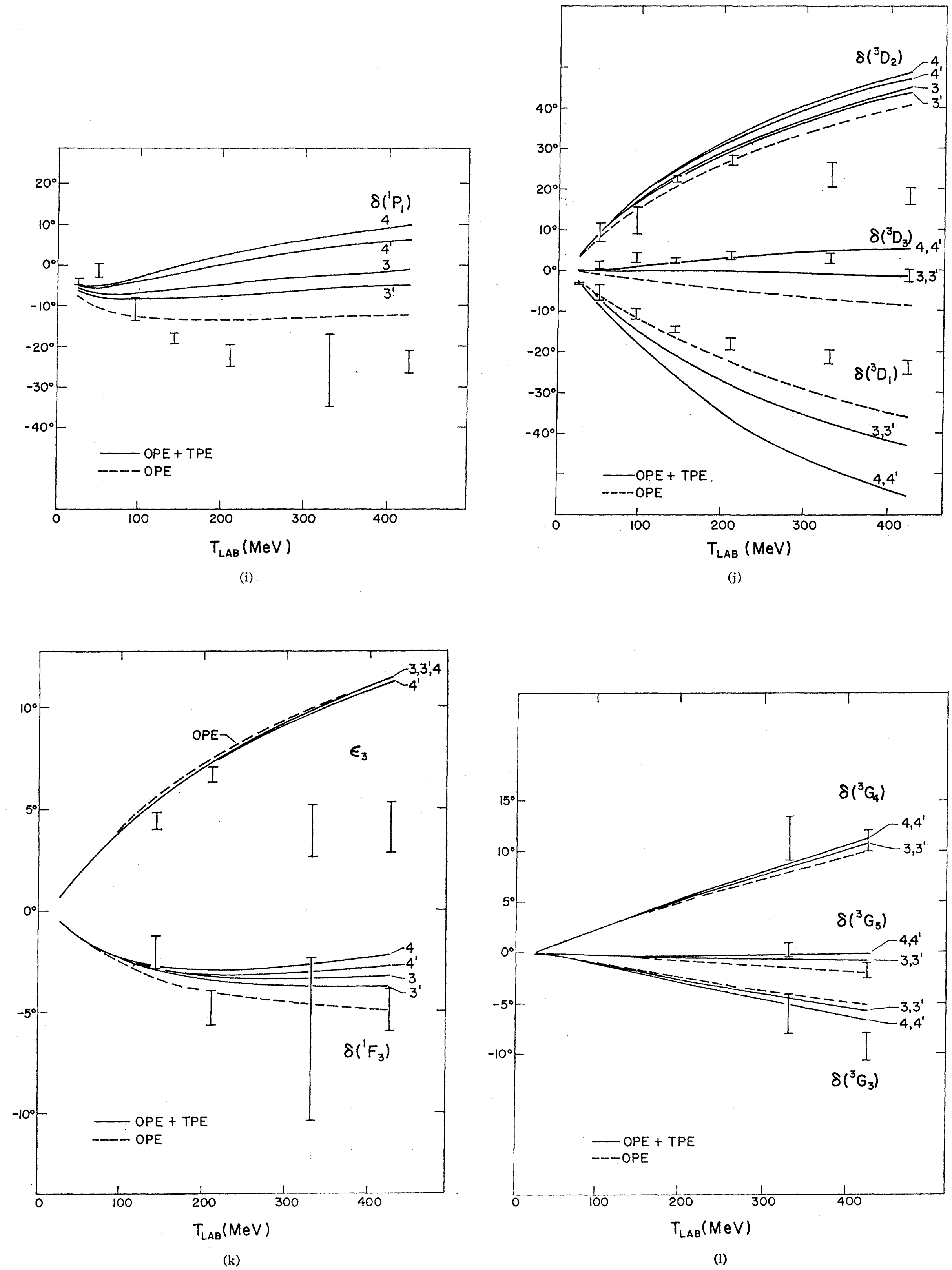


FIG. 5. (*continued*)

TABLE I. TPE phase parameters (in degrees) for NN $T=0$ scattering.

