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Bubble collisions in the very early universe

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It is believed that first-order phase transitions occurred in the very early universe when the temperature dropped below the grand-unification and Weinberg-Salam energies. Bubbles of the new, broken-symmetry phase would have formed surrounded by the symmetric phase. The energy released in the phase transition would have caused the walls of the bubbles to accelerate outwards. We study what happens when the walls collide with each other. We find that the energy in the walls would not be thermalized for a considerable time. In the inflationary-universe scenario, in which the bubble nucleation rate is low, thermalization could not occur until long after the baryon and nucleosynthesis eras and would not be complete. We also investigate the formation of primordial black holes in bubble collisions. The Weinberg-Salam phase transition is not likely to produce black holes but, under certain circumstances, the grand-unified phase transition might give rise to black holes of 10^3 g.

I. INTRODUCTION

In the currently accepted unified theories, phase transitions are predicted to occur at temperatures of the orders of the energies at which the various interactions are unified. Thus in the grand unified theories such as SU(5) (Ref. 1) or SO(10) (Refs. 2 and 3) there would be phase transitions at temperatures of the order of the grand-unification energy $\sim 10^{15}$ GeV and at the electroweak-unification energy $\sim 10^2$ GeV. The phase transitions are likely to be first order (see Ref. 4 and references therein). The reason is that in the unified, symmetric phase all the vector particles are massless and therefore have thermal fluctuations of the order of $T^2/12$ for each degree of freedom. These thermal fluctuations help to maintain the expectation values of the Higgs fields in the symmetric phase. On the other hand, if the Higgs fields were to develop expectation values which broke the symmetry from a group G to a subgroup K, the vector particles corresponding to G/K would develop masses and their thermal fluctuations would be greatly reduced. This in turn makes it easier for the Higgs fields to have nonsymmetric expectation values. One can thus have two phases at the same temperature which are locally stable against small changes in the expectation values of the Higgs fields. One of the phases will in general have a lower energy density than the other and so quantum tunneling will occur giving a first-order phase transition with the release of energy.

The manner in which such phase transitions would occur has been described by Coleman.[5] At a certain time t_1 quantum fluctuations would cause a bubble of the new phase to form in a medium which was elsewhere in the old phase. The most probable bubble would be a $t=$constant section through a solution of the classical scalar field equation in Euclidean space with an effective potential which includes quantum corrections and the effects of thermal fluctuations of the scalar and vector fields. At times $t > t_1$, the expectation values of the scalar fields would evolve according to the classical field equations in Lorentzian space with the time-symmetric data at t_1 given by the Euclidean "bubble" solution. The bubble of the new phase would expand into the old phase until it collided with other expanding bubbles of the new phase.

It is thought that such bubble formation might have occurred in the early universe when the temperature dropped below the grand-unification temperature and maybe again when the temperature dropped below the electroweak-unification temperature. The energy released by the bubble formation would be transformed into kinetic energy of the bubble walls and would cause the walls to expand outwards with uniform acceleration. It is generally assumed[6,7] that when the bubble walls eventually collide with those of another bubble, the kinetic energy will be dissipated and the universe will be reheated to almost the temperature it had before the phase transition. The purpose of this paper is to examine the collision of bubbles and to see to what extent this assumption is justified.

If the bubble nucleation rate N per unit four-volume is large compared to H^4 where $H=\dot{R}/R$ is the expansion rate of the universe, one can neglect the spatial curvature of the universe to the first approximation and treat the bubbles as in flat spacetime. We have used analytical and numerical techniques to study the collision of two bubbles in this situation. If the two bubbles correspond to significantly different members of the family G/K of broken-symmetry phases, most of the energy of the bubble walls will be converted into a pair of waves in which the phase of the scalar field rotates from its value in the two bubbles to a compromise value. These phase waves will propagate at the speed of light. The energy in them will eventually be thermalized by their coupling to the gauge field but this may take a long time. On the other hand, if the phases in the two bubbles are nearly aligned, most of the energy of the walls is not dissipated in the collision and the walls pass through each other and start accelerating back towards each other. The region to the future of the collision surface and between the walls will again be in the old symmetric phase and this phase would continue to exist for quite a time as the walls oscillate backwards and forwards with increasing frequency. On each collision they will lose some fraction of their kinetic energy and their energy will also be reduced by the expansion of the bubbles. Eventually they will disappear leaving only the new broken-symmetry phase. In either situation thermalization may not occur for some time. This could have a big effect on the possible generation of baryon symmetry by *CP*-noninvariant interactions.

It is generally assumed that the broken-symmetry phases of the different bubbles are uncorrelated because they are spacelike separated when they form and because the particle horizon of standard Friedmann-Robertson-Walker cosmological models prevent casual influences from propagating very far. However, this assumption would seem to lead to the production of far too many magnetic monopoles of the grand-unification mass.[8,9] A possible mechanism for correlating the phases in different bubbles would be the noncausal interactions which must be present if spacetime has a foamlike structure on scales of the Planck length.[10–12]

If the bubble nucleation rate N per unit four-volume were small compared to H^4 one would have the "inflationary-universe" scenario.[6,7] The universe would get stuck in the symmetric phase and the expansion would cause it to supercool. Eventually, the thermal energy density would be reduced below the difference in energy densities of the two phases. Because one does not want to have a large cosmological term in the broken-symmetry phase, one has to adjust the normalization of energy density so that it is small in this phase. One then gets a large postive (i.e., repulsive) cosmological term in the symmetric phase.

This would cause the universe to expand exponentially and to approach the de Sitter metric. When bubbles of the broken-symmetry phase did form eventually, the acceleration of their walls towards each other would be counteracted by the exponential expansion of the universe. It will be shown that they could never collide with more than a bounded center-of-mass energy. Despite this there would be enough energy in the walls to reheat the universe to a temperature near the critical temperature if the energy could be thermalized and spread uniformly over space. However, the probability distribution of the sizes of bubbles would be very flat which would mean that very large bubbles would dominate. The speed of light would prevent the energy released in the collisions of the walls of these very large bubbles from being spread over the volume occupied by the bubbles until long after the conventional baryon and nucleosynthesis eras and maybe even the present day. It would seem that the inflationary universe could not account for the ratios of baryons to photons and helium to hydrogen or for the large-scale homogeneity and isotropy of the universe today.

