Glories—and other degenerate points of the action

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We present a path-integral computation of the cross section at, and near, glory angles. We treat glory as an example of a generalized WKB method for solving problems where the critical points of the action are degenerate. We then classify the degeneracies of the rainbow, glory, and conservation-law type.

I. INTRODUCTION

WKB approximations can be obtained by making a stationary-phase approximation of the path-integral representation of the wave function. The phase is the action; its critical points are the solutions of the corresponding classical problem. The standard WKB approximations break down if[1] (i) the integrand is not of compact support and (ii) the critical points are degenerate.

Recently Schulman[2] has solved exactly a problem (the "knife-edge problem") where the integrand is not of compact support, and has computed the boundary contributions when the critical point falls outside the range of integration. In this paper we generalize the standard WKB approximation to cases where the critical points are degenerate. We have already come across this situation in the study of rainbows[3] and in the study of conservation laws in path integration.[4] In this paper we analyze another case of degenerate critical points: glory scattering. Each of these three cases corresponds to a different type of degeneracy and can serve as an example for a classification of degenerate critical points.

We discuss first WKB cross sections in general because we want to compute glory scattering from the generalized WKB approximation. We then compute WKB cross sections at and near glory angles. The result is an explicit formula (33) in terms of the initial momentum p_a, the impact parameter B, the scattering angle θ, and the derivative $dB/d\theta$. In conclusion, we classify degenerate critical points and display glories, rainbows, and conservation laws as examples of different degeneracies.

II. CLASSICAL CROSS SECTIONS AND WKB CROSS SECTIONS

A. Summary

The classical cross section for the scattering of a flow of particles by an axial-symmetric potential is

$$\sigma_{\rm cl}(\Omega)d\Omega=(B\,dB/\sin\theta\,d\theta)d\Omega\ , \tag{1}$$

and *under some conditions* the WKB cross section is equal to the classical one:

$$\sigma_{\rm WKB}(\Omega)=\sigma_{\rm cl}(\Omega)\ . \tag{2}$$

The classical cross section is infinite for

(i) $\theta=0$, $\theta=\pi$, i.e., forward and backward glories;

(ii) $d\theta/db=0$, i.e., rainbows.

However, the semiclassical approximation, whether generalized WKB (glories) or post WKB (rainbows), is finite.

B. The set up

The incoming particles have momentum p_a parallel to the axis of symmetry of the potential. The surface of impact is the annulus centered at the axis of symmetry of area $2\pi B\,dB$. The exit area at a distance R from the scatterer is $R^2d\Omega=2\pi R^2\sin\theta\,d\theta$. The potential is assumed to be of compact support, the incoming and outgoing particles propagating freely in the distant past $t\leq t_a$ and in the distant future $t\geq t_b$. We shall use both Cartesian coordinates (x^α) and cylindrical coordinates (z,ρ,φ) for a point $x\in\mathbb{R}^3$, $x^1=\rho\cos\varphi$, $x^2=\rho\sin\varphi$, $x^3=z$.

C. The classical (or geometrical) cross section

The classical (or geometrical) cross section is, by definition, the ratio of the number of particles hitting the area $R^2d\Omega$ per unit time to the number of incident particles per unit time per unit area (see Fig. 1).[5]

D. The WKB cross section

The WKB cross section is obtained either from the WKB approximation of the wave function or from the WKB approximation of the probability amplitude for a momentum-to-momentum transition.

1. *Cross section from wave function*

Consider an initial wave function, at time $t=t_a$,

$$\phi(x)=\exp[iS_0(x)/\hbar]T(x)\ , \tag{3}$$

where S_0 and T are arbitrary well-behaved functions on the configuration space of the system. Consider the flow

