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Statistical mechanics of instantons in quantum chromodynamics

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We develop a field-theoretic method to investigate the statistical mechanics associated with the instanton-antiinstanton ensemble in Yang-Mills gauge theory. This consists in exact rewriting of the partition function in terms of collective fields and incorporation of short-range forces. In the framework of this effective field theory we clarify the meaning of the continuum medium approximation introduced recently by Callan, Dashen, and Gross. Furthermore, we present a direct treatment of the first-order liquid-gas phase transition which is ultimately responsible for bag creation in quantum chromodynamics.

I. INTRODUCTION

The basic difficulty in understanding quantum chromodynamics stems from the fact that the standard perturbation theory applies only in the large-momentum region and does not teach us anything about the physical spectrum of the theory. Confinement is a nonperturbative phenomenon and is partially understood in the framework of strong-coupling lattice gauge theories. However, there the basic problem then comes in taking the continuum limit in accordance with asymptotic freedom.

In the last few years considerable progress has been made in understanding field theories through the semiclassical path-integral method. The hope is that in the Yange-Mills theory also the semiclassical method represents an improvement over the ordinary perturbation expansion and consequently may offer some insight into the spectrum of the theory. The Euclidean Yang-Mills equations possess instanton solutions, and it has been shown that this configuration plays an important role in understanding the correct vacuum state and the resolution of the U(1) problem.[1,2]

Most of the literature on instantons deals with investigating the properties of single-instanton configurations and then translating this into properties of the ensemble. In this procedure no account of the instanton-antiinstanton interactions is given and this is the basis of the so-called "dilute-gas" approximation, which is valid only when the density of instantons is extremely small. In the real vacuum that is certainly not the case and consequently the incorporation of interactions must be done.

The importance of interactions in the example of the three-dimensional Georgi-Glashow model was demonstrated in the work by Polyakov.[3] There the instanton interactions are Coulombic, and this had the important effect of producing the Debye screening and consequently in generating the mass for the photon.

Now in quantum chromodynamics (QCD) the interaction between an instanton and an antiinstanton is dipolar, and in recent papers Callan, Dashen, and Gross[4,5] pointed out how the incorporation of these effects leads to very important physical consequences. First of all they showed in the framework of a certain approximation that the vacuum is now a paramagnetic medium with a rather large permeability. This then produces a large finite renormalization of the coupling constant and causes the β function to grow rapidly. Furthermore, they argued that in the presence of strong external electric fields the vacuum suffers a first-order phase transition.[5,6] This effect may represent an important step in understanding the baglike nature of hadrons.

The methods employed in deriving the above results consist in approximating the instanton-antiinstanton statistical mechanics by a macroscopic continuum medium and then using mean-field-type methods to calculate its properties. For example, the expression for the permeability μ as a function of the density was obtained by generalizing the well-known Onsager method to the instanton case. This formula for the permeability is then used in the argument for the first-order phase transition.

The purpose of the present paper is to develop a method for investigating directly the statistical mechanics of the interacting instanton-antiinstanton ensemble. This consists in representing the partition function in terms of collective fields and subsequently employing standard field-theoretic methods and approximations.

Based on this collective-field formalism we then proceed to understand and clarify the following important questions:

(1) the degree and nature of approximations involved in the continuum medium approach,

(2) the question about the existence of a first-

order phase transition in direct computation of the instanton-antiinstanton partition function.

After clarifying the above questions we are able in the present approach to develop improved approximation schemes and a complete computational procedure for the instanton-antiinstanton ensemble.

Before proceeding to our calculations let us comment on the recent objection by Witten,[8] who argued in the framework of the two-dimensional σ model[7] that instantions do not survive quantum fluctuations and as such have no relevance to the quantum theory. In a separate paper,[9] we reconsider this problem and demonstrate that this is not the case. What we find is that instantons are in fact present at the quantum level.