One might hope that in the intermediate situation in which the bubble nucleation rate is of the same order as H^4, one might get sufficient inhomogeneities to account for galaxies without having too much to be compatible with observation. We shall show that in such a situation a significant fraction of the universe would collapse to form primordial black holes. In the case of the grand-unified phase transition this would not be in conflict with observation because the black holes would have been of the order of 10^3 g and would have evaporated very early on. On the other hand, any black holes produced in the Weinberg-Salam phase transition would have masses of the order of 10^{28} g and would still be around at the present time. The requirement that their mass density should not exceed the critical density places a very stringent limit on the number that can have been produced. In turn this implies that the nucleation rate in the Weinberg-Salam phase transition must have been high compared to H^4. The production of black holes by first-order phase transitions has been considered previously but only under the assumption of spherical symmetry which will not hold.[13]

There is also a possible scenario in which the bubble nucleation rate is negligible compared to H^4 until the temperature drops to some value at which the potential barrier to bubble formation becomes small or zero. This situation could occur at the Weinberg-Salam phase transition because of chiral-symmetry breaking.[14] However it seems[15] that NH^{-4} would rise on a time scale much shorter than the Hubble time H^{-1}. Thus the bubbles would not have expanded very far before everywhere slid into the broken-symmetry phase. On the other hand, the chances of creating primordial black holes at the grand-unified phase transition seem rather better because one would not expect NH^{-4} to change so rapidly compared to the much shorter expansion time scale.

The plan of this paper is as follows. In Sec. II we describe the formation of bubbles in first-order phase transitions. The case of high nucleation rate is considered in Sec. III. We show that the collision of two bubbles can be reduced to the study of a complex scalar field coupled to a U(1) gauge field in one space and one time dimension and we use analytic approximations to study the details of the collision. Our results are compared with numerical calculations in Sec. IV. The inflationary-universe case of low nucleation rate is studied in Sec. V and the production of primordial black holes is considered in Sec. VI.

II. PHASE TRANSITIONS

We shall consider a Higgs scalar field Φ which is in the fundamental representation of a Yang-Mills group G with a potential A_μ.

The effective Lagrangian for the scalar field will have the form

$$L_\Phi = -(D\Phi)^\dagger \cdot (D\Phi) - V(\Phi) , \tag{1}$$

where D is the gauge-covariant derivative and $V(\Phi)$ is the effective potential including quantum and thermal corrections. The detailed form of $V(\Phi)$ will depend on the particular model in question but in order to have a general discussion we shall assume that $V(\Phi)$ has two local minima, an isolated one at $\Phi=\Phi_1$ which is invariant under the full group and a degenerate one at $\Phi=\Phi_2$ which is invariant only under a subgroup K. In fact there will be a whole degenerate family G/K of such local minima. In order to obtain a first-order phase transition we shall assume

$$V(\Phi_1)-V(\Phi_2)=\epsilon^4 , \tag{2}$$

where ϵ has the dimension of energy in units in which $c=h=1$. We shall assume that the effective height of the potential barrier between the two minima is ξ^4 where $\xi^4 >> \epsilon^4$.

The quantum tunneling from the symmetric phase Φ_1 to the broken-symmetry phase Φ_2 occurs through a "bubble," a solution in Euclidean space of the classical field equations derived from (1) which has a spherical region of the new phase Φ_2 surrounded by the old phase Φ_1. The radius of the spherical region will be of order $\xi^2 v\epsilon^{-4}$ where $v=|\Phi_1-\Phi_2|$. The thickness of the transition region or bubble wall separating the regions Φ_1 and Φ_2 is of order $\frac{3}{2}v\xi^{-2}$, and the action per unit area of the wall $\sigma\sim\frac{1}{3}\xi^2 v$. The rate of bubble formation per unit four-volume is

$$N\sim\epsilon^4 e^{-B} , \tag{3}$$

where $B\sim\xi^8 v^4\epsilon^{-12}$ is the difference between the action of the bubble solution and that of a solution in which $\Phi=\Phi_1$ everywhere.

If one looks at the situation in Lorentzian, as opposed to Euclidean, space, the idea is that the quantum fluctuations lead to a scalar field configuration on a surface $t=t_1$ which is the same as that on an equatorial slice through the Euclidean bubble solution. At times $t>t_1$, the scalar field, or rather its expectation value, evolves according to the classical field equations with the initial data given at $t=t_1$. The bubble wall accelerates uniformly outwards into the old $\Phi=\Phi_1$ phase. The phase-transition energy released by the expansion of the bubble is converted into kinetic energy of the wall. The solution is invariant under Lorentz transformations about the point $t=t_1$ at the center of the bubble. The scalar field in Lorentzian space is in fact the analytic continuation of the bubble solution in Euclidean space.

The expectation value of the gauge field A_μ will be zero during the formation and expansion of the bubble. The gauge field will affect the bubble only through the quantum corrections it produces in the effective potential $V(\Phi)$. Eventually, however, the bubble wall will collide with that of another bubble. If the value of Φ inside the second bubble corresponds to a different member of the family G/K of degenerate minima of $V(\Phi)$, the collision of the bubbles will generate a nonzero expectation value of A_μ which in turn will affect the evolution of Φ. We shall consider this in more detail in Sec. III.

We shall assume that the universe was spatially homogeneous and isotropic before the phase transition and contained predominantly massless particles whose interactions could be neglected and which were in thermal equilibrium. The total energy density is then

$$\rho=\frac{\pi^2}{30}g(T)T^4+\epsilon^4 , \tag{4}$$

where $g(T)$ is a numerical factor which depends on the number of boson and fermion degrees of freedom.

It is reasonable to assume that the first term in (4) representing the thermal energy, dominates the second term, representing the latent heat at the critical temperature T_c.

By the Einstein equation,

$$H^2=\frac{8\pi}{3{m_P}^2}(\rho+\mathscr{E}) , \tag{5}$$

where $H=\dot{R}/R$ is the expansion rate, m_P is the Planck mass, and $\mathscr{E}$ is the expansion energy per unit volume. As the universe expands, T will be proportional to R^{-1} and $\mathscr{E}$ will be proportional to R^{-2}. From measurements of the expansion rate and the deceleration parameter the present value $|\mathscr{E}_0|/\rho_0$ is less than 10^4 where ρ_0 is the present density of the microwave background radiation. Thus the expansion energy can be neglected at early times such that $R/R_0<10^{-2}$.

