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Relativistic oscillator in a Bethe-Salpeter model: Nonexistence of discrete bound states

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We investigate the solutions of the covariant Bethe-Salpeter equation for bound states of a scalar quark-antiquark system with an effective confining kernel which in coordinate four-space behaves as R^2. Using a suitable set of new variables we separate the equation into one-dimensional equations and study the eigenvalue problem. We find that the covariant bound-state Bethe-Salpeter equation with the above kernel possesses only a continuous spectrum.

I. INTRODUCTION

Recent developments in non-Abelian gauge theories suggest that quantum chromodynamics may be the correct theory of strong interactions. In these theories, the fundamental constituents of hadrons, i.e., quarks, which are the color sources, interact with the non-Abelian gauge fields in a fashion such that the quarks are confined: it is energetically impossible to separate the source fields to infinity. One also hopes that the dynamics is such as to generate a hadron spectrum with a linearly rising Regge trajectory.

The use of the relativistic Bethe-Salpeter (BS) formalism to describe this spectrum is an attractive possibility. Over the past few years, color confinement models have been proposed where quarks are bound in a long-range potential, which behaves[1] as R^λ $(\lambda>0)$ for large quark separation where $R(x)=(x_0{}^2-\vec{x}^2)^{1/2}$. In this paper we investigate the use of a relativistic harmonic-oscillator kernel $(\sim R^2)$ in a BS formalism for scalar quarks in conjunction with the Schwinger-Dyson equations for the dressed quark propagators and show that no discrete spectrum consistent with a rising Regge trajectory exists. For phenomenological applications one has to take into account complications arising from the spin of the quarks; this has, however, not been considered in this paper.

For the BS kernel, we choose the ansatz

$$K(x,x')=\beta R^2(x-x'),\quad \beta>0 \tag{1}$$

which, in momentum space, is given by

$$K(k,k')=-(2\pi)^4\beta\Box_k\delta^{(4)}(k-k') . \tag{2}$$

The Schwinger-Dyson equation for the dressed quark propagator is

$$\Delta_F{}^{-1}(k)=Z\Delta_0{}^{-1}(k)+\frac{1}{(2\pi)^4}\int d^4k'K(k,k')\Delta_F(k') , \tag{3}$$

where Z is the wave-function renormalization constant and

$$\Delta_0(k)=\frac{1}{k^2-m_0{}^2}$$

is the free quark propagator for bare mass m_0. Equation (3) with $K(k,k')$ given by (2) has been studied by Alabiso and Schierholz[2] in detail. Their results are

$$\text{(i) for } m_0\to 0,\quad Z=1,\quad \Delta_F=\frac{1}{k^2} , \tag{4}$$

$$\text{(ii) for } Z=0,\quad \Delta_F(k)=\left[\frac{1}{3\beta}\right]^{1/2}(-k^2)^{1/2} , \tag{5}$$

$$\text{(iii) for } m_0{}^2\to\infty,\quad Zm_0{}^2\to\delta \text{ a finite constant},\quad \Delta_F(k)=-1/\delta . \tag{6}$$

We show in the following that in our model a discrete spectrum corresponding to a rising Regge trajectory cannot be obtained with any of the solutions of Alabiso and Schierholz.[3]

II. THE BS EQUATION IN MOMENTUM SPACE AND REDUCTION TO ONE-DIMENSIONAL FORM

Consider the BS equation for scalar quarks interacting through (1) written in the form

$$\Delta_F^{-1}\left[\frac{P}{2}+p\right]\Delta_F^{-1}\left[\frac{P}{2}-p\right]\phi(p) = -\frac{1}{(2\pi)^4}\int K(p-p')\phi(p')d^4p' , \quad (7)$$

where $\phi(p)$ is the two-body BS amplitude in momentum space; p and P are the relative and total four-momenta, respectively. The major problem in dealing with (7) is that of separation of variables and its reduction to a set of one-dimensional equations. Starting with Salpeter and Bethe,[4] the simplest procedure for achieving this end is the use of the instantaneous approximation with consequent loss of manifest covariance. In the present instance with K given by (2) we show that this separation can be achieved with the retention of manifest covariance. Choosing, for the present, (4) for $\Delta_F(k)$ (the bare quark mass is taken to be zero) and a frame of reference such that $P^2=E^2>0$ we obtain the differential equation

$$\Box_p\phi(p)=\frac{1}{\beta}\left[\left[\frac{E}{2}+p_0\right]^2-\vec{p}^{\,2}\right]\left[\left[\frac{E}{2}-p_0\right]^2-\vec{p}^{\,2}\right]\phi(p) . \quad (8)$$