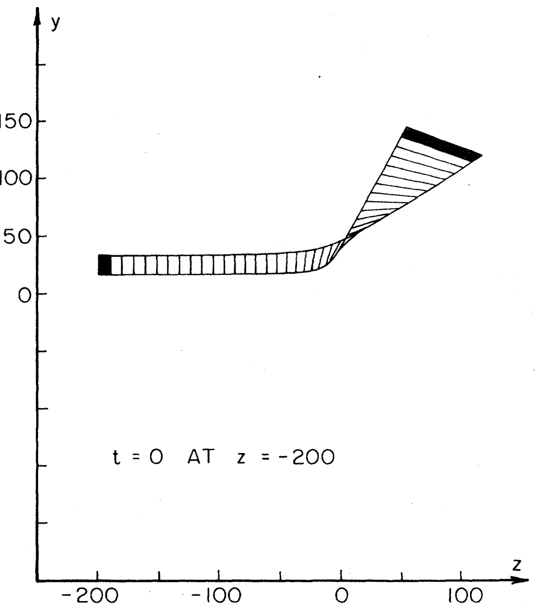

FIG. 1. Consider the flow of Coulomb paths $q(t,B)$ (Figs. 1 and 3) parametrized so that $t=0$ and $z=-200$ for all values of the impact parameter B. Let $h(t,B)$ be the Jacobi field along $q(t,B)$. The particles which, at time $t_a=0$, hit the line element dB defined by the segment of $h(0,B)$ beginning at $q(0,B)$ and ending at $q(0,B+dB)$ hit at time t_b the segment of $h(t_b,B)$ limited by the same classical paths. The integral curve $h(0,B)$ of the Jacobi flow is approximately perpendicular to $q(t,B)$ at $q(0,B)$ because at $z=-200$ the paths are approximately parallel. However, $h(t_b,B)$ is not perpendicular to $q(t,B)$ at $q(t_b,B)$. Thus, all the particles hitting dB at t_a do not hit at the same time the exit area which in the definition of the cross section is assumed to be perpendicular to the classical trajectories. But if we consider the number of particles within the surface of area $v\,dt\,dB$ at time t_a, it is the same as the number of particles with a surface of area $v\,dt\,R d\theta$ at time t_b regardless of how slanted the Jacobi fields are at time t_b, and the definition of cross section makes good sense. (Computer graphics by Alice Young.)

of classical paths

$$\{q(t,a,p_a):a\in N\subset \mathbb{R}^3\} \tag{4}$$

defined by their initial positions and initial momenta:

$$q(t_a,a,p_a)=a,\;\; p_a=\nabla S_0(a)\;. \tag{5}$$

In general, this flow generates a local group of transformations:

$$\begin{aligned}&\{\Phi_{t-t_a}:t\in T\equiv[t_a,t_b]\}\;,\\ &\Phi_{t-t_a}:N\to \mathbb{R}^3 \;\text{ by } a\to q(t,a,p_a)\;.\end{aligned} \tag{6}$$

Given $x\in\mathbb{R}^3$, there may be a unique path $\bar{q}$ in the flow such that $\bar{q}(t)=x$, i.e., there may be a path $\bar{q}$ such that

$$\bar{q}(s)=\Phi_{s-t_a}\circ\Phi^{-1}_{t-t_a}(x),\;\; s\in T\;. \tag{7}$$

Condition (a). The "inverse flow" $\Phi^{-1}_{t-t_a}$ is defined at x (see Fig. 2). If condition (a) is satisfied, the WKB approximation of the wave function is

$$\begin{aligned}\psi_{\mathrm{WKB}}(x,t)=&\,|\det_{\alpha\beta}K^{\alpha}_{(\beta)}(t,t_a)|^{1/2}\\ &\times\exp[iS(x,t)/\hbar]T(\Phi^{-1}_{t-t_a}(x))\;,\end{aligned} \tag{8}$$

where $S(x,t)$ is the solution of the Hamilton-Jacobi equation with Cauchy data S_0:

$$S(x,t)=S_0(\bar{q}(t_a))+\int_{t_a}^{t}L(\bar{q}(s),\dot{\bar{q}}(s))ds\;, \tag{9}$$

and where the Van Vleck determinant is constructed from the n Jacobi fields $\{K_{(\beta)}(s,t_a);\beta=1,\ldots,n\}$ along $\bar{q}$ defined by the classical flow $\{q(t,a,p_a)\}_a$:

$$K^{\alpha}_{(\beta)}(t,t_a)=\partial q^{\alpha}(t,a,p_a)/\partial a^{\beta}\,|_{a=\phi^{-1}_{t-t_a}(x)}\;. \tag{10}$$