III. HIGH NUCLEATION RATE

In the case of a high nucleation rate, the average separation between bubbles $\sim N^{-1/4}$ is much less than the expansion time or the horizon size $\sim H^{-1}$. In this situation one can ignore the expansion and curvature of the universe and treat the bubbles as if in flat spacetime. A single bubble is invariant under a Lorentz group O(3,1) about its center. When one considers two bubbles, the line adjoining their centers is a preferred axis (say the x axis). The situation is therefore invariant under an O(2,1) group consisting of Lorentz boosts in the y and z directions and spatial rotations in the yz plane. It is convenient to define new coordinates s,ψ,θ by

$$\begin{aligned} t&=s\cosh\psi ,\\ y&=s\sinh\psi\sin\theta ,\\ z&=s\sinh\psi\cos\theta , \end{aligned} \tag{6}$$

so that $s^2=t^2-y^2-z^2$. In these coordinates the flat-space metric takes the form

$$-ds^2+s^2(d\psi^2+\sinh^2\psi\, d\theta^2)+dx^2 . \tag{7}$$

The solution representing the colliding bubbles will be independent of the coordinates ψ and θ. In order to specify the initial data on the surface $s=0$ for the two-bubble collision one must first pick a suitable gauge for the Yang-Mills field such as $A_s=0$ or $\partial^\mu A_\mu=0$. The initial data for the Yang-Mills field are then $A_\mu=0$, and $\partial A_\mu/\partial s=0$. The initial data for the scalar field consist of the suitably smoothed data for two single bubbles separated by a distance $2b$ along the x axis. In general the values Φ_L and Φ_R in the left- and right-hand bubbles will correspond to different members of the family G/K of degenerate minima of $V(\Phi)$. Then one can find a U(1) subgroup W of G such that

$$\Phi_L=e^{ig\alpha w}\Phi_R , \tag{8}$$

where g is the coupling constant of the Yang-Mills group and w is the generator of W normalized so that $e^{2\pi i g w}=1$. The subgroup W is defined up to conjugacy in K provided that $e^{ig\alpha w}$ is not in the center of G.

The expectation value of the Yang-Mills field A_μ is initially zero and will remain nearly so as the bubble walls accelerate towards each other along the x axis. After they collide the Yang-Mills potentials A_x and A_s will develop nonzero expectation values which will lie in the U(1) subgroup generated by w. There is thus no loss of generality in the two-bubble collision in restricting attention to the Abelian Higgs model consisting of a single complex scalar field ϕ coupled to an electromagnetic field. The symmetric phase will correspond to a local minimum of $V(\phi)$ at $\phi=0$ and the broken-symmetry phase will correspond to a circle of degenerate minima at $|\phi|=v$. If one takes the origin of the coordinate to lie midway between the bubbles, the initial data will satisfy the conditions

$$\begin{aligned} \phi(-x)&=e^{i\alpha}\bar{\phi}(x) ,\\ A_x(x)&=A_x(x) ,\\ A_s(-x)&=-A_s(x) . \end{aligned} \tag{9}$$

The conditions (9) will remain true at subsequent values of s because the field equations are invariant under these symmetries.

Let S^a be the vector $\partial/\partial s$, i.e., the unit vector normal to the surface $s=$const and let X^a be the vector $\partial/\partial x$, i.e., the unit vector normal to the surface $x=$const. Then

$$\int_{\Sigma_2} T^{ab}S_a d\Sigma_b - \int_{\Sigma_1} T^{ab}S_a d\Sigma_b = \int_V (T^{ab}S_a)_{;b}dv \ , \tag{10}$$

where V is the region between the $s=$const surfaces Σ_1 and Σ_2. From this it follows that

$$\frac{\partial}{\partial s}\left(s^2 \int T^{ab}S_aS_b dx\right) = -s^2 \int T^{ab}S_{a;b}dx = -s\int T^{ab}k_{ab}dx \ , \tag{11}$$

where $k_{ab}=g_{ab}+S_aS_b-X_aX_b$ is the projection operator into the two-space orthogonal to S^a and X^a. One can regard Eq. (11) as the expression of the conservation of energy, the component $T^{ab}S_aS_b$ in the coordinate system s,x,ψ,θ. The factor of s^2 on the left-hand side represents the fact that the energy is diluted by being spread out over a larger surface while the term $-sT^{ab}k_{ab}$ represents the work done by the pressure forces because of the expansion of the coordinate system in the directions spanned by k_{ab}.

Because of the reflection symmetry (9) it is sufficient to consider only the region $x\geq 0$. If one adopts the thin-wall approximation to the bubble, then up until the time that the bubbles collide the energy-momentum tensor in the region $0\leq x\leq b$ can be represented as

$$T^{ab} = -\epsilon^4 g^{ab}\theta(\bar{x}(s)-x) - \sigma(g^{ab}-V^aV^b)[1+(V^s)^2]^{-1/2}\delta(\bar{x}(x)-x) \ . \tag{12}$$

The first term represents the vacuum energy-momentum in the old phase outside the wall of the right-hand bubble which is at $\bar{x}(s)$. The second term is the energy-momentum of the wall regarded as a thin shell where V^a is the unit vector in the sx plane orthogonal to the wall and σ is the energy density per unit area of the wall. Because V^a is orthogonal to the wall of the bubble

$$V^s = -\bar{x}'(s)\{1-[\bar{x}'(s)]^2\}^{-1/2} \ ,$$
$$V^x = -\{1-[\bar{x}'(s)]^2\}^{-1/2} \ . \tag{13}$$

Using (12) and (13) in (11), one finds

$$\sigma s\bar{x}'' = -2\sigma\bar{x}'(1-\bar{x}'^2) - s\epsilon^4(1-\bar{x}'^2)^{3/2} \ . \tag{14}$$

Equation (14) can be integrated to give

$$\bar{x}' = \pm\frac{s}{(a^2+s^2)^{1/2}} \ , \tag{15}$$

where

$$a = \frac{3\sigma}{\epsilon^4}\frac{s^3}{s_0{}^3-s^3} \tag{16}$$

and s_0 is a constant of integration. In the interval $0\leq s\leq s_1=[b^2-(3\sigma/\epsilon^4)^2]^{1/2}$ the solution has $s_0=0$, $a=3\sigma/\epsilon^4$, and

$$(\bar{x}-b)^2 - s^2 = (3\sigma/\epsilon^4)^2 \ . \tag{17}$$

This corresponds to the usual single-bubble solution in which the left-hand wall of the right-hand bubble accelerates towards the left with a uniform acceleration of $\epsilon^4/3\sigma$. By the reflection symmetry (9) the right-hand wall of the left-hand bubble will be accelerated towards the right in a similar way (Fig. 1). They will collide at $x=0$ when $s=s_1$. If the separation of the bubbles, $2b$, is large compared to their radius, $3\sigma/\epsilon^4$, the walls will be highly relativistic when they collide and will have a γ factor $\epsilon^4 b/3\sigma$. The kinetic energy of the walls will be concentrated in a small region around the x axis of thickness $\frac{3}{2}\nu^2/b\epsilon^4$, the proper thickness of the wall $\sim\nu^2/2\sigma$ divided by the γ factor because of the Lorentz contraction.

