# Comment on Fujikawa's path-integral derivation of the chiral anomaly

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It is shown that the path-integral derivation of the axial anomaly, when generalized correctly to the case of parity-violating gauge theories, gives the same result as the use of standard gauge-invariant regulators.

In an elegant series of papers[1-3] Fujikawa has argued that anomalous Ward identities can be understood in the path-integral formulation of quantum field theory as a consequence of the fact that the integration measure is not invariant with respect to the relevant transformations on the fields. In this Comment we concentrate on the chiral anomaly in gauge theories with fermions; it is recommended that the reader consult Ref. 2 (referred to henceforward as F) wherein Fujikawa gives a lucid and detailed analysis of this case.

Consider a single massless Dirac fermion $\psi$, with vector and axial-vector current sources $j_\mu^V$ and $j_\mu^A$. If we use a gauge-invariant regularization procedure (i.e., one which maintains vector-current conservation) then the following anomalous Ward identities are readily verifiable at one loop (and extendable to all orders by the Adler-Bardeen theorem[4]):

$$k^\mu \langle j_\mu^A(k) j_\nu^V(p) j_\lambda^V(q) \rangle = \frac{1}{2\pi^2} \epsilon_{\nu\lambda\rho\sigma} p^\rho q^\sigma \quad , \tag{1}$$

$$k^\mu \langle j_\mu^A(k) j_\nu^A(p) j_\lambda^A(q) \rangle = \frac{1}{6\pi^2} \epsilon_{\nu\lambda\rho\sigma} p^\rho q^\sigma \quad , \tag{2}$$

where

$$j_\mu^V(x) = \bar\psi \gamma_\mu \psi, \quad j_\mu^A(x) = \bar\psi \gamma_\mu \gamma_5 \psi \quad .$$

The factor $\frac{1}{3}$ difference between (1) and (2) is readily understood on the basis of the Bose symmetry present in the latter case.[5]

It was pointed out in Ref. 7 that in Sec. IV of F, which deals with parity-violating gauge couplings, a conclusion is reached which is inconsistent with Eqs. (1) and (2). Since this casts doubt on the validity of Fujikawa's analysis, it seemed to us imperative to reconsider the parity-violating case from his point of view. We have done so, and find in fact that a careful application of Fujikawa's method leads to results perfectly consistent with (1) and (2). We consider a theory defined by[8]

$$\mathscr{L} = i\bar\psi \not{D} \psi + j_\alpha^A A^\alpha \quad ,$$

where

$$\not{D} \equiv \gamma^\alpha(\partial_\alpha - \gamma_5 A_\alpha) \quad . \tag{3}$$

For simplicity we take the Abelian case, a massless fermion, and a purely axial-vector gauge coupling to a field $A_\alpha$, which may be regarded as an external source.

Following Fujikawa, we see that under the local chiral transformation

$$\psi(x) \to e^{i\alpha(x)\gamma_5} \psi(x) \quad . \tag{4}$$

The integration measure

$$d\mu = d\psi\, d\bar\psi \tag{5}$$

changes to

$$d\mu \to d\mu \exp\left[-2i \int dx\, \alpha(x) A(x)\right] \quad , \tag{6}$$

where $A(x)$ is essentially the trace of $\gamma_5$ in the function space defined by the eigenfunctions of the operator $D$. Thus

$$A(x) = \sum_n \phi_n(x)^\dagger \gamma_5 \phi_n(x) \quad , \tag{7}$$

where

$$\not{D} \phi_n(x) = \lambda_n \phi_n(x) \tag{8}$$

and

$$\int d^4x\, \phi_n(x)^\dagger \phi_m(x) = \delta_{nm} \quad . \tag{9}$$

(Note that our $\phi_n$ differ from those defined in F.)

$A(x)$ is ill-defined and requires regularization. This is achieved by inserting a factor

$$e^{-\lambda_n^2/M^2} \to e^{-\not{D}^2/M^2} \quad .$$

[In fact any smooth function $f(\not{D}^2/M^2)$ will lead to the same result as long as $f(z)$ and its derivatives approach zero sufficiently rapidly as $z \to \infty$ and $f(0) = 1$. The proof of this given by F is easily generalized to our case.]