To separate (8) into one-dimensional equations, we set

$$\phi(p)=\frac{\psi(p_s,p_0)}{p_s}Y_{lm}(\theta_p,\phi_p) , \quad (9)$$

where $p_s{}^2=\vec{p}^{\,2}$, and obtain

$$\left[\left[\frac{\partial^2}{\partial p_0{}^2}-\frac{\partial^2}{\partial p_s{}^2}\right]+\frac{l(l+1)}{p_s{}^2}\right]\psi(p_s,p_0)=\lambda\left[\left[\frac{E}{2}+p_0\right]^2-p_s{}^2\right]\left[\left[\frac{E}{2}-p_0\right]^2-p_s{}^2\right]\psi(p_s,p_0) , \quad (10)$$

where $\lambda=1/\beta$. We introduce a new set of variables ξ and η defined through

$$p_0=E\cosh\xi\cosh\eta ,$$
$$p_s=E\sinh\xi\sinh\eta , \quad (11)$$
$$0\le\xi,\ \ \eta<\infty ,$$

and set

$$\psi(\xi,\eta)=v(\xi)u(\eta) . \quad (12)$$

Equation (10) now separates into the following pair of equations:

$$\left[\frac{\partial^2}{2\xi^2}-\frac{l(l+1)}{\sinh^2\xi}+\kappa\right]v(\xi)=\lambda E^6(\tfrac{9}{16}\sinh^2\xi+\tfrac{3}{2}\sinh^4\xi+\sinh^6\xi)v(\xi) \quad (13)$$

and

$$\left[\frac{\partial^2}{\partial\eta^2}-\frac{l(l+1)}{\sinh^2\eta}+\kappa\right]u(\eta)=\lambda E^6(\tfrac{9}{16}\sinh^2\eta+\tfrac{3}{2}\sinh^4\eta+\sinh^6\eta)u(\eta) . \quad (14)$$

Here κ stands for the separation constant. From (11) we see that the (ξ,η) plane maps the (p_0,p_s) plane only in the domain $0\le p_s<\infty$, $E\le p_0<\infty$. This causes no essential difficulty: since $p_0=E$ is not a singular point of (10), analytic continuation allows us to extend the solutions to the region $0\le p_0\le E$.[5]

III. THE EIGENVALUE PROBLEM AND THE NATURE OF THE SPECTRUM

The eigenvalue problem is now defined by Eqs. (13) and (14) supplemented by the boundary conditions

$$v(\xi)\,|_{\xi\to 0}=u(\eta)\,|_{\eta\to 0}=0 ,$$
$$v(\xi)\,|_{|\xi|\to\infty}=u(\eta)\,|_{|\eta|\to\infty}=0 . \quad (15)$$

It would appear that the four quantum numbers characterizing the eigenstates of the relativistic bound state are provided by the quantum conditions resulting from the ξ,η equations in addition to quantum numbers l and m which specify the angular momentum. Thus we expect that, say, from Eq. (13) the separation constant κ is quantized which when used in conjunction with (14) will lead to quantized energy levels for each value of the confining strength λ. Unfortunately, Eqs. (13) and (14) for $v(\xi)$ and $u(\eta)$ and the associated boundary conditions are identical; both the equations lead to identical quantization formulas. As a result, the energy levels cannot be expressed in terms of two quantum numbers. Stated differently one of the two quantities E and κ remains unquantized. Without loss of generality this may be taken to be E; there is thus a continuous spectrum and we have no discrete bound states.

We emphasize that this result holds in spite of the circumstance that Eqs. (13) and (14) separately possess a discrete spectrum. Consider for instance the s-wave problem. We set (13) in the form

$$\left[\frac{\partial^2}{\partial\xi^2}+\kappa-q(\xi)\right]v(\xi)=0 \tag{16}$$

with

$$q(\xi)\equiv\lambda E^6(\tfrac{9}{16}\sinh^2\xi+\tfrac{3}{2}\sinh^4\xi+\sinh^6\xi),\quad \lambda,E>0\,. \tag{17}$$

Following Titchmarsh[6] who has studied a wide class of eigenvalue problems, we see from (17) that since $q(\xi)$ has no singularity in the range of interest and becomes monotonically unbounded as $\xi\to\infty$, κ has a positive discrete spectrum. However, the same is identically true of Eq. (14) with the same quantization relation and the two equations taken together imply that E can only have a continuous spectrum.