Consider first the case where $\alpha=0$ (in which the

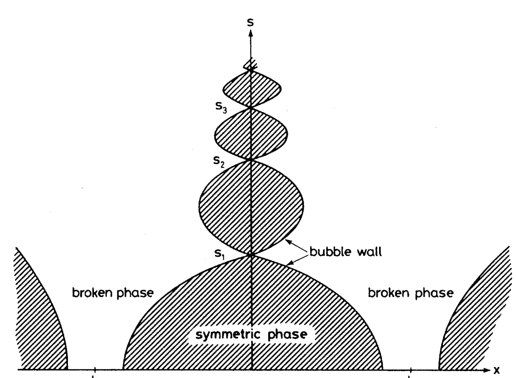

FIG. 1. The collision of two bubbles in the case $\alpha=0$. The dashed region denotes the old, symmetric phase outside the bubbles which nucleate at $x=b$ and $x=-b$ on $s=0$.

field ϕ can be taken to be real and the electromagnetic field will be zero at all times). In this situation the only way to conserve energy in the collision region is for the region to emit two walls with almost the same velocities as those of the original walls. One wall will move to the right and the other to the left away from $x=0$, $s=s_1$. The regions ahead of the walls will be in the new broken-symmetry phase while the region between them will be in the old symmetric phase (Fig. 1). The pressure difference between the new and the old phases and the tension in the walls will cause the walls to accelerate towards each other with an initial acceleration which is five times the acceleration $\epsilon^4/3\sigma$ of the original walls. The walls will be brought to rest at $s=2^{1/3}s_1$ and they will collide again at $s=(2^{4/3}-1)s_1$. One will get a series of oscillations with shorter and shorter periods and collisions at $s_n \sim 1.4n^{1/2}$. After a time $s_l \sim \frac{1}{4}(1.4)^2 b^2\epsilon^4/\sigma$ the velocity of the walls will become nonrelativistic and they will disappear. The time s_l is a lower bound for the time taken to come to equilibrium. It can be longer than the expansion time H^{-1} even though the average bubble separation $2b$ is small compared to H^{-1}.

In the case that the values of ϕ in the two bubbles differ by a nonzero phase angle α, the electromagnetic field will still be nearly zero up until the first collision and it can be neglected in the actual collision provided $g << 2b\epsilon^4/3v^3$ which will normally be the case. The kinetic energy of the incoming walls will now be shared between two outgoing walls which correspond to a change in $|\phi|$ from zero to v and two outgoing phase waves in which $|\phi|$ remains unchanged at $|\phi|=v$ but in which the phase of ϕ changes from zero on the right or from α on the left to the compromise value of $\alpha/2$ in the center [we are assuming $0<\alpha<\pi$ (Fig. 2)].

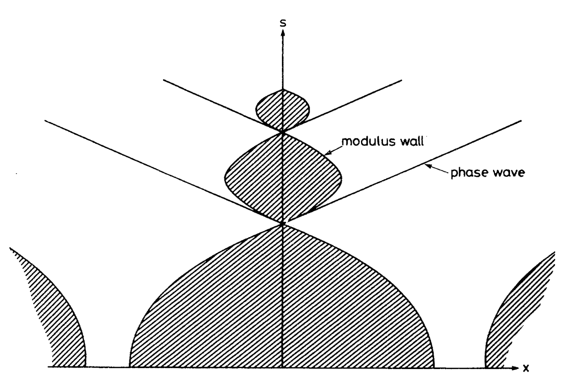

FIG. 2. The collision of two bubbles when $0<\alpha<1$. The kinetic energy of the walls is shared in each collision between a new pair of walls and a phase wave which travels outwards at the speed of light.

The energy of the incoming wall per unit area is $\frac{1}{3}b\epsilon^4$. The energy per unit area of the outgoing modulus wall is $\sigma(1-\bar{x}'^2)^{-1/2}$ where $\bar{x}'$ is the initial velocity of the wall. The energy of the phase wave is $2|\Delta\phi|^2/d$ where $|\Delta\phi|=\frac{1}{2}v\alpha$ is the change in ϕ across the phase wave and d is the thickness of the phase wave. The thickness d will be equal to the thickness $\frac{3}{2}v^2/b\epsilon^4$ of the incoming wall. The energy in the phase wave is therefore $\frac{1}{3}b\epsilon^4\alpha^2$. If α is less than about 1, the energy in the phase wave is less than that of the incoming wall and therefore the collision will give rise to both a phase wave and a modulus wall with velocity given by

$$\bar{x}'=\left[1-\frac{9\sigma^2}{(\epsilon^4 b)^2(1-\alpha^2)^2}\right]^{1/2}. \tag{18}$$

At a value of α of the order of 1, the energy in the phase wave will be equal to the energy of the incoming walls and the outgoing modulus walls will disppear. For values of α greater than 1, the energy of the incoming walls would be smaller than that of the phase waves if the latter had the thickness $\frac{3}{2}v^2/b\epsilon^4$. What will happen in this case is that the phase waves will have to have a greater thickness but they will take up nearly all the energy and not leave anything over for a modulus wall. Finally, if α is nearly equal to π, the field ϕ will be nearly real but now antisymmetric under reflection in $x=0$. In this case the phase wave will disappear because the field will not be able to decide whether to compromise on $\alpha=\frac{1}{2}\pi$ or $\alpha=\frac{3}{2}\pi$. Instead one will get a modulus wall with nearly the same velocity as in the $\alpha=0$ case.