So the expression to be evaluated is

$$A(x) = \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 e^{-ikx} e^{-(\not{D}/M)^2} e^{ikx} \quad . \tag{10}$$

The only (but essential) difference between (10) and Eq. (2.15) of F is the presence of the $\gamma_5$ in $\not{D}$, which renders the simplification of (10) more tedious. After some straightforward algebra, one finds

$$A(x) = \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 e^{-k^2/M^2} \exp\left[-\frac{1}{2M^2}\{4ik^\mu\partial_\mu + 2(\partial^\mu + \gamma_5 A^\mu)(\partial_\mu - \gamma_5 A_\mu) - \tfrac{1}{2}\gamma_5[\gamma^\mu,\gamma^\nu](F_{\mu\nu} - 4A_\mu\partial_\nu - 4iA_\mu k_\nu)\}\right] \quad . \tag{11}$$

 

Then expanding the second exponential and performing the Dirac trace in the limit $M \to \infty$ we get (see *Note added*)

$$A(x) = \left(\frac{-1}{16\pi^2}\right)\frac{1}{3}{}^*FF \ , \tag{12}$$

where ${}^*FF = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Comparison with Eq. (2.15) of F reveals the presence of the required $\frac{1}{3}$ factor. It is now simple to obtain the Ward-Takahashi identity (2).

The reader may care to repeat the calculation for the case when $A_\mu$ is a matrix; the result is

$$A^a(x) = -\frac{1}{96\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\lambda_A^a\{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - [A_\mu, A_\nu][A_\alpha, A_\beta]\} \ , \tag{13}$$

where $A^\mu = \sum_a \lambda_A^a A_a^\mu$.

The case when there are both vector and axial-vector sources[9] is more tedious but also straightforward. The appropriate regulator is $e^{-\not{D}^2/M^2}$, where $\not{D} \equiv \gamma^\mu D_\mu$ and $D_\mu = \partial_\mu - V_\mu - A_\mu\gamma_5$. Then the general result may be written as

$$A^a(x) = -\frac{1}{96\pi^2}\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}\lambda_A^a(\bar{D}_\kappa D_\lambda \bar{D}_\mu D_\nu + D_\kappa \bar{D}_\lambda D_\mu D_\nu + D_\kappa D_\lambda \bar{D}_\mu D_\nu) \ , \tag{14}$$

where $\bar{D}_\mu \equiv \partial_\mu - V_\mu + A_\mu\gamma_5$. This formula provides a compact representation of the full set of anomalies associated with the triangle, box, and pentagon diagrams of perturbation theory. Performing the Dirac algebra we obtain

$$A^a(x) = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\lambda_A^a[\tfrac{1}{2}G_{\mu\nu}^V G_{\alpha\beta}^V + \tfrac{1}{6}G_{\mu\nu}^A G_{\alpha\beta}^A + \tfrac{4}{3}(A_\mu A_\nu G_{\alpha\beta}^V + G_{\mu\nu}^V A_\alpha A_\beta + A_\mu G_{\nu\alpha}^V A_\beta) + \tfrac{16}{3}A_\mu A_\nu A_\alpha A_\beta] \ , \tag{15}$$

where

$$G_{\mu\nu}^V = \partial_\mu V_\nu - \partial_\nu V_\mu - [V_\mu, V_\nu] - [A_\mu, A_\nu]$$

and

$$G_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu - [V_\mu, A_\nu] - [A_\mu, V_\nu] \ .$$

Equation (15) leads to the usual expression for the anomalies.[9] [See, for example, Adler's[5] Eq. (178).]