The organization of this paper is as follows. In Sec. II we develop the collective-field representation for the partition function of the dipole gas and resolve certain technical difficulties which make this transition nontrivial.

In Sec. III we apply the developed method to the Yang-Mills problem. Realizing the necessity for short-range repulsive forces we describe how they are incorporated in the collective-field formalism. Then we proceed and show how the Onsager-type continuum-medium calculations correspond to neglecting in this field theory all interaction terms, i.e., terms higher than quadratic in the fields. We describe also a variational method which is capable of producing a much better approximation.

In Sec. IV we consider the question of a first-order phase transition in the instanton medium. We demonstrate how in direct field-theoretic expansion of the partition function one obtains a Van der Waals behavior and a liquid-to-gas phase transition.

II. DEVELOPMENT OF THE FORMALISM: THREE-DIMENSIONAL DIPOLE GAS

In this section we develop the field-theoretic representation for the dipole-gas partition function. Since this is not quite straightforward and there are some nontrivial technical points to be understood, we shall consider for pedogogical purposes the three-dimensional dipole gas in what follows. After solving the formal problems and obtaining a workable field-theoretic representation we proceed to answer some of the questions raised in the Introduction about the continuum-medium approach. We shall show how this approximation corresponds to dropping in our nonlinear field theory (which is exact) all interaction terms, i.e., terms higher than quadratic in the fields.

The idea of replacing the many-body integrations by a functional integral is very well known in statistical mechanics. This method is useful for treatment of long-range forces and especially relevant in the case of Coulomb interactions. One version of this procedure (called the random-field method) goes as follows.[12] Consider the grand-partition function

$$Z(\beta)=\sum_{N=0}^{\infty}\frac{1}{N!}z^N\int dx_1\cdots dx_N\exp\left[-\beta\sum_{i<j}V(x_i,x_j)\right], \tag{2.1}$$

where $V(x,y)$ is a long-range two-body potential. Writing

$$\sum_{i<j}V(x_i,x_j)=\frac{1}{2}\left[\sum_{i,j}^{N}V(x_i,x_j)-\sum_{i=1}^{N}V(x_i,x_i)\right], \tag{2.2}$$

and employing the Gaussian integration method

$$\int[d\phi]\exp\left[-\tfrac{1}{2}\int dx\,dy\,\phi(x)V^{-1}(x,y)\phi(y)+i\beta\sum_i\phi(x_i)\right]=\exp\left[-\frac{\beta}{2}\sum_{i,j}V(x_i,x_j)\right], \tag{2.3}$$

we introduce a scalar field $\phi(x)$. Performing the summation in (3.1) one obtains for the partition function

$$Z(\beta)=\int[d\phi]\exp\left[-\tfrac{1}{2}\phi V^{-1}\phi+z\int dx\,e^{\beta V(x,x)/2}e^{i\sqrt{\beta}\phi(x)}\right], \tag{2.4}$$

which is the functional integral representation.

For example, in the case of Coulomb interactions where

$$V(x,y)=\frac{e^2}{|x-y|},\qquad V^{-1}(x,y)=-\frac{1}{4\pi e^2}\nabla_x^2\delta(x-y) \tag{2.5}$$

this method gives the following ϕ-field Lagrangian:

$$L=\tfrac{1}{2}(\nabla\phi)^2-z\exp(ie\sqrt{4\pi\beta}\,\phi). \tag{2.6}$$

Now one develops the perturbation expansion by simply expanding the exponential in (2.6). The reason this approach is so useful lies in the fact that already in the first approximation one obtains the Debye-Hückel theory since the propagator $(-\nabla^2+4\pi\beta e^2)^{-1}$ has a mass. Consequently, there are no infrared divergences in higher orders of this expansion.

Now for the case of dipolar interactions it is straightforward to apply the above method and

obtain formally an effective derivative-coupling Lagrangian. However, as we shall show below certain difficulties arise when one attempts to use this Lagrangian and formulate a perturbation theory. In fact the straightforward way of doing this is incorrect, and for these reasons we shall abandon this method and formulate a different approach free of difficulties.