If the coupling constant g of the electromagnetic field is small, the phase waves will spread out at the speed of light in the sx plane. The thickness of the phase wave will increase as s^2 because of the expansion in the ψ and θ directions. The modulus walls do not carry an electric charge but the phase waves will correspond to positively and negatively charged sheets with charge $gv^2\alpha$ per unit area spreading out from the collision. They will create an electric field $gv^2\alpha$ behind the phase waves. In most of this region the scalar field will be in the broken-symmetry state $|\phi|=v$ and so the photon will have a mass $gv\sqrt{2}$. This mass will give rise to a skin depth $(gv\sqrt{2})^{-1}$ as in the Meissner effect. If the skin depth is small compared to the separa-

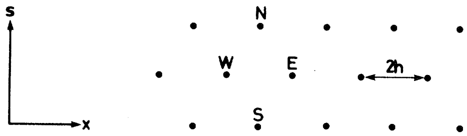

FIG. 3. The diamond-shaped lattice used for the numerical integration of the scalar-electromagnetic field equations. The d'Alembertians were evaluated at the center of each diamond-shaped set of points $NWES$.

tion b, which is likely to be true in the cases of physical interest, the electric field will be confined to a small region behind the phase wave and will not affect the modulus walls which will oscillate as in the $\alpha=0$ case but with reduced initial velocity. On the other hand, if the skin depth is large compared to b, the electric field will be nearly uniform and constant behind the phase waves. The nonzero vector potential will cause the phase of the modulus walls to rotate. When the modulus walls collide again they will have a phase difference and so they will give rise to a second phase wave. This will cause the modulus walls to be damped rapidly and to disappear in a short time.

IV. NUMERICAL RESULTS

In order to check the conclusions of Sec. III we ran numerical calculations of the coupled classical scalar and electromagnetic field equations with the initial data corresponding to two bubbles. The Lagrangian was

$$L=-\tfrac{1}{4}F^2-(D\phi)^\dagger(D\phi)-V(\phi)\ . \tag{19}$$

The effective potential $V(\phi)$ should be a function of $|\phi|^2$ which has a local minimum at the origin and a circle of degenerate global minima away from the origin. We chose the simplest such $V(\phi)$, a cubic in $|\phi|^2$. The presence of $|\phi|^6$ terms in the fundamental Lagrangian would make the theory nonrenormalizable. However the quantum and the thermal corrections will introduce terms like $|\phi|^6$ and higher orders. We have neglected these higher-order terms because the results should not depend sensitively on the exact form of the potential but only on the difference in height of the two minima and the height and width of the barrier between them. We can neglect the constant term in $V(\phi)$ except for gravitational effects which we are not considering. Of the remaining three parameters in $V(\phi)$, two can be set to unity by scaling ϕ and the coordinates. It is therefore sufficient to consider a complex field u with electric charge e and a potential $V(u)$ of the form

$$(|u|^2+\beta)(|u|^2-1)^2\ , \tag{20}$$

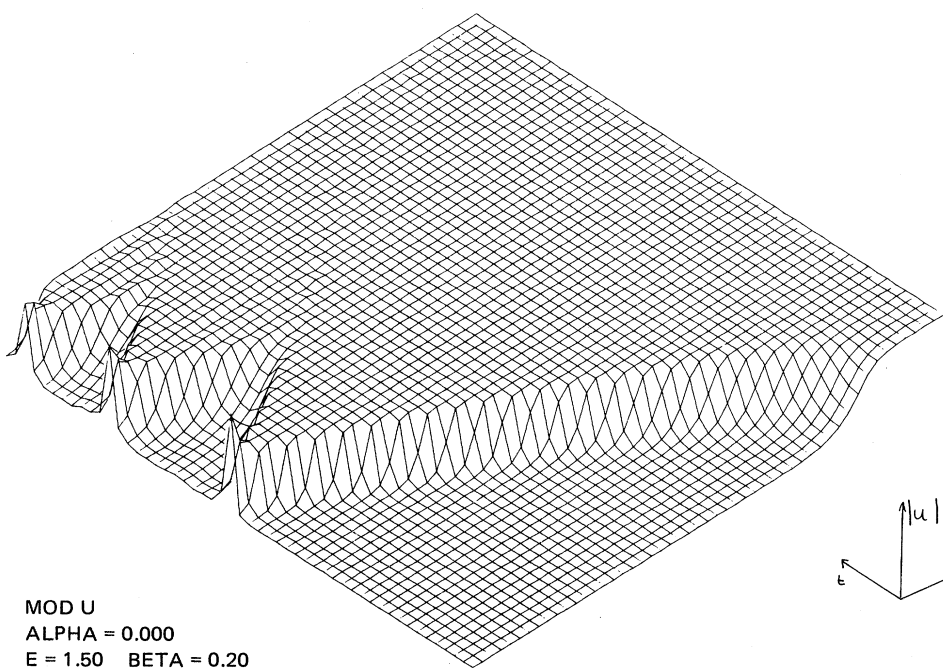

FIG. 4. The modulus of u is here plotted over a range of 60 units in s and 30 units in x for the collision of a bubble wall moving from right to left with another (not shown) moving from the left.

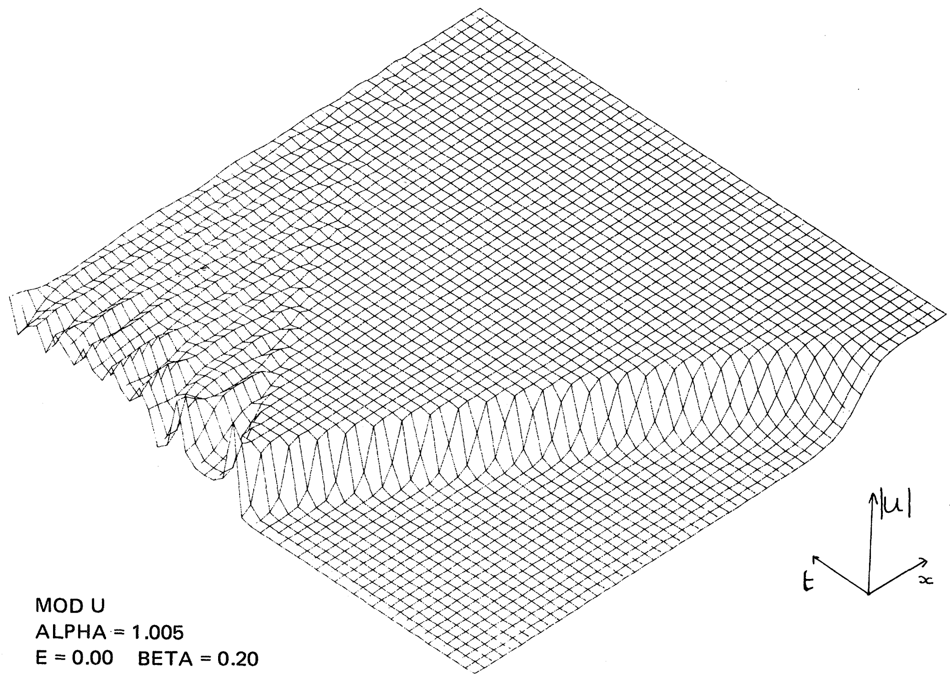

FIG. 5. The modulus of u when the phase difference $\alpha \approx 1$.