The discrete spectrum for Eqs. (13) and (14) separately may be exhibited explicitly in the weak-coupling limit. Under the transformations

$$v(\xi)=\left[\frac{\sinh\xi}{\xi}\right]^{-l} f(\xi)\,, \tag{18}$$

$$\xi=\lambda^{-1/4}y \tag{19}$$

in the weak-coupling limit ($\lambda\to\infty$), we obtain from (13) the eigenvalue equation

$$\frac{\partial^2 f}{\partial y^2}+\left[\frac{4\nu^2}{3E^3\sqrt{\lambda}}-y^2-\frac{(l+\frac{1}{2})^2-\frac{1}{4}}{y^2}\right]f=0\,, \tag{20}$$

where

$$\nu^2=\frac{\kappa+(l+1)^2}{4}\,.$$

The solution of (20) satisfying the boundary conditions (15) are the associated Laguerre functions:

$$f_{n,l}(y)=y^{l+1}e^{-y^2/2}L_n^{l+1/2}(y^2) \tag{21}$$

with the quantized energy levels given by

$$E^3=\frac{\kappa+(l+1)^2}{3\sqrt{\lambda}(4n+2l+3)},\quad n=0,1,2,\dots\,, \tag{22}$$

where we choose $\kappa>0$. Since the η equation gives an identical relation, κ remains unknown and one obtains a continuous spectrum for E.

We therefore arrive at the main result of our discussion: the covariant BS equation for bound states of the scalar quark-antiquark system with an effective kernel which has the structure of an oscillator potential in four space

$$K(p,k)=-\beta\Box_p\delta^{(4)}(p-k),\quad \beta>0$$

does not possess a discrete spectrum.

IV. CONCLUSION

We conclude by noting the effect of a few variations in the model which bear on this result.

(A) Our result is not altered if bare quark propagators are used in the BS equation and the bare-quark mass is finite. In this case Eq. (13) is replaced by

$$\left[\frac{\partial^2}{\partial\xi^2}-\frac{l(l+1)}{\sinh^2\xi}+\kappa\right]v(\xi)=\lambda E^6[(\mu^2-1)\sinh^2\xi+(2\mu^2-1)\sinh^4\xi+\sinh^6\xi]v(\xi) \tag{23}$$

with

$$\mu^2=\frac{m^2}{E^2}+\frac{5}{4}$$

with an identical equation for η. The structure of (23) is identical to (13) and the result that the eigenvalue spectrum is continuous remains unaltered.

(B) We may consider the use of (5) for the dressed quark propagator. In this case Eq. (8) is replaced by

$$\Box_p\phi(p)=\frac{\lambda'\phi(p)}{[(\frac{1}{2}P+p)^2(\frac{1}{2}P-p)^2]^{1/2}}\ . \tag{24}$$

From (24) we note that λ' is dimensionless. If further the bare quark mass is taken to be zero, the energy eigenvalue remains indeterminate as there is no energy scale present in the theory. Indeed, using

$$\begin{aligned} p_0&=\tfrac{1}{2}E\cosh\xi\cosh\eta\ ,\\ p_s&=\tfrac{1}{2}E\sinh\xi\sinh\eta\ , \end{aligned} \tag{25}$$

E is scaled away from (24) altogether. The same continues to be the true even if one considers the quarks to be massive.

The use of propagator (6) is of no physical interest since it leads to the free Klein-Gordon equation for $\phi(p)$.

(C) The use of a momentum-space cutoff function to reduce the continuum to a discrete spectrum was suggested by Goldstein[7] and Green[8] for the two-nucleon BS problem. Thus using

$$\Box_p[(p-k)^2]^{-1}=-i(2\pi)^2\delta^{(4)}(p-k)\ , \tag{26}$$

we replace Eq. (8), namely,

$$\phi(p)=\frac{i\lambda}{(2\pi)^2}\int\frac{d^4k}{(p-k)^2}\left[\left[\frac{E}{2}+k_0\right]^2-\vec{k}^2\right]\times\left[\left[\frac{E}{2}-k_0\right]^2-\vec{k}^2\right]\phi(k) \tag{27}$$

by

$$\phi(p)=\frac{i\lambda}{(2\pi)^2}\int\frac{d^4k}{(p-k)^2}\left[\left[\frac{E}{2}+k_0\right]^2-\vec{k}^2\right]\times\left[\left[\frac{E}{2}-k_0\right]^2-\vec{k}^2\right]C(k^2,\Omega)\phi(k)\ , \tag{28}$$

where

$$\begin{aligned} C(k^2,\Omega)&=1 \text{ for } k^2<\Omega\\ &=0 \text{ for } k^2>\Omega\ . \end{aligned} \tag{29}$$