To explain the above comments let us consider the three-dimensional dipole-dipole interaction

$$V(1,2) = -(3\vec{p}_1\cdot\hat{r}\,\vec{p}_2\cdot\hat{r} - \vec{p}_1\cdot\vec{p}_2)\frac{1}{|\vec{x}_1-\vec{x}_2|^3}, \quad (2.7)$$

where $\hat{r} = (\vec{x}_1-\vec{x}_2)/|\vec{x}_1-\vec{x}_2|$. In order to apply the Gaussian method one writes this in the form

$$V(1,2) = p^2\hat{n}_1\cdot\nabla_{x_1}\hat{n}_2\cdot\nabla_{x_2}\frac{1}{|\vec{x}_1-\vec{x}_2|}, \quad (2.8)$$

where $\nabla_x = \partial/\partial\vec{x}$ and $\hat{n}_i$ are unit vectors. The grand canonical partition function reads

$$Z(\beta) = \sum_{N=0}^{\infty}\frac{z^N}{N!}\int d(1)\cdots d(N)\prod_i e^{\beta V(\hat{n}_i)/2}\exp\left[-\frac{\beta}{2}\sum_{i,j}V(i,j)\right] \quad (2.9)$$

with $d(i) = d\vec{x}_i d\hat{n}_i$ and $V(\hat{n}) \equiv V(\hat{n},\hat{n})$. Now employing the random-field method, the partition function can be written as

$$Z(\beta) = \int[d\phi]\exp\left[-\int d\vec{x}\left(\tfrac{1}{2}\phi(-\nabla^2)\phi - z\int d\hat{n}\, e^{\beta V(\hat{n})/2}\exp[i(4\pi\beta p^2)^{1/2}\hat{n}\cdot\nabla\phi(x)]\right)\right]. \quad (2.10)$$

This is a nonlinear field theory with derivative couplings. Concerning (2.10) we mention that a treatment of dipolar interactions along these lines has been considered in Ref. 13 by Förster in connection with instantons of the two-dimensional O(3) σ model. There, at the classical level, instantons do not interact among themselves and the instanton-antiinstanton interaction is dipolar. To account for the first fact in Ref. 13 it was found necessary to introduce dipoles with imaginary dipole moments and then the resulting Lagrangian represents a slight generalization of (2.10).

To proceed in this approach one expands the exponential in a (2.10) and then the quadratic term reads

$$\int d\vec{x}\,\tfrac{1}{2}[\phi(-\nabla^2)\phi + (4\pi/3)\beta p^2 z(\nabla\phi)^2]. \quad (2.11a)$$

Now it looks natural to write this as

$$\tfrac{1}{2}(1+\alpha)\int\phi(-\nabla^2)\phi, \quad (2.11b)$$

with $\alpha = (4\pi/3)\beta p^2 z$, and conclude that the ϕ-field propagator equals

$$\langle\phi(x)\phi(x)\rangle_0 = \frac{-1}{(1+\alpha)}\left\langle x\left|\frac{1}{\nabla^2}\right|y\right\rangle. \quad (2.12)$$

Unfortunately as we shall demonstrate later, this straightforward procedure is erroneous. Before doing this, let us still explain how one is to compute the dielectric constant in the field-theoretic formulation and see what result follows if one uses (2.12).