E (MeV)	TPE[a]	TPE[b]	TPE[a]	TPE[b]	TPE[a]	TPE[b]	TPE[a]	TPE[b]
	$^1P_1$		$^3D_1$		$^3D_2$		$^3D_3$	
20	2.084	1.725	0.085	0.076	0.253	0.238	0.161	0.151
50	4.968	3.923	−1.500	−1.558	1.180	1.082	0.957	0.899
100	8.548	6.549	−4.836	−5.002	2.644	2.330	2.799	2.636
150	11.120	8.405	−7.439	−7.708	3.710	3.148	4.619	4.364
200	13.026	9.759	−9.284	−9.639	4.481	3.669	6.222	5.895
300	15.582	11.524	−11.473	−11.947	5.508	4.214	8.755	8.343
400	17.135	12.534	−12.526	−13.064	6.162	4.419	10.586	10.144
	ϵ_3		$^1F_3$		$^3G_3$		$^3G_4$	
20	−0.003	−0.004	0.007	0.006	0.001	0.001	0.001	0.001
50	−0.033	−0.035	0.070	0.057	−0.006	−0.006	0.018	0.016
100	−0.100	−0.110	0.286	0.220	−0.094	−0.102	0.101	0.090
150	−0.142	−0.165	0.580	0.435	−0.250	−0.272	0.227	0.196
200	−0.154	−0.194	0.906	0.669	−0.428	−0.470	0.371	0.311
300	−0.116	−0.195	1.563	1.135	−0.769	−0.857	0.669	0.536
400	−0.040	−0.160	2.169	1.557	−1.406	−1.181	0.953	0.733
	$^3G_5$		ϵ_5		$^1H_5$			
20	0.001	0.001	0.000	0.000	0.000	0.000		
50	0.015	0.014	0.000	0.000	0.001	0.001		
100	0.106	0.098	−0.004	−0.004	0.013	0.010		
150	0.273	0.253	−0.010	−0.011	0.040	0.030		
200	0.490	0.455	−0.015	−0.018	0.081	0.060		
300	0.989	0.920	−0.020	−0.028	0.196	0.142		
400	1.500	1.401	−0.018	−0.032	0.338	0.240		

[a] N pole only in $\pi\pi \to \bar{N}N$.
[b] Both N and $\Delta(1236)$ poles in $\pi\pi \to \bar{N}N$.

TABLE II. TPE phase parameters (in degrees) for NN $T=1$ scattering.

E (MeV)	TPE[a]	TPE[b]	TPE[a]	TPE[b]	TPE[a]	TPE[b]	TPE[a]	TPE[b]
	$^3P_0$		$^3P_1$		$^3P_2$		ϵ_2	
20	0.065	−0.059	2.053	2.115	1.866	1.804	−0.002	0.000
50	−2.536	−2.911	5.421	5.699	5.427	5.275	0.002	0.012
100	−4.637	−5.422	9.462	10.281	10.359	10.138	0.054	0.089
150	−5.743	−6.945	12.165	13.642	14.013	13.781	0.139	0.208
200	−6.584	−8.233	14.064	16.263	16.762	16.542	0.236	0.343
300	−8.145	−10.824	16.472	20.255	20.538	20.370	0.428	0.621
400	−9.739	−13.671	17.838	23.376	22.933	22.815	0.597	0.888
	$^1D_2$		$^3F_2$		$^3F_3$		$^3F_4$	
20	0.095	0.095	0.005	0.005	0.006	0.006	0.005	0.005
50	0.561	0.567	0.030	0.027	0.068	0.066	0.068	0.064
100	1.688	1.737	0.119	0.107	0.308	0.306	0.329	0.310
150	2.872	3.004	0.248	0.230	0.643	0.651	0.713	0.676
200	3.970	4.214	0.393	0.374	1.012	1.045	1.151	1.097
300	5.804	6.335	0.675	0.672	1.744	1.874	2.054	1.976
400	7.206	8.080	0.908	0.942	2.406	2.689	2.903	2.813
	ϵ_4		$^1G_4$		$^3H_4$		$^3H_5$	
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.008	0.008	0.001	0.001	0.001	0.001
100	0.001	0.002	0.063	0.063	0.007	0.006	0.013	0.013
150	0.005	0.008	0.170	0.171	0.022	0.020	0.043	0.042
200	0.012	0.018	0.314	0.320	0.045	0.042	0.090	0.089
300	0.034	0.048	0.662	0.692	0.109	0.105	0.221	0.225
400	0.059	0.086	1.033	1.111	0.185	0.185	0.380	0.400
	$^3H_6$							
20	0.000	0.000						
50	0.001	0.001						
100	0.014	0.013						
150	0.047	0.044						
200	0.101	0.094						
300	0.257	0.242						
400	0.453	0.429						