Remark. If the initial wave function (3) is a plane wave of momentum p_a, the wave function $\psi(x,t)$ is the probability amplitude for the momentum-to-position transition $\mathcal{K}(x,t;p_a,t_a)$ which we shall abbreviate to $\mathcal{K}(x;p_a)$. In this case, the wave function ψ is equal to $\mathcal{K}$:

$$\psi(x,t)\equiv\mathcal{K}(x;p_a) \tag{11}$$

for ϕ a plane wave of momentum p_a.

For $a=\Phi^{-1}_{t-t_a}(x)$, assumed to be unique, the WKB approximation of the current vector density of a particle of mass m is

$$j_{\mathrm{WKB}}(x,t)=\frac{1}{m}\nabla_x(S(x,t))\,|\det_{\alpha\beta}K^{\alpha}_{(\beta)}(t,t_a)|^{-1}\,|T(a)|^2$$

and the WKB cross section is, for t_b in the distant future and t_a in the distant past,

$$\sigma_{\mathrm{WKB}}(\Omega)d\Omega=\frac{|j_{\mathrm{WKB}}(x(R,\Omega),t_b)|^2}{|j(a,t_a)|^2}R^2d\Omega\;, \tag{12}$$

$$\begin{aligned}\sigma_{\mathrm{WKB}}(\Omega)=&\,|\det_{\alpha\beta}K^{\alpha}_{(\beta)}(t_b,t_a)|^{-1}\\ &\times\frac{|\nabla_x S(x(R,\Omega)t_b)|}{|\nabla_a S_0(a)|}\;.\end{aligned} \tag{13}$$

Let $\tau(q(t,a,p_a))$ be the image of the volume element $\tau(a)$ under the transformation Φ_{t-t_a}, then by (10)

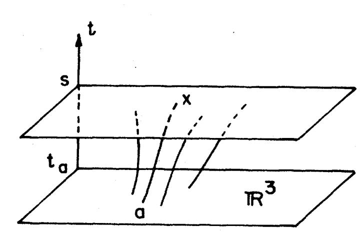

FIG. 2. A classical flow. The path $\bar{q}$ in this flow which is such that $\bar{q}(s)=x$ is given by $\bar{q}(s)=\Phi_{s-t_a}\circ\Phi^{-1}_{t-t_a}(x)$ and $\Phi^{-1}_{t-t_a}(x)=a$.

$$|\det_{\alpha\beta} K^{\alpha}_{(\beta)}(t_b,t_a)|^{-1}=\tau(a)/\tau(\bar{q}(t,a,p_a)) \tag{14}$$

and

$$\sigma_{\rm WKB}(\Omega)=R^2\times(\text{area of impact})/(\text{area of exit})\ . \tag{15}$$

In this case, $\sigma_{\rm WKB}=\sigma_{\rm cl}$. It is clear that this result depends on the validity of (8), hence on condition (a). If in the given flow there is more than one classical path going through x at time t, $\psi_{\rm WKB}(x,t)$ is a sum of terms analogous to the right-hand side of (8). If the several classical paths defined by (x,t) are "sufficiently distinct" then $\sigma_{\rm WKB}$ is equal to $\sigma_{\rm cl}$. However, this is not so if wave packets propagating along these classical paths interfere with each other. Such quantum interferences "soften up" the classical cross sections.