To define and compute the dielectric constant ϵ we introduce into the system two external charges (q_1, q_2) located at points $(\vec{y}_1, \vec{y}_2)$, respectively. The charge-dipole interaction in the vacuum reads

$$V_{q\hat{p}} = -qp\hat{n}\cdot\nabla_x\frac{1}{|x-y|}. \quad (2.13)$$

In the partition function we therefore have the additional interaction terms

$$\Delta V = \frac{q_1q_2}{|\vec{y}_1-\vec{y}_2|} - p\sum_{i=1}^{N}\hat{n}_i\cdot\nabla_i\sum_{a=1}^{2}\frac{q_a}{|\vec{x}_i-\vec{y}_a|}, \quad (2.14)$$

and after the Gaussian integration we derive

$$Z(\beta,q_1,q_2) = \exp\left(-\beta\frac{q_1q_2}{|\vec{y}_1-\vec{y}_2|}\right)\times\int[d\phi]\exp\left[-\int dx\, L(\phi,j)\right], \quad (2.15)$$

where

$$L(\phi,j) = \tfrac{1}{2}[\nabla(\phi-j)]^2 + z\int d\hat{n}\exp[i(4\pi\beta p^2)^{1/2}\hat{n}\cdot\nabla\phi], \quad (2.16)$$

with the source

$$j(x) = \frac{\beta p}{(-4\pi\beta p^2)^{1/2}}\sum_{a=1}^{2}\frac{q_a}{|\vec{x}-\vec{y}_a|} \quad (2.17)$$

representing the effect of external charges.

Now the effective charge-charge interaction in this dipolar medium is obtained from the free energy

$$\beta V_{\text{eff}}(y_1,y_2) = -\ln Z(\beta,q_1q_2). \quad (2.18)$$

The dielectric constraint is given by the asymp-

totic behavior of this effective interaction, i.e.,

$$\lim_{|y_1-y_2|\to\infty} V_{\text{eff}} = \frac{1}{\epsilon}\frac{q_1 q_2}{|y_1-y_2|}. \tag{2.19}$$

Let us now consider the quadratic approximation

$$L_{(2)} = \tfrac{1}{2}(1+\alpha)(\nabla\phi)^2 - \nabla\phi\cdot\nabla j + \tfrac{1}{2}(\nabla j)^2. \tag{2.20}$$

Use of the ϕ-field propagator (3.12) then gives

$$\beta V_{\text{eff}} = \beta\frac{q_1 q_2}{|y_1-y_2|} - \frac{1}{2}\left(\frac{\alpha}{1+\alpha}\right)\int j\nabla^2 j, \tag{2.21}$$

which reduces to

$$V_{\text{eff}} = \frac{1}{1+\alpha}\frac{q_1 q_2}{|y_1-y_2|}. \tag{2.22}$$

From this we read the dielectric constant to be $\epsilon = 1+\alpha$ and this is what one obtains in the so-called dilute-gas approximation. The above result makes sense and certainly does not lead one to suspect the method, however, the following argument does.

Consider the convolution

$$C(1,2) = \int dx_3 dn_3 \frac{(3\hat{n}_1\cdot\hat{x}_{13}\hat{n}_3\cdot\hat{x}_{13} - \hat{n}_1\cdot\hat{n}_3)}{|x_1-x_3|^3} \times \frac{(3\hat{n}_3\cdot\hat{x}_{32}\hat{n}_2\cdot\hat{x}_{32} - \hat{n}_3\cdot\hat{n}_2)}{|x_3-x_2|^3}. \tag{2.23}$$

A naive way to evaluate this convolution consists in employing (2.8) and partially integrating to write

$$C(1,2) = -\int dx_3 n_1\cdot\nabla_{x_1}\frac{1}{|x_1-x_3|} \times \int dn_3 (n_3\cdot\nabla_{x_3})^2 n_2\cdot\nabla_{x_2}\frac{1}{|x_3-x_2|}, \tag{2.24}$$

which becomes

$$\left(\frac{4\pi}{3}\right)\int dx_3 n_1\cdot\nabla_1\frac{1}{|x_1-x_3|}n_2\cdot\nabla_2\delta(\vec{x}_3-\vec{x}_2), \tag{2.25}$$

and then after another partial integration the result reads

$$\left(\frac{4\pi}{3}\right)\hat{n}_1\cdot\nabla_1\hat{n}_2\cdot\nabla_2\frac{1}{|x_1-x_2|}, \tag{2.26}$$

which is a rescaled dipolar interaction.