[a] N pole only in $\pi\pi \to \bar{N}N$.
[b] Both N and $\Delta(1236)$ poles in $\pi\pi \to \bar{N}N$.

to the phase shifts given by Durso[25]), it appears to give the right amount for the physical value of $\Gamma_\Delta=120$ MeV. (Reference 28 gives $\Gamma_\Delta=120\pm2$ MeV.) A glance at the graphs in Sec. III A will show this general attraction.

Two critical phase shifts are $^1D_2$ and $^3D_1$. Since ω and ρ exchange contribute very little to the $^1D_2$ phase shift, it should be given by OPE+TPE (if our model is to be OPE+TPE+ω exchange+ρ exchange). Since OPE is much too small, that leaves only TPE to account for most of the phase shift. The graph for $^1D_2$ does show that the TPE contribution is just about right.

The $^3D_1$ phase shift requires a large negative contribution in a pole model, if π, ρ, and ω exchange are present. It requires an even more negative contribution if an ϵ exchange is present. We note that the graphs show the TPE contribution to be large and negative, as required.

As a check on the results of this paper, we calculated the TPE contribution for $\Gamma_\Delta=140$ MeV, $M=938$ MeV/c^2, $g_{NN\pi}{}^2=14.4$, $\mu=135$ MeV/c^2, $M=1236$ MeV/c^2, and $u_c=(3\mu)^2$, $(4\mu)^2$. The results were expected to be identical to Durso's CGLN TPE contribution[25] for all phase shifts other than $^3P_0$, $^3P_1$, and $^1S_0$. We found agreement on all the phase shifts given in his paper, namely, D waves through H waves.

For $J\geq2$, the TPE contribution is similar to that reported by Barker and Haracz,[32] even though their model is different. Their TPE contribution to $^3D_1$ is also strongly negative. We find our TPE contribution is also similar to that of Wortman,[33] for $J\geq2$.

V. CONCLUSIONS

Adding the $\Delta(1236)$ pole term to the N pole term in the $\pi\pi\to\bar{N}N$ amplitude has little effect on the TPE contribution to $NN\to NN$ scattering. This is for lab kinetic energies below 425 MeV, for a cutoff in momentum transfer $(3\mu)^2\leq u_c\leq(4\mu)^2$, and for $\Gamma_\Delta=120$ MeV.

The TPE contribution cannot be characterized as resembling an isoscalar scalar meson exchange alone, because there is a strong repulsion in the $^3D_1$ partial wave. An $I=0$, $J=0$ meson exchange would contribute positively to all NN phase shifts.

A quick glance over the graphs shows that OPE +TPE is an improvement over OPE alone, for the $T=1$ phase parameters, for $u_c=(3\mu)^2$. OPE+TPE is not an improvement over OPE alone, for the $T=0$ phase parameters. However, we would expect that the TPE contribution would considerably improve an OBE pole model. The general attraction plus the negative $^3D_1$ contribution from TPE is what is lacking in an OBE pole model consisting of π, ρ, and ω exchanges.[34]

[32] B. M. Barker and R. D. Haracz, Phys. Rev. **186**, 1624 (1969).
[33] W. R. Wortman, Phys. Rev. **176**, 1762 (1968).
[34] This was pointed out to me by Professor Ronald A. Bryan.