It is of some interest to understand the origin of the difference between our correct calculation and the incorrect result obtained in Sec. IV of F. It can be traced to the regulator $\exp(-\not{D}^2/M^2)$, where we chose $\not{D} = \gamma^\mu(\partial_\mu - V_\mu - A_\mu\gamma_5)$ and Fujikawa chose $\not{D} = \gamma^\mu(\partial_\mu - V_\mu)$. Both are invariant under local vectorial gauge transformations, and neither are invariant under local axial gauge transformations. So the issue *cannot* be decided on the basis of gauge symmetries alone. We believe it rests on the need to define the theory as the limit of an appropriately regulated (i.e., cutoff) theory. Consider, for example, the Pauli-Villars method [see the discussion following (2.31) of F]. One introduces another fermionic field $\chi$ of mass $M$ which couples to the vector and axial-vector fields in precisely the same way as $\psi$, but $\chi$ is assumed to obey Bose statistics. In path-integral language, we assume that the full measure $d\psi\, d\bar{\psi}\, d\chi\, d\bar{\chi}$ is then *invariant* under simultaneous chiral transformations of $\psi$ and $\chi$, analogous to Adler's statement[5] that the regulated theory obeys the naive Ward identity. Performing the chiral transformation and then integrating out the $\chi$ field (as a bosonic variable) produces the factor $A(x)$ again, but regulated by the function $(1+\not{D}^2/M^2)^{-1}$ instead of $\exp(-\not{D}^2/M^2)$. As already described, the result is independent of the form of regulator function employed, but not of the choice of $\not{D}$. This derivation leaves no doubt, however, that the choice we made is necessary. Had the Pauli-Villars field $\chi$ not been taken to couple precisely like $\psi$, then not all graphs in perturbation theory would be rendered finite, i.e., regulated. It would be enlightening to see *directly*, without recourse to perturbation theory, that the definition of the fermionic path integral requires this.

Note that, since no integration over the gauge fields is involved and since the anomaly is finite, there is no need to discuss renormalization for the purposes of this argument.

If an anomalous current is coupled to a dynamical gauge field, it leads in general to a nonrenormalizable theory. This is usually dealt with by arranging cancellation of the anomalies by appropriate choices of fermion representations. In such a theory, one may consider additional currents which are not gauged, as in discussions of $\pi^0$ decay[5] or baryon decay.[10] In the latter case, the requirement that the electroweak theory be renormalizable implies that the resulting anomaly be associated with the (vector) baryon current. The Pauli-Villars method used above is not consistent with this, and appropriate local counterterms must be added or else another class of regulators consistent with baryonic $V$-$A$ symmetry must be devised.[11]

To recapitulate, a careful analysis of the axial anomaly vindicates Fujikawa's methodology and correctly gives the full set of anomalies [Eq. (15)] for an arbitrary gauge theory. This is, to our mind, conceptually simple and yields all anomalies without resort to diagrammatic analysis.

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*Note added.* Since this paper was submitted, we have received a number of other papers dealing with various aspects of this issue. For purposes of comparison and, we hope, clarification, we will add a few remarks. In our Comment, we consider vector and axial-vector currents coupled to external (nondynamical) sources $A_\mu$ and $V_\mu$. We introduce a path-integral regularization method which we show to be equivalent to the addition of fermion fields quantized as bosons but coupled identically to the external sources (Pauli-Villars regularization). The path-integral anomaly calculation must therefore yield the same result as the perturbative evaluation with this method,[12] and we checked this for the relevant ${}^*FF$ term in the anomaly. We failed to note the occurrence of additional "normal parity" contributions to Eq. (15). These can be removed by additional local counterterms. Reference 13 emphasizes this point [see their Eq. (13)] and generalizes the discussion to scalar and tensor sources. It has also come to our attention that a path-integral evaluation of the anomaly using a zeta-function regularization method has been previously performed by Balachandran, Marmo, Nair, and Trahern.[14]

There may remain some confusion over what must be done if the set of fields $A_\mu^a$ are the dynamical vector gauge fields of an Abelian or non-Abelian gauge theory with parity-violating couplings as, for example, in the $SU_2 \otimes U_1$ electroweak theory. As is well known, renormalizability requires that the fermion currents $j_{5\mu}^a = \bar{\psi}\gamma_\mu T_L^a \psi$ coupled to $A_\mu^a$ be conserved. [For simplicity we assume left-handed couplings: $T_L^a = T^a \frac{1}{2}(1-\gamma_5)$, where $T^a$ are the appropriate representations of the group generators.] This is achieved by virtue of cancellations between the contributions of various fermions, as expressed by the condition

$$\mathrm{Tr}\, T^a \{T^b, T^c\} = 0 \ . \tag{A1}$$

The Pauli-Villars method explicitly breaks axial-vector conservation because of the mass term. Consequently, it is not surprising that renormalization using this regularization method requires the addition of local counterterms to ensure axial-vector conservation.