We can distinguish two different situations:

(i) The Euler-Lagrange equation together with the boundary conditions $(p_a=\nabla_a S_0(a),t_a)$ and (x,t) has a discrete set of roots (i.e., of solutions), say two for simplicity. As x approaches a point x_0 on the caustic (i.e., the envelope of the classical flow) the two roots coalesce into a double root and the quantum interferences begin to be felt at the WKB level. The roots of the Euler-Lagrange equation with the above boundary conditions are the critical points of the action defined on the space of arbitrary continuous paths, all of which have the same boundary conditions. If the root is a multiple root the critical point is degenerate; the Jacobi operator has null eigenvectors. However, in general the contribution of the third variation produces an Airy function which makes the wave function finite at the caustic. Rainbows are examples of such a situation.

(ii) The Euler-Lagrange equation together with a set of boundary conditions has a continuous one-parameter set of roots. For example, at backward glory scattering, the boundary conditions $\{(p_a,t_a),(-p_a,t_b)\}$ determine a one-parameter family of classical paths. Indeed the particles with impact parameter B, such that

$$\theta(B)=\pi\ ,$$

exit at $(x^1=B\cos\varphi,\ x^2=B\sin\varphi,\ x^3=z)$ for any x sufficiently far from the scatterer. Thus, the boundary conditions $\{(p_a,t_a,(-p_a,t_b)\}$ define a family of classical paths parametrized by φ. It is the existence of this continuous one-parameter family of classical paths, or rather the interference of the wave packets along these paths, which makes $\sigma_{\rm WKB}$ (at glory) finite. On the other hand, the boundary conditions $\{(p_a,t_a),(x,t_b)\}$ define only one classical path. Hence, to compute the interference effects which make σ_{WKB} (at glory) finite we have to compute the cross section from the momentum-to-momentum transition amplitude.

2. *Cross section from momentum-to-momentum amplitude*

Let $\mathscr{K}(p_b,t_b;p_a,t_a)\equiv\mathscr{K}(p_b;p_a)$ be the probability amplitude that the particle with momentum p_a at t_a be found with momentum p_b at t_b. The cross section $\sigma(\Omega)$ is the transition probability per unit time per unit incident flux into the direction Ω. Hence,[6] we divide $|\mathscr{K}(p_b;p_a)|^2$ by (t_b-t_a) and by $|p_a|/m$, and sum $|\mathscr{K}(p_b;p_a)|^2$ over all states having direction $\hat{p}$. Since the sum over all p states is the integral over p with respect to the measure $d^3p/(2\pi\hbar)^3$ we obtain

$$\sigma(\Omega)d\Omega=\frac{d\Omega}{t_b-t_a}\frac{m}{|p_a|}\int_0^\infty\frac{|p_b|^2}{(2\pi\hbar)^3}dp_b\,|\mathscr{K}(p_b;p_a)|^2\ . \tag{16}$$

We shall check that *away from glories and rainbows* the WKB approximation of (16) is equal to the geometrical cross section (1): It has been shown[4,7] that

$$\mathscr{K}_{\rm WKB}(p_b;p_a)=2\pi\hbar\delta(|p_b|-|p_a|)\hat{\mathscr{K}}_{\rm WKB}(p_b;p_a)=2\pi\hbar\frac{|p_b|}{m}\delta(E_b-E_a)\hat{\mathscr{K}}_{\rm WKB}(p_b;p_a) \tag{17}$$

with

$$\hat{\mathscr{K}}_{\rm WKB}(p_b;p_a)=2\pi i\hbar\exp\left[\frac{i}{\hbar}S(p_b,t_b;p_a,t_a)\right]\frac{1}{|p_b|}\left[\frac{B\,dB}{\sin\theta\,d\theta}\right]^{1/2}\ . \tag{18}$$

As usual

$$[2\pi\hbar\delta(E_b-E_a)]^2=\lim_{t_b,t_a\to\infty}2\pi\hbar\delta(E_b-E_a)(t_b-t_a)=\lim_{t_b,t_a\to\infty}2\pi\hbar\frac{m}{|p_b|}\delta(|p_b|-|p_a|)(t_b-t_a) \tag{19}$$

and (16)—(19) give

$$\sigma(\Omega)=B\,dB/\sin\theta\,d\theta\ . \tag{20}$$

III. WKB CROSS SECTION FOR GLORY SCATTERING

A. At glory angle

We must reexamine the calculation of $\mathscr{K}_{\rm WKB}(p_b;p_a)$ for glory scattering, say backward glory to be specific. In Ref. 4 we computed

$$\mathscr{K}_{\rm WKB}(p_b,t_b;p_a,t_a)=\text{s.p.a.}\int_M\mathscr{K}_{\rm WKB}(p_bt_b;x,t)\mathscr{K}_{\rm WKB}(x,t;p_a,t_a)dx \tag{21}$$