ACKNOWLEDGMENTS

Thanks are due to Professor Ronald A. Bryan, for enlightenment on the structure of the one-boson-exchange contribution and for constructive criticism, and to Professor John W. Durso for a helpful conversation.

APPENDIX

We have approximated the pion-nucleon amplitude $\mathfrak{F}$ as a sum of N and $\Delta(1236)$ pole terms [Eq. (2.8)]. This approximation and our choice of cutoff u_c in the dispersion integral of Eq. (2.18) are both motivated by a consideration of the kinematics of pion-nucleon scattering. The Mandelstam variables s', t', and u' for pion-nucleon scattering (Fig. 6) are defined as follows:

$\pi N\to\pi N$ s' channel,
$$s'=(p_1'+p_2')^2=(p_3'+p_4')^2\,, \tag{A1a}$$
$\pi\pi\to\bar{N}N$ t' channel,
$$t'=(p_1'-p_3')^2=(p_4'-p_2')^2\,, \tag{A1b}$$
$\pi N\to\pi N$ u' channel,
$$u'=(p_1'-p_4')^2=(p_3'-p_2')^2\,, \tag{A1c}$$
where the 4-momenta $p_1'-p_4'$ are as shown in Fig. 6,
$$p_1'^2=p_3'^2=\mu^2\,,\quad p_2'^2=p_4'^2=M^2\,,$$
and
$$s'+t'+u'=2M^2+2\mu^2\,.$$

The Mandelstam variables s', t', and u' for the pion-nucleon amplitude $\mathfrak{F}(\hat{\imath},\hat{I})$, in terms of previously defined variables [Eqs. (2.2) and (2.5)] are
$$t'=u\,, \tag{A2a}$$
$$s'=-\bar{p}^2-q^2+2\bar{p}q(\hat{\imath}\cdot\hat{I})\,, \tag{A2b}$$
$$u'=-\bar{p}^2-q^2-2\bar{p}q(\hat{\imath}\cdot\hat{I})\,. \tag{A2c}$$
The variables s', t', and u' for $\mathfrak{F}(\hat{f},\hat{I})$ are given by the same equations [Eqs. (A2)], except that $\hat{\imath}$ is replaced by $\hat{f}$.

In the unitarity integral [Eq. (2.9)], u and $\bar{\theta}$ are held fixed, and θ and φ are allowed to vary over the physical values. This corresponds to integration over all angles for the intermediate $\pi\pi$ state. The integration region

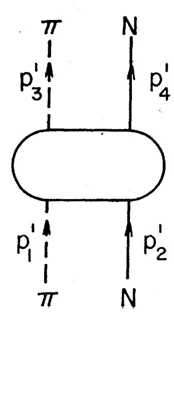


FIG. 6. $\pi N\to\pi N$ kinematics.

(region 1) for $\mathfrak{F}(\hat{\imath},\hat{I})$, in the unitarity integral, is then

$$s'=M^2+\mu^2-\tfrac{1}{2}t'+i(t'-4\mu^2)^{1/2}(M^2-\tfrac{1}{4}t')^{1/2}\cos\theta,\quad 0\leq\theta\leq\pi. \tag{A3}$$

The integration region (region 2) for $\mathfrak{F}^*(\hat{f},\hat{I})$, in the unitarity integral, is

$$s'=M^2+\mu^2-\tfrac{1}{2}t'+i(t'-4\mu^2)^{1/2}(M^2-\tfrac{1}{4}t')^{1/2}\times[\bar{z}\cos\theta+i(\bar{z}^2-1)^{1/2}\sin\theta\cos\varphi],\quad 0\leq\theta\leq\pi,\quad 0\leq\varphi\leq2\pi, \tag{A4}$$

where

$$\bar{z}=2s/(4M^2-t')-1. \tag{A5}$$

Regions 1 and 2, projected onto the real s', t', u' plane, are shown on the Mandelstam plot (Fig. 7), for a choice of $T_{\text{lab}}=100$ MeV and $t'=(3.5\mu)^2$. [T_{lab} is the lab kinetic energy for nucleon-nucleon scattering $=(s-4M^2)/2M$.] Region 1 is a point; region 2 is a straight line.