However, this result is erroneous since, owing to the highly long-range nature of the dipole-dipole potential, surface terms in partial integrations do not vanish and give a nonzero contribution, which was missed in the above calculation. To obtain the correct answer the simplest way is to avoid partial integrations and evaluate (2.23) directly. First, for the Fourier transform

$$\int e^{i\vec{k}\cdot\vec{x}}\frac{n_1\cdot n_2 - 3\hat{n}_1\cdot\hat{x}\hat{n}_2\cdot\hat{x}}{|x|^3}d\vec{x}, \tag{2.27}$$

a direct integration gives

$$V(\vec{k}) = \left(\frac{4\pi}{3}\right)(3\hat{n}_1\cdot\hat{k}\hat{n}_2\cdot\hat{k} - \hat{n}_1\cdot\hat{n}_2). \tag{2.28}$$

Then for the convolution we have

$$\int\frac{d\vec{k}}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\left[\int d\hat{n}_3 v(\vec{k},13)v(\vec{k},32)\right], \tag{2.29}$$

which gives the result

$$\int\frac{d\vec{k}}{(2\pi)^3}e^{-ik\cdot(x_1-x_2)}\frac{1}{3}\left(\frac{4\pi}{3}\right)^2[3n_1\cdot\hat{k}n_2\cdot\hat{k} + n_1\cdot n_2], \tag{2.30}$$

obviously different from (2.26).

Having exemplified the fact that partial integrations with neglect of surface terms are dangerous in the case of dipolar interactions,[14,15] we now understand why the propagator (2.12) is not correct. Namely, the step from (2.11a) to (2.11b) involved a partial integration and to see more explicitly what is happening, consider evaluating the propagator by treating the second term in (2.11a) as a vertex and summing the series of insertions. A general term in this series reads

$$\int\left(\prod_{i=1}^{m}dx_i dn_i\right)n_1\cdot\nabla_1\frac{1}{|x-x_1|}n_1\cdot\nabla_1 n_2\cdot\nabla_2\frac{1}{|x_1-x_2|}\cdots n_m\cdot\nabla_m\frac{1}{|x_m-y|}, \tag{2.31}$$

and obviously what we have are nothing but convolutions of dipolar interactions. Doing partial integrations naively corresponds to the propagator (2.12), while the correct answer should be obtained by using iteratively the convolution (2.30). As will be shown later, this has the consequence that for the dielectric constant, one obtains the Clausius-Mosotti formula instead of the dilute-gas formula.

In order to avoid temptations such as (2.11b) we shall now describe a slightly different formalism which is more convenient when dealing with dipolar interactions. In this derivation we shall avoid doing any partial integrations and the field which we introduce will have a physical meaning of a density variable. This procedure is related to the so-called collective-field formalism which was developed originally by Gross, Bohm, and

Pines[16] and also by Bogoliubov and Zubarev.[17]

Let us consider the partition function

$$Z_N(\beta) = \int d(1)\cdots d(N) \exp\left[-\beta \sum_{i<i} V(i,j)\right]. \quad (2.32)$$

We introduce the dipole moment density variable

$$\vec{S}(x) = \sum_{i=1}^{N} \hat{n}_i \delta^{(3)}(\vec{x} - \vec{x}_i) \quad (2.33)$$

through the following δ-function integral:

$$\int [d\vec{S}(x)] \prod_x \delta^{(3)}\left[\vec{S}(x) - \sum_{i=1}^{N} \hat{n}_i \delta(x - x_i)\right] = 1. \quad (2.34)$$

This is inserted into (2.32) and then the interaction term reads

$$\int d\vec{x}\, d\vec{y}\, \vec{S}(x) \cdot \nabla_x \vec{S}(y) \cdot \nabla_y \frac{1}{|x-y|} \equiv \int d\vec{x}\, d\vec{y}\, v[\vec{S}(x), \vec{S}(y)] \quad (2.35)$$

plus an infinite counterterm which comes from identical points. To the order we will be working, it is not needed but we shall keep it in mind.