(where s.p.a. stands for stationary-phase approximation, and $t_a \le t \le t_b$ is arbitrary) and we analyzed the stationary-phase method when the set of critical points form a submanifold in R^3. Such a situation occurs, for instance, when the initial and final momenta of the corresponding classical system are constrained by conservation laws. A similar situation occurs at glory scattering but with an additional feature. Whereas in the case of conservation laws the action function depends linearly on the variable conjugate to the conserved quantity, here the action function does not depend on it, hence the need to reexamine the calculation of $\mathscr{K}_{\mathrm{WKB}}(p_b;p_a)$. For simplicity we take the configuration space M to be R^3 and we choose the arbitrary value of t in the integral equal to t_b. Then

$$\mathscr{K}_{\mathrm{WKB}}(p_b,t_b;x,t_b)=\exp\left[-\frac{i}{\hbar}p_{b\alpha}x^\alpha\right]$$

and we can write

$$\mathscr{K}_{\mathrm{WKB}}(p_b,t_b;p_a,t_a)=\text{s.p.a.}\int_{\mathsf{R}^3}\exp\left[\frac{i}{\hbar}f(x)\right]g(x)dx\ , \tag{22}$$

where, by (11), (8), and (14),

$$f(x)=-p_{b\alpha}x^\alpha+S(x,t_b;p_a,t_a)\ ,$$

$$g(x)=|\tau(x)/\tau(a)|^{1/2}$$

and where $x=\bar{q}(t_b,a,p_a)$ and $a=\bar{q}(t_a,a,p_a)$ for $\bar{q}$ defined by (7).

Basically, the method developed in Ref. 4 consists of making a change of variable which diagonalizes the Hessian $\partial^2 f(x)/\partial x^\beta$. One can expect that cylindrical coordinates (z,ρ,φ) will diagonalize the Hessian of f [i.e., the Hessian of $S(x,t_b;p_a,t_a)$] at glory scattering. Let

$$f(x^\alpha(x,\rho,\varphi))\equiv\hat{f}(z,\rho,\varphi)$$

and similarly for all careted functions. Let (p_z,p_ρ,p_φ) be the corresponding momenta:

$$p_1=p_\rho\cos\varphi-p_\varphi\sin\varphi/\rho\ ,$$

$$p_2=p_\rho\sin\varphi+p_\varphi\cos\varphi/\rho\ ,$$

$$p_3=p_z\ .$$

Then

$$\hat{f}(z,\rho,\varphi)=-zp_{bz}-\rho P_{b\rho}+\hat{S}(z,\rho,t_b;p_a,t_a)\ .$$

Note that $\hat{f}(z,\rho,\varphi)$ is independent of φ. This is the mathematical feature which distinguishes glories from conservation laws.[4] The critical points $\{x_0\}$ of f are the points whose coordinates solve the set of equations:

$$0=\partial\hat{f}/\partial z_0=-p_{bz}+p_z(t_b)\ , \tag{23a}$$

$$0=\partial\hat{f}/\partial\rho_0=-p_{b\rho}+p_\rho(t_b)\ , \tag{23b}$$

$$0=\partial\hat{f}/\partial\varphi_0\ , \tag{23c}$$

where $p(t)$ is the momentum along the classical path with boundary conditions $\{(p_a,t_a),(x_0,t_b)\}$.