where

$$\beta = \tfrac{9}{4}\epsilon^4/\xi^4 ,$$

$$e = \tfrac{3}{2} g v^2/\xi^2 ,$$

$$u = \phi/v .$$

The scalar and electromagnetic field equations were integrated numerically in the Lorentz gauge using an algorithm that was developed on a Commodore Pet with the assistance of R. G. Hawking and then run on an IBM 370. It used an sx lattice

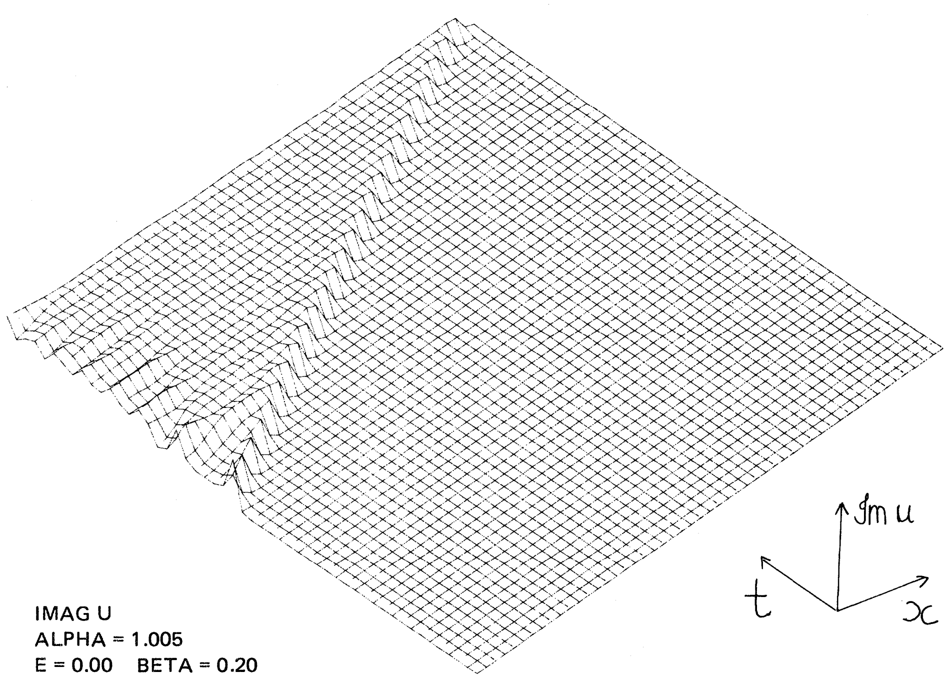

FIg. 6. The imaginary part of u for the above case $\alpha \approx 1$. The right-hand bubble (shown) has been chosen to be real at $s=0$.

of the form shown in Fig. 3. The d'Alembertian

$$\frac{\partial^2}{\partial s^2}+\frac{2}{s}\frac{\partial}{\partial s}-\frac{\partial^2}{\partial x^2} \tag{21}$$

on the scalar field u at the center of each diamond was approximated by

$$\frac{1}{h^2}(u_N+u_S-u_E-u_W)+\frac{1}{sh}(u_N-u_S)\ , \tag{22}$$

where $2h$ is the lattice spacing. The value of u at the center of the diamond was approximated by

$$\tfrac{1}{2}(u_E+u_W)\ . \tag{23}$$

Similar approximations were made for the d'Alembertian on the vector potential and for the gradients of u. This algorithm, unlike others that were tried, was numerically stable and did not introduce numerical viscosity. As a check on the accuracy, $\partial_\mu A^\mu$ was evaluated and found to remain small. Figure 4 shows the numerical results for the case $\alpha=0$. Only the positive x axis is shown. The fields at negative values of x are given by the conditions (9). One sees the wall of the right-hand bubble accelerating towards the left. When it reaches $x=0$, it collides with a similar wall moving to the right from negative values of x. The two walls pass through each other and then start accelerating back towards each other and collide several more times. The regions between the walls are in the symmetric phase ($\phi=0$) and the regions outside the walls are in the broken phase ($\phi=\nu$).

Figures 5 and 6 show the real and the imaginary parts of u, respectively, for the case $\alpha\approx 1$ and $e=0$. Note the phase wave that arises from the collision which is responsible for carrying away most of the energy leaving an amount insufficeint to cause the scalar field to return to the symmetric phase.

Figures 7 and 8 show the real part and the modulus of u in the case $\alpha=\frac{1}{4}\pi$ and $e=1.5$. The variations in the real part of u which do not occur in the modulus are mainly gauge-induced rotations of the phase of u. The electric field is small except near the first phase wave.

V. LOW NUCLEATION RATE

If the nucleation rate N is small compared to H^4, the universe will expand by a large factor and the temperature will drop a long way below the critical temperature before any bubbles form. The energy-momentum tensor will approach the form $-\epsilon^4 g_{\mu\nu}$ which arises from the energy density of the unbroken symmetric phase. This will act like a cosmological constant and will cause the universe