The partition function (2.32) now becomes

$$Z_N(\beta) = \int [d\vec{S}(x)] J_N[\vec{S}(\cdot)] \times \exp\left\{-\frac{\beta p^2}{2} \int dx\, dy\, v[\vec{S}(x), \vec{S}(x)]\right\}, \quad (2.36)$$

namely, it is expressed as a functional integral over a three-component scalar field $S^a(x)$.

The Jacobian J_N present in (2.36) arises from the fact that we have made a change from N-body integrations to a functional integral and it is given by

$$J_N[S] \equiv \int d(1)\cdots d(N) \prod_x \delta\left(\vec{S}(x) - \sum_{i=1}^{N} \hat{n}_i \delta(x - x_i)\right). \quad (2.37)$$

To write this in the form which allows for explicit evaluation one introduces a new field $\vec{\phi}(x)$ to raise the δ condition and then

$$J_N[\vec{S}] = \int [d\vec{\phi}(x)] \exp\left(-i \int \vec{S} \cdot \vec{\phi}\right) \times \left[\int d\vec{x} \int d\hat{n}\, e^{i\hat{n}\cdot\vec{\phi}(x)}\right]^N. \quad (2.38)$$

For the grand-partition function the summation over N exponentiates the expression in the brackets and consequently the collective-field representation reads

$$Z(\beta) = \int [d\vec{S}\, d\vec{\phi}] \exp\left\{-\frac{\beta p^2}{2} \int dx\, dy\, v[S(x), S(x)] + \int dx \left[iS(x)\cdot\phi(x) + z \int d\hat{n}\, e^{i\hat{n}\cdot\vec{\phi}(x)}\right]\right\}. \quad (2.39)$$

In comparison with the earlier random-field representation, here we have two fields $\vec{S}$ and $\vec{\phi}$. Now, since $\vec{S}$ appears quadratically it can in principle be integrated out giving a Lagrangian for the field $\vec{\phi}$. This differs from (2.10), most importantly in the fact that the present scalar field $\vec{\phi}$ has three components while in (2.10) we had a single field ϕ.

Concerning the computation of the dielectric constant we see that a coupling of two external charges produces the following addition term in the exponent of (2.39):

$$\Delta A = \beta \frac{q_1 q_2}{|y_1 - y_2|} + \beta p \int d\vec{x}\, \vec{S}(x) \cdot \vec{q}(x), \quad (2.40)$$

where

$$\vec{q}(x) = \sum_{a=1}^{2} q_a \frac{(\vec{x} - \vec{y}_a)}{|\vec{x} - \vec{y}_a|^3}. \quad (2.41)$$

The effective charge-charge interaction is then given by

$$V_{\text{eff}} = \frac{q_1 q_2}{|y_1 - y_2|} - \tfrac{1}{2}\beta p^2 \int dx\, dy\, q^i(x) G_{ij}(x,y) q^j(x), \quad (2.42)$$

where G_{ij} represents the exact S-field Green's function, i.e.,

$$G_{ij}(x,y) = \langle S_i(x) S_j(x)\rangle, \quad i,j = 1,2,3. \quad (2.43)$$

Obviously one can only calculate this propagator and the dielectric constant in some approximation and the simplest possibility is to expand perturbatively in $(\beta p^2)^{1/2}$.