At backward glory $p_b=-p_a$, and any point x_0 with coordinates

$$z_0 \text{ negative and far from the scatter assumed be at } z=0\ , \tag{24a}$$

$$\rho_0=B, \text{ the glory parameter defined by } \theta(B)=\pi\ , \tag{24b}$$

$$\varphi_0 \text{ arbitrary} \tag{24c}$$

is a critical point. It is the arbitrariness of z_0 and φ_0 which requires a more careful handling of the stationary-phase method than when the critical points are isolated zeros of the action functional $S(q)=\int_{t_a}^{t_b}L(q(s),\dot{q}(s))ds$.

We compute first the second derivatives of $\hat{S}$ and check that the proposed change of coordinates has diagonalized the Hessian of $\hat{S}$:

$$\frac{\partial^2\hat{S}}{\partial z^2}=\frac{\partial p_z(t_b)}{\partial z}=0$$

for any far away z_0. Note that all the higher derivatives of $\hat{S}$ with respect to z also vanish at any far away z_0:

$$\frac{\partial^2\hat{S}}{\partial\rho^2}=\frac{\partial p_\rho(t_b)}{\partial\rho}\ .$$

Since $p_\rho=p_1\cos\varphi+p_2\sin\varphi$ and φ is constant, we can set $p_\rho=p_1$; at large distance $p_1=|p|\sin\theta$ and close to glory angle $\sin\theta=\rho/R$; hence, $\partial^2\hat{S}/\partial\rho^2$ (at glory) $=|p|/R$. All derivatives of $\hat{S}$ with respect to φ vanish. All mixed derivatives vanish.

Finally the expansion of $\hat{f}(z,\rho,\varphi)$ around (z_0,B,φ_0) is

$$\hat{f}(z,\rho,\varphi)=-z_0p_{bz}-Bp_{bz}+\hat{S}(z_0,B,t_b;p_a,t_a)+\tfrac{1}{2}(\rho-B)^2\frac{|p_a|}{R}+O((\rho-B)^3)\ . \tag{25}$$

The following is a heuristic argument justified by a previous calculation[4] which, in particular, avoids the need of having to justify the limits of integration of z_0.

For t_b sufficiently large,

$$\hat{S}(z_0,B,t_b;p_a,t_a)=z_0p_z(t_b)+\hat{\hat{S}}(B,t_b;p_a,t_a)\ . \tag{26}$$

Moreover, at large distance and close to the glory angle (see Fig. 3)

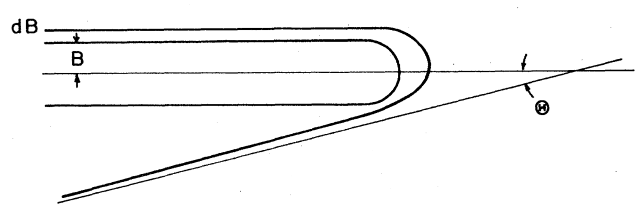

FIG. 3. At glory impact parameter B, particles in the volume element $2\pi B\,dB\,v\delta t$ are scattered in the volume element $2\pi BR\,d\theta\,v\delta t$.

$$\hat{g}(z_0,B,\varphi)=|(\text{volume element at exit})/(\text{volume element at entry})|^{-1/2}\,,$$

$$\hat{g}(z_0,B,\varphi)=\left|\frac{2\pi BR\,d\theta}{2\pi B\,dB}\right|^{-1/2}=\left|\frac{R\,d\theta}{dB}\right|^{-1/2}. \tag{27}$$

Finally (22) and (25)—(27) give for glory scattering (where $p_a=-p_b$)