In Ref. 15, Fujikawa addresses the case of dynamical gauge fields $A_\mu^a$ and investigates the anomaly associated with the insertion of some *external* current $j_{5\mu}^R = \bar{\psi}\gamma_\mu\gamma_5 R\psi$, for some matrix $R$. Assuming that the gauge theory has been renormalized to maintain conservation of currents coupled to dynamical fields, then the anomaly to be associated with $j_{5\mu}^R$ is

$$\partial^\mu j_{5\mu}^R = \frac{-1}{32\pi^2} \mathrm{Tr} R \{F_{\mu\nu}(A), {}^*F_{\mu\nu}(A)\} \ , \tag{A2}$$

where $A_\mu = A_\mu^a T^a$, Fujikawa's regularization method is constructed to yield this result by manifestly preserving the existing gauge invariance. It does not provide a regularization method for all the graphs of the gauge theory but is adequate for graphs involving an even number of axial vertices. We believe that his procedure is most simply understood in Pauli-Villars form. Consider the Lagrangian

$$L = -\tfrac{1}{4}\mathrm{Tr} F_{\mu\nu}{}^2 + \bar{\psi}(i\not\partial + e\not{A}^a T_L^a)\psi + \bar{\chi}(i\not\partial + e\not{A}^a T^a + M)\chi \ , \tag{A3}$$

where $\chi$ is the Pauli-Villars fermion. Note that $L$ is *invariant* under the infinitesimal transformation

$$\begin{aligned} \psi &\to \psi + igT_L^a \psi\alpha^a \ , \\ \chi &\to \chi + igT^a\chi\alpha^a \ , \\ A_\mu^a &\to A_\mu^a + \partial_\mu\alpha^a - gf^{abc}\alpha^b A_\mu^c \ , \end{aligned} \tag{A4}$$

because $A_\mu^a$ is taken to couple *vectorially* to the Pauli-Villars field. Of course, the measure $d\mu = D\bar{\psi}\, D\psi\, D\bar{\chi}\, D\chi$ is invariant under this transformation by virtue of Eq. (A1) and the (assumed) validity of the Adler-Bardeen theorem. However, our interest is in the response to another chiral transformation of the form

$$\begin{aligned} \psi &\to e^{i\alpha R\gamma_5}\psi \ , \\ \chi &\to e^{i\alpha R\gamma_5}\chi \ . \end{aligned} \tag{A5}$$

The measure $d\mu$ is invariant under this transformation, but the Lagrangian $L$ is not because of the Pauli-Villars term. Consequently, the current $j_{5\mu}^R$ is not conserved. In the limit $M \to \infty$, its divergence can be shown to be given by Eq. (A2). Thus Fujikawa's method has the technical advantage of providing a regularization for external sources consistent with the dynamical gauge symmetry. If $R$ happens to be proportional to an element of the Lie algebra of the gauge group, it does not yield an incorrect result since the right-hand side of Eq. (A2) then vanishes, because of the renormalizability requirement expressed by Eq. (A1).

Clearly, for calculating anomalous contributions to external currents in parity-violating theories, Fujikawa's procedure is the appropriate (and convenient) one. However, for an analysis of anomalies in general our method is appropriate, since the various graphs (such as $AAA$) are individually regularized without the necessity of Eq. (A1).

With regard to the question of Hermiticity, emphasized in Ref. 15, we remark that, in Minkowski space, the fermion integration leads to a term in the effective action of the form $\pm \mathrm{tr} \ln(\not{k} + \not{V} + \not{A}\gamma_5 + M)$ whose *overall* sign is opposite for Fermi-Dirac and Pauli-Villars fields. This does not effect Hemiticity, and, regardless of whether an axial coupling is present, remains real under continuation to Euclidean space. Euclidean positivity is another matter, regardless of the presence or absence of $\gamma_5$ in the coupling of the Pauli-Villars regulator.

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