The quadratic part of the action in (2.39) reads

$$\frac{\beta p^2}{2} \int dx\, dy\, v[S(x), S(y)] + \int dx \left(-iS\cdot\phi + \frac{z}{2\times 3}\vec{\phi}^2\right). \quad (2.44)$$

Since we only need the S propagator one can eliminate the field ϕ to obtain

$$A_{(2)} = \frac{\beta p^2}{2} \int v[\vec{S}(x), \vec{S}(y)] + \int dx \frac{3}{2z} \vec{S}^2. \quad (2.45)$$

Using the expression (2.28) we see that the momentum-space propagator $G_{ab}(k)$ is obtained by inverting the matrix

$$\left(\frac{4\pi}{3}\beta p^2\right)(3\hat{k}_a\hat{k}_b-\delta_{ab})+\frac{3}{z}\delta_{ab}. \tag{2.46}$$

Making the ansatz

$$\tilde{G}_{ab}(k)=\tfrac{1}{3}z(c_1\delta_{ab}+c_2\hat{k}_a\hat{k}_b), \tag{2.47}$$

we easily find for the coefficients

$$c_1=\frac{1}{1-y},\quad c_2=\frac{-3y}{(1+2y)(1-y)},$$

where

$$y=\frac{4\pi}{9}\beta p^2 z.$$

Let us now evaluate the dielectric constant in this quadratic approximation. The Fourier transform of the effective potential (2.42) reads

$$\tilde{V}_{\text{eff}}(k)=4\pi q_1q_2\frac{1}{k^2}-\beta p^2q_1q_2\sum_{a,b=1}^{3}\phi_a(k)\tilde{G}_{ab}(k)\phi_b(-k), \tag{2.48}$$

where $\vec{\phi}(k)=4\pi i\hat{k}/k$ is the momentum-space charge-dipole interaction (2.41). Substituting the above expression for the propagator we obtain

$$\tilde{V}_{\text{eff}}(k)=\frac{1-y}{1+2y}\frac{4\pi q_1q_2}{k^2}. \tag{2.49}$$

Consequently, in this first approximation the dielectric constant reads

$$\epsilon=\frac{1+2y}{1-y}\quad\text{or}\quad\frac{\epsilon-1}{\epsilon+2}=y. \tag{2.50}$$

This is precisely the well-known Clausius-Mosotti formula which is usually derived using the Lorentz field as the local field in the continuum-medium approximation. We see therefore that this continuum-medium result corresponds in neglecting in the present exact field-theoretic formulation all terms higher than quadratic.

Concerning the Clausius-Mosotti formula one notices the obvious defect that is predicts $\epsilon=\infty$ for a physical density, i.e., $y=1$. One would think that this may be an artifact of the quadratic approximation. However, that is not so and the solution of this difficulty lies in the inclusion of repulsive short-range forces.

Another well-known expression for the dielectric constant is the Onsager formula

$$\frac{(\epsilon-1)(\epsilon+2)}{9\epsilon}=y, \tag{2.51}$$

which does not suffer from the above-mentioned problem. However, in a sense, both these relations are equally good since when comparing with numerical Monte Carlo computations it turns out that they both deviate considerably at higher densities. Consequently, ϵ_{cM} and ϵ_0 represent empirically an upper and lower bound for the exact dielectric constant.

A much better expression for ϵ can be obtained by a variational method which will be explained in the following section, after the incorporation of short-range forces is achieved.

III. INSTANTONS IN QCD: A COLLECTIVE FIELD APPROACH

Let us now start with an investigation of instantons and their physical effects in four-dimensional Yang-Mills field theory. The purpose of this section is twofold: First, we develop the analog field-theoretic representation of the instanton-antiinstanton ensemble, and second, in the framework of this formalism, we clarify the nature of the Onsager-type continuum-medium approximation which was introduced and developed by Callan, Dashen, and Gross.[3,4]