$$\begin{aligned}\mathscr{K}(-p_a,t_b;p_a,t_a)&=\exp\left[\frac{i}{\hbar}[Bp_{ap}+\hat{S}(b,t_b;p_a,t_a)]\right]\left|\frac{dB}{R\,d\theta}\right|^{1/2}\\&\quad\times\int_{-\infty}^{+\infty}dz_0\exp\left[\frac{i}{\hbar}z_0[p_{az}+p_z(t_b)]\right]\int_0^{\infty}\exp\left[\frac{i}{2\hbar}(\rho-B)^2\frac{|p_a|}{R}\right]\rho\,d\rho\int_0^{2\pi}d\varphi_0\\&=\exp\left[\frac{i}{\hbar}S(-p_{ap},t_b;p_{ap}t_a)\right]\left|\frac{dB}{R\,d\theta}\right|^{1/2}2\pi\hbar\delta(p_{az}+p_z(t_b))(2\pi i\hbar R/|p_a|)^{1/2}B2\pi\\&=\exp\left[\frac{i}{\hbar}S(-p_{ap},t_b;p_{ap},t_a)\right]2\pi\hbar\left|\frac{p_a}{m}\right|\delta(E_b-E_a)\left|\frac{2\pi i\hbar B^2\,dB}{|p_a|\,d\theta}\right|^{1/2}2\pi\,.\end{aligned} \tag{28}$$

We can now complete the calculation of $\sigma_{\rm WKB}$ (at glory). According to (16)

$$\sigma_{WKB}(\Omega)=\frac{2\pi}{\hbar}|p_a|B^2\frac{dB}{d\theta}\,, \tag{29}$$

which is finite as long as $\hbar\neq0$. This expression is to be compared to the geometrical cross section

$$\begin{aligned}\sigma_{\rm cl}(\Omega)&=\lim_{R\to\infty}R^2\frac{2\pi B\,dB}{2\pi R^2\sin\theta\,d\theta}\\&=\lim_{R\to\infty}R\,dB/d\theta\,.\end{aligned} \tag{30}$$

The key element of the generalized WKB method is the integration over z_0 and φ_0 [satisfying conditions (23a) and (23c)], i.e., the integral over z and φ is approximated by an integral over z_0 and φ_0.

B. Near glory angle

Near the glory impact parameter B, there are, for each value of φ two paths which exit at the same scattering angle θ, one with impact parameter $B+\delta B$ deflecting a little less than a glory path, and one with impact parameter $-B+\delta B$ deflecting a little more than a glory path. For this discussion it is convenient to give an algebraic value to the impact parameter (see Fig. 4).

As θ approaches π these two paths coalesce geometrically, but not dynamically, since one hits the impact area where the other exits. Nevertheless, the wave packets propagating along these two paths interfere already at the WKB approximation. We shall compute $\mathscr{K}_{\rm WKB}(p_b;p_a)$ for p_b near glory. Let

$$p_b=-p_a-\Delta\,,$$

where

$$\Delta\equiv p_b\sin\theta\simeq p_b\rho/R\,.$$

We shall expand

$$f(x)=-p_{b\alpha}x^\alpha+S(x,t_b;p_a,t_a)\,,$$

not around its critical points x_0 such that $f'(x_0)=0$, but around the glory critical points x_g:

$$\begin{aligned}&f_g(x)=p_{a\alpha}x^\alpha+S(x,t_b;p_a,t_a)\,,\quad f_g'(x_g)=0\,,\\&f(x)=f_g(x)+\Delta_\alpha x^\alpha\\&\quad=f_g(x_g)+\Delta_\alpha x_g^\alpha+\frac{1}{2}\frac{\partial^2S}{\partial x_g^\alpha\partial x_g^\beta}(x-x_g)^\alpha(x-x_g)^\beta\\&\qquad+O((x-x_g)^3)+\Delta_\alpha(x-x_g)^\alpha\,.\end{aligned} \tag{31}$$

In cylindrical coordinates, $\{x_g^\alpha\}=\{B\cos\varphi,B\sin\varphi,z_g\}$ [whereas $f_g(x)$ is independent of φ], and the new term $\Delta_\alpha x_g^\alpha$ is not independent of φ. Indeed we can choose the Cartesian coordinate of Δ to be $(0,\Delta,0)$, and then

$$\Delta_\alpha x_g^\alpha=|p_b|\sin\theta\,B\cos\varphi\,. \tag{32}$$

The calculation of $\mathscr{K}_{\rm WKB}(-p_a-\Delta;p_a)$ proceeds as the calculation of $\mathscr{K}_{\rm WKB}(-p_a;p_a)$ but $\int_0^{2\pi}d\varphi$ is replaced by