So for a given T_{lab} and cutoff u_c, the region for which a knowledge of the pion-nucleon scattering amplitude is required is (after projection onto the real s', t', u' plane) bounded by a curve and a line. Call this region 3. In Fig. 7, this region is shaded in, for a choice of $T_{\text{lab}}=100$ MeV and $u_c=(3.5\mu)^2$. In general, the bounding curve is given by

$$[s'-(M^2+\mu^2-\tfrac{1}{2}t')]^2=(t'-4\mu^2)(M^2-\tfrac{1}{4}t')(\bar{z}_c^2-1), \tag{A6a}$$

where

$$\bar{z}_c=(4M^2+4MT_{\text{lab}}+t')/(4M^2-t'), \tag{A6b}$$

and the bounding straight line is given by

$$t'=u_c. \tag{A7}$$

We see, in Fig. 7, that the nearest singularities to the region of integration (region 3) are the nucleon poles in the two $\pi N\to\pi N$ channels. (The lines labeled $s'=M^2$ and $u'=M^2$ correspond to the position of the nucleon poles.) The $\Delta(1236)$ poles ($s'=M_\Delta{}^2$, $u'=M_\Delta{}^2$) are farther away. If nearby singularities do indeed dominate the scattering amplitude, then the approximation of $\pi\pi\to\bar{N}N$ by the nucleon-pole term alone should give a first-order approximation to the u discontinuity

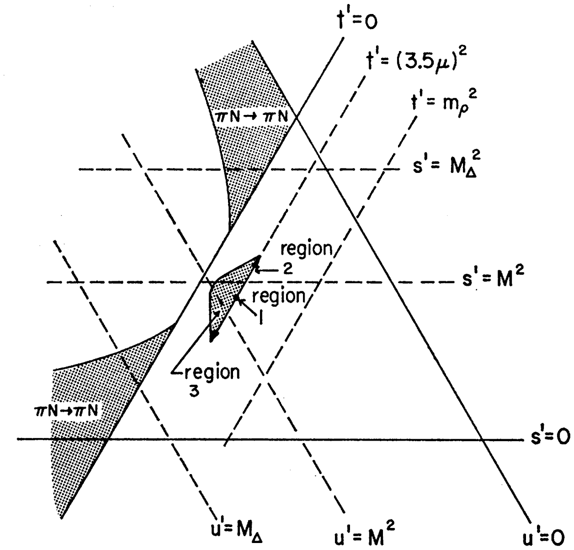


FIG. 7. Mandelstam plot for pion-nucleon scattering amplitude. The region for which a knowledge of the pion-nucleon scattering amplitude is required, in the unitarity integral of (2.9), is (after projection onto the real s', t', u' plane) the shaded region (region 3). Region 3 is drawn for a choice of $T_{\text{lab}}=100$ MeV and $u_c=(3.5\mu)^2$.

of the $NN\to NN$ amplitude, at least up to the 3π intermediate state threshold.

Inclusion of the $\Delta(1236)$ pole in $\pi\pi\to\bar{N}N$ should give a second-order approximation. We include the $\Delta(1236)$ pole, because we note that the πN S-wave scattering lengths are given badly by a model of $\pi N\to\pi N$ consisting of the nucleon-pole term only. A zero-width $\Delta(1236)$ pole (CGLN approximation[24]) in $\pi N\to\pi N$ has an effect on the scattering lengths of the same order of magnitude as the effect of the N-pole term, and tending to cancel it, as required.

Since we have not reproduced the $\rho(765)$ resonance in our approximation for the pion-nucleon scattering amplitude, we suggest a cutoff in t' well below $t'=m_\rho{}^2\approx(5.5\mu)^2$. Furthermore, for $t'>(3\mu)^2$, unitarity would require additional terms in Eq. (2.9), representing three-pion-exchange effects. We conclude that a reasonable value for the cutoff u_c would be $(3\mu)^2\leq u_c\leq(4\mu)^2$.