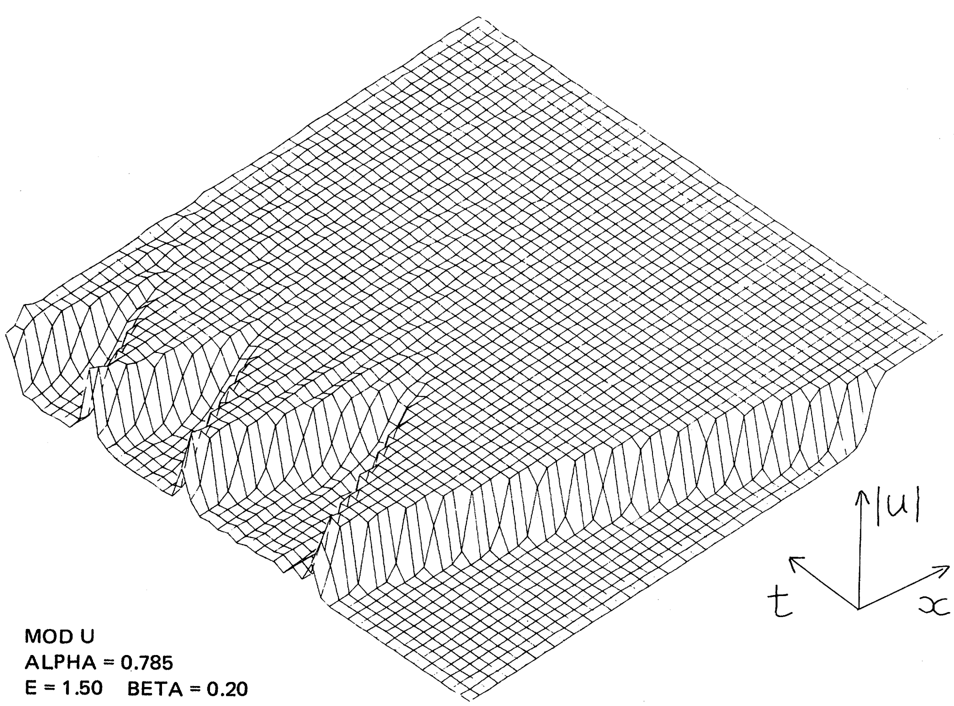

FIG. 7. The modulus of u for $\alpha=\frac{1}{4}\pi$ and coupling constant $e=1.5$. The s range here is from 20 to 100 units.

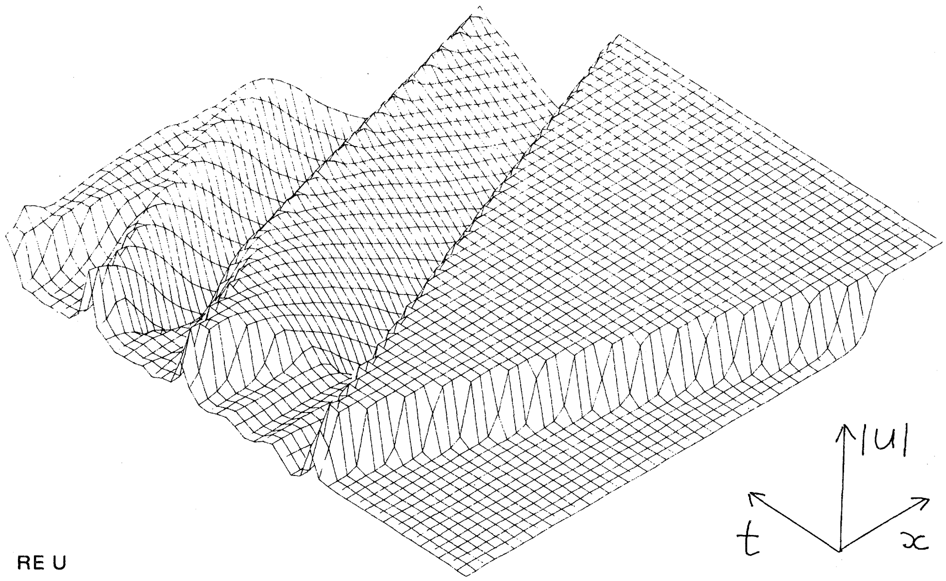

FIG. 8. The real part of u for the above case $\alpha=\frac{1}{4}\pi$, $e=1.5$.

to expand exponentially like de Sitter space.

The metric of de Sitter space can be written in the form

$$ds^2=-dt^2 +e^{2Ht}[dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)] \; . \tag{24}$$

Let $\eta=-H^{-1}e^{-Ht}$. Then

$$ds^2=H^{-2}\eta^{-2}[-d\eta^2+dr^2 +r^2(d\theta^2+\sin^2\theta\, d\phi^2)] \; . \tag{25}$$

In other words, the space is conformal to the region of flat space below the plane $\eta=0$ with a conformal factor $\Omega=(H\eta)^{-1}$.

Consider a bubble centered at the point p, $r=0$, $\eta=\eta_p$, with initial proper radius R_0. The wall will obey the equation

$$r^2-(\eta-\eta_p)^2=R_0{}^2H^2\eta_p{}^2 \tag{26}$$

(Fig. 9). The region outside the wall will have the original de Sitter metric while the region inside will be almost flat space. The bubble will never collide with another bubble which is centered in the region

$$r+\eta>\eta_p \tag{27}$$

outside the event horizon of the point p [there is a small correction to (27) from the finite initial sizes of the bubbles]. Bubbles which are centered at points just inside the event horizon of p will collide with the bubble from p at very late physical times t. However, the γ factor of the colliding walls in the local center-of-mass frame will never be bigger than

$$(R_0H)^{-1}(1+R_0{}^2H^2)^{1/2} \; . \tag{28}$$

The $R_0{}^2H^2$ term can usually be neglected. The bubbles can grow very large before they collide but the energy per unit area of the wall which is released in the collision is bounded by (28). One might therefore think that an initially spatially flat $k=0$ model could convert itself into a highly unbound $k=-1$ model. However this is not the case. One can see this by supposing that there was some time t at which every bubble had just collided with other bubbles at every point on its wall. On a

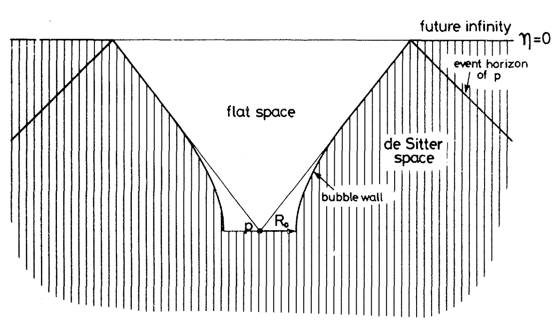

FIG. 9. A conformal diagram of bubble formation in de Sitter space. The metric inside the bubble will be almost flat.

length scale large compared to the typical bubble size, the geometry of this spacelike surface would be flat although it would not be flat on the scale of individual bubbles. Thus the large-scale structure of the resultant universe will be that of a $k=0$ model but with the irregularities on the scale of the bubbles.