$$\int_0^{2\pi}\exp\left[\frac{i}{\hbar}p_b\sin\theta\,B\cos\varphi\right]d\varphi=2\pi J_0\left[\frac{|p_b|}{\hbar}B\sin\theta\right]$$

and

$$\sigma_{\rm WKB}(\text{near glory})=\frac{2\pi}{\hbar}|p_a|B^2\frac{dB}{d\theta}J_0{}^2\left[\frac{|p_b|}{\hbar}B\sin\theta\right]. \tag{33}$$

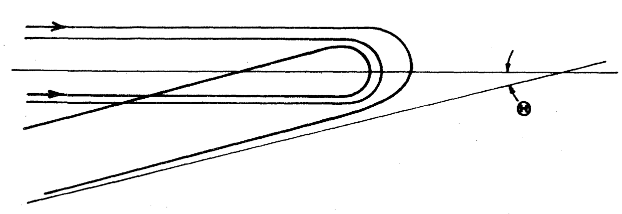

FIG. 4. Near glory particles with impact parameter $B+\delta B$ and $-B+\delta B$ exit with approximately the same angles, namely, π, by terms of order $(\delta\theta)^3$.

This formula is also valid at glory angle.

Formula (33) is identical with the formula derived by Ford and Wheeler[8] via partial wave decomposition. Arriving at the same result by two entirely different methods strengthens both methods: It justifies the several unrelated approximations made when using partial wave decomposition, and it shows that path integration is a practical tool for scattering theory. In addition to being a more direct approach requiring only one expansion in power of $\sqrt{\hbar}$, the path-integration method underscores and clarifies the roles of the *degenerate* critical points of the action.

IV. CONCLUSION

The action functional $\mathscr{S}$ of a system is a map from X into R, where X is the space of paths in the configuration space (or phase space) of the system which satisfy appropriate initial and final conditions. The critical points of $\mathscr{S}$ are the solutions of the Euler-Lagrange (or associated Hamilton-Jacobi) equations. A critical point is k-fold degenerate if there are k nonzero Jacobi fields along the path defined by the critical point which have zero boundary conditions. The path integral for the semiclassical expansion of the momentum propagator $K(p_a,p_b)$ in scattering theory involves the space of paths defined by the initial and final momenta p_a and p_b. When the classical path defined by these boundary conditions is a nondegenerate critical point of $\mathscr{S}$, the WKB propagator is well defined and given by[17]

$$K_{\mathrm{WKB}}(p_b,t_b;p_a,t_a)=(2\pi i\hbar)^{n/2}\,|\det L_{\alpha\beta}(t_a,t_b)|^{1/2}\times\exp\left[\frac{i}{\hbar}S(p_b,t_b;p_a,t_a)\right],$$

where S is the action function and the matrix L is given by

$$L_{\alpha\beta}(t_a;t_b)=\frac{\partial p_\alpha(t_b;a,t_a)}{\partial a^\beta}\quad[\text{with } a=q(t_a)]\ .$$

The determinant of L vanishes if (and only if) the classical path is a degenerate critical point. There are several types of degeneracy which arise in scattering theory:

(1) The Hamilton-Jacobi equations have a multiple root, as in rainbow scattering.[3]

(2) The Hamilton-Jacobi equations have a one-parameter family of solutions, as is the case of glory scattering where

$$\frac{\partial p}{\partial \varphi}=0\ .$$

(3) The action S is invariant under a continuous group of transformations. An example is the conservation-law restriction on $|p|$ in the WKB approximation where the condition

$$\frac{\partial p(t)}{\partial z}=0\quad(\text{for } t \text{ sufficiently large})$$

implies that the WKB propagator K_{WKB} is proportional to

$$\delta(|p_a|-|p_b|)\ .$$

Path integration gives finite semiclassical cross sections when standard WKB approximations break down.[3,4,8,9] It also leads to a classification of the degeneracies in terms of degenerate critical points of the action.

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