Equation (27) can be equivalently written as

$$\Box_p\phi(p)=\lambda\left[\left[\frac{E}{2}+p_0\right]^2-p_s{}^2\right]\times\left[\left[\frac{E}{2}-p_0\right]^2-p_s{}^2\right]\phi(p)C(p^2,\Omega)\ , \tag{30}$$

which upon use of the transformations (11) reduces to

$$\left[\frac{\partial^2}{\partial\xi^2}-\frac{l(l+1)}{\sinh^2\xi}+\kappa_1\right]v_<(\xi)=\lambda E^6(\tfrac{9}{16}\sinh^2\xi+\tfrac{3}{2}\sinh^4\xi+\sinh^6\xi)v_<(\xi)\ \text{ for } \xi<\xi_0$$

and (31)

$$\left[\frac{\partial^2}{\partial\xi^2}-\frac{l(l+1)}{\sinh^2\xi}+\kappa_2\right]v_>(\xi)=0\ \text{ for } \xi>\xi_0\ ,$$

ξ_0 being related to Ω. An identical pair of equations result for $u_<(\eta)$ and $u_>(\eta)$. κ_1 and κ_2 are, in general, independent separation constants. The boundary conditions now are

$$\begin{aligned} &v_<(\xi)=0,\ \ \xi=0\ ,\\ &\frac{d}{d\xi}\ln v^{\rm I}_<(\xi)\bigg|_{\xi=\xi_0}=\frac{d}{d\xi}\ln v^{\rm II}_>(\xi)\bigg|_{\xi=\xi_0}\ , \end{aligned} \tag{32}$$

where $v^{\rm I}_<(\xi)$ and $v^{\rm II}_>(\xi)$ are the solutions of (31) regular at $\xi=0$ and $\xi=\infty$, respectively. A similar pair of equations obtain for $u(\eta)$. Since the separation constants κ_1 and κ_2 are, in general, different the energy levels E would depend not only on the cutoff parameters ξ_0 and η_0 but also on one of the constants, say κ_2, which must generally assume continuous values. Our earlier result thus remains unaltered: the general eigenvalue problem even with a momentum-space cutoff does not possess a discrete spectrum.

(D) It may be remembered that the relativistic hydrogen-atom problem studied in the BS formalism by Wick and Cutkosky[9] possesses a discrete spectrum. The analog of Eq. (27) studied by Cutkosky is

$$\phi(p)=i\pi^2 g^2 \int d^4k \frac{1}{(p-k)^2}\Delta_F\left[\frac{P}{2}+k\right] \times \Delta_F\left[\frac{P}{2}-k\right]\phi(k) . \qquad (33)$$

The essential difference between (27) and (33) lies in the fact that while (33) can also be separated into one-dimensional equations the resulting analogs of (13) and (14) are distinct. Indeed, they are, in a separable set of variables α and β,

$$\frac{\partial^2 g}{\partial\beta^2}+\left[\kappa-\frac{l(l+1)}{\sin^2\beta}\right]g=0 ,$$
$$\frac{\partial^2 f}{\partial\alpha^2}+\left[\kappa-\frac{\lambda}{E^2-m^2\cos^2\alpha}\right]f=0 . \qquad (34)$$

Two distinct quantization relations follow from (34); the energy levels and eigenvalues are characterized by four quantum numbers which possess the appropriate nonrelativistic limit.

(E) It may be argued that our result is to be expected on somewhat general grounds. A configuration—space-time kernel $\sim\beta R^2$ is not positive definite over its entire domain: discrete confined states thus "leak" away in a fashion not possible for a system subject to the three-dimensional oscillator potential. One way of removing this difficulty is to have the kernel vanish not only on the light cone (as the one considered by us does) but also for spacelike R^2 (Ref. 1). Experience with potentials with compact configuration-space support in Schrödinger theory suggests that this procedure may reduce the continuum to a discrete spectrum.

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[5]If we set $y=\sinh^2\xi$ and $v(\xi)=y^{(l+1)/2}g(y)$ Eq. (12) reduces to

$$y(y+1)\frac{\partial^2 g}{\partial y^2}+[(l+\tfrac{3}{2})+(l+2)y]\frac{\partial g}{\partial y} -\lambda E^6 y(y+\tfrac{3}{4})^2 g+\frac{\kappa+(l+1)^2}{4}g=0 .$$

An identical equation follows through the substitution of $\sinh^2\eta=x$ and $u(\eta)=x^{(l+1)/2}f(x)$. The physical significance of the variables x and y can be easily seen by writing them in terms of p_0 and p_s, i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4E^2}[(p_0{}^2-E^2+2Ep_s)^{1/2} \pm(p_0{}^2-E^2-2Ep_s)^{1/2}]^2 .$$

Notice that to cover all physical values of p_0 and p_s, x and y vary over complex values.

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