Of course the bubbles do not all collide at the same time. Consider a point q on a future-directed null geodesic from p. The proper four-volume of the past of q to the future of some initial surface $t=t_0$ is

$$\tfrac{4}{3}\pi H^{-3}(t_q-t_0) \tag{29}$$

plus terms which are small when $t_1-t_0 >> H^{-1}$. Thus the probability that there is not another bubble in the past of the q given that there is not one in the past of p is

$$\exp[-\tfrac{4}{3}\pi NH^{-3}(t_q-t_p)] \tag{30}$$

(the finite initial size R_0 has been neglected). The proper radius of the bubble at time t_q is approximately

$$H^{-1}e^{H(t_q-t_p)}\ . \tag{31}$$

Thus the probability that the bubble reaches a proper radius $X>x$ in a given direction before it collides with another bubble is

$$P(X>x)=(Hx)^{-4\pi NH^{-4}/3} \tag{32}$$

In the inflationary-universe scenario[6] $NH^{-4}<<1$. This means that the probability distribution of bubble sizes is very flat. For example it is suggested that $\frac{4}{3}\pi NH^{-4}$ might be of the order $\frac{1}{50}$. The quantity Hx is about 10^{20} for the comoving region occupied by a galaxy at the present time. Thus about one in three bubbles would grow to a size larger than that which would correspond to a galaxy now. This might seem attractive as an explanation of the origin of the fluctuations that led to galaxy formation. However there are serious problems. The large size of these regions would mean that they would occupy most of the universe. The energy released by these wall collisions could not thermalize and distribute itself more or less uniformly over a region until light had time to cross the region. In a Friedmann universe the time at which light can cross a comoving region the size of a galaxy is about 10^8 sec at which stage the temperature would have fallen to about 10^6 K, way below the temperature at which baryon and nucleosynthesis are supposed to take place. Thus one would not have the usual explanations of the baryon and helium abundances. Moreover, thermalization would not be complete at this late epoch and so the spectrum of the microwave background would be distorted.[17] One would also expect bubbles of an even larger size to occur and to produce significant anisotropies in the microwave background. Thus it seems that an inflationary-universe scenario with $NH^{-4}<<1$ can be ruled out. A model in which $NH^{-4}\sim 1$ might however provide enough inhomogeneity to account for galaxies without being in conflict with observations in other ways. We shall discuss the formation of primordial black holes in such a model in the next section.

VI. BLACK-HOLE FORMATION

In this section we shall consider the question of whether the collision of several bubbles can create a sufficient local concentration of energy to cause gravitational collapse and the formation of a primordial black hole. The most favorable situation is that in which the vacuum energy density ϵ^4 of the symmetric phase dominates over the thermal energy density because in this case the density contrast produced by the bubbles is greatest. Such a situation could come about either because of a fairly low nucleation rate or because, by chance, bubbles did not form in a certain region of spacetime. We shall therefore consider bubble formation in de Sitter space. The probabilitiy that no bubble has nucleated in the past of a point p is approximately

$$e^{-4\pi NH^{-3}(t_p-t_0)/3}\ , \tag{33}$$

where t_0 is the time at which the nucleation could have started. Consider a ball B of proper radius

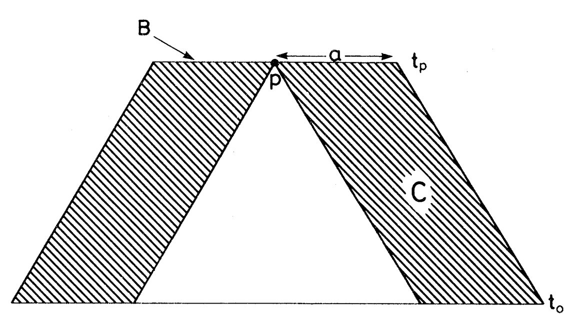

FIG. 10. The region C lies to the past of the ball B of radius a about the point p but not to the past of p. The walls of bubbles that nucleate in C will pass through B.

$a < H^{-1}$ about the point p in the surface $t = t_p$ (Fig. 10). Let C be the region to the past of B which does not lie to the past of p. The volume of C is of the order of

$$\tfrac{4}{3}\pi H^{-3} a \ . \tag{34}$$

If a bubble nucleates in the region C, its wall is likely to reach the limiting γ factor of $H^{-1}R_0^{-1}$ by the time $t = t_p$. Because there are no bubbles to the past of p, it can have collided with another bubble wall only recently, if at all. Its energy per unit area will therefore be about

$$\gamma\sigma = \tfrac{1}{3}\epsilon^4 H^{-1} = \frac{m_P^2}{8\pi} H \ . \tag{35}$$

If there were n such bubbles which nucleated in the region C, the energy in their walls within B would be

$$\tfrac{1}{8} n a^2 H m_P^2 \ , \tag{36}$$

One would expect this energy to cause gravitational collapse if it were bigger than $\frac{1}{2} a m_P^2$. Thus one would expect a black hole to form if

$$n > \frac{4}{aH} \ . \tag{37}$$

It is reasonable to use the above criterion for gravitational collapse only for regions whose size a is small compared to the Hubble radius H^{-1}. In such cases one can neglect the expansion and curvature of the Universe. We shall assume that the criterion is roughly valid for $a \leq \frac{1}{2}H^{-1}$. Thus one might expect a black hole to form for

$$n \geq 8 \tag{38}$$

The probability that eight bubble walls collide in a ball B with radius $\frac{1}{2}H^{-1}$ is the product of the probability that no bubbles nucleate in the past of p with the probability that eight bubbles nucleate in C. This is of the order

$$e^{-4\pi N H^{-3}(t_p - t_0)/3} (\tfrac{2}{3}\pi N H^{-4})^8 \ . \tag{39}$$

The mass of the black hole would be about

$$\tfrac{1}{4} H^{-1} m_P^2 \ . \tag{40}$$

If $\frac{4}{3}\pi N H^{-4} < 1$, one can choose t_p so that the exponent is of order 1. The probability of forming black holes will be appreciable unless $\frac{4}{3}\pi N H^{-4} << 1$. However this case is ruled out by the considerations of the last section. In the case of the grand-unified phase transition, the black holes formed would be of the order of 10^3 g. They would evaporate in a time of order 10^{-18} sec and so would not produce any observable effect except for possibility altering the baryon-to-photon ratio. On the other hand, black holes produced in the Weinberg-Salam phase transition would have masses of the order of 10^{30} g, about the mass of Jupiter. Such black holes would not evaporate significantly and therefore would still be around at the present day. They would be very difficult to detect apart from their contribution to the average mass density of the universe. The requirement that they should not cause the deceleration parameter to be large puts a limit of about one in 10^{14} on the probability of forming black holes in the Weinberg-Salam era. The only way to achieve this would be for the Weinberg-Salam phase transition to have a high nucleation rate such that the probability of a region expanding for a time of order H^{-1} is very small.

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