First, we introduce the notation and briefly summarize some results of Ref. 4. The basic building blocks of the semiclassical approximation to the Euclidean functional integral are the instanton-antiinstanton configurations, which for large separations read

$$A_\mu^a(N_+,N_-)=\sum_{i=1}^{N_+}\frac{2\rho_i{}^2R_{ab}(i)\bar{\eta}_{b\mu\nu}(x-x_i)^\nu}{(x-x_i)^2[(x-x_i)^2+\rho_i{}^2]}+\sum_{j=1}^{N_-}\frac{2\bar{\rho}_j{}^2\bar{R}_{ab}(j)\eta_{b\mu\nu}(x-\bar{x}_j)^\nu}{(x-\bar{x}_j)^2[(x-\bar{x}_j)^2+\bar{\rho}_j{}^2]}. \tag{3.1}$$

Here R_{ab} and $\bar{R}_{ab}$ are internal-symmetry matrices describing the orientation of instantons and antiinstantons, and we choose to consider the SU(2) case in what follows. Furthermore, (x_i,ρ_i) and $(\bar{x}_i,\bar{\rho}_i)$ are the positions and scales. We integrate over the positions but for notational simplicity keep the scales fixed to $(\rho,\bar{\rho})$. This can be thought of as some average scales.

Now the instanton-antiinstanton approximation to the Euclidean path integral reads

$$Z=\sum\frac{z(\rho)^{N_+}z(\rho)^{N_-}}{N_+!N_-!}\int\prod_{i=1}^{N_+}(dx_i\,dR_i)\prod_{j=1}^{N_-}(d\bar{x}_j\,d\bar{R}_j)e^{-S_{\text{int}}}. \tag{3.2}$$

The fugacities z and $\bar{z}$ are derived from the instanton action and also the quantum correction of 't Hooft.[18] S_{int} represents the total interaction energy and involves only the interaction between instantons and antiinstantons,

$$S_{\text{int}} = \frac{1}{g^2}\sum_{i=1}^{N_+}\sum_{j=1}^{N_-} V(i,j), \tag{3.3}$$

where the potential reads

$$V(1,2) = 8\pi^2 D^a_{\mu\nu} D^a_{\mu'\nu'} T_{\mu\mu'} T_{\nu\nu'} 1/r^4, \qquad T_{\mu\mu'} = \delta_{\mu\mu'} - q\hat{r}_\mu \hat{r}_{\mu'}. \tag{3.4}$$

This is a dipole-dipole interaction and the dipole moments are given by

$$D^a_{\mu\nu} = -\rho^2 R_{ab}\eta_{b\mu\nu}, \qquad \bar{D}^a_{\mu\nu} = -\bar{\rho}^2 \bar{R}_{ab}\eta_{b\mu\nu}, \qquad a = 1,2,3 \tag{3.5}$$

for the instanton and the antiinstanton, respectively. Obviously ρ^2 and $\bar{\rho}^2$ play the role of dipole moments and g^2 represents the temperature.

The above dipoles one can term non-Abelian since their orientation involves an O(3) matrix. The Abelian version would read

$$D_{\mu\nu} = -\rho^2 \hat{n}_a \eta_{a\mu\nu}, \qquad \bar{D}_{\mu\nu} = -\bar{\rho}^2 \hat{m}_a \eta_{a\mu\nu}, \tag{3.6}$$

with $\hat{n}$ and $\hat{m}$ being unit vectors. The discussion of Ref. 4 was almost entirely Abelian and we mention that in low orders (i.e., when only two orientational matrices are contracted) there is indeed no difference between the two, but in higher orders differences arise. Furthermore, when coupling quarks it is certainly important to consider the non-Abelian version and not the Abelian.

Let us now briefly summarize the continuum-medium approach of Callan, Dashen, and Gross. Since it is rather difficult to deal directly with the statistical-mechanics problem associated with the instanton-antiinstanton ensemble (3.2), they proposed to think in macroscopic terms and compute the properties of the instanton gas using mean-field techniques. In view of the dipole-dipole interaction (3.4) this is a paramagnetic continuum medium and one defines the permeability through

$$\mu = B_{\mu\nu}/H_{\mu\nu}, \tag{3.7}$$

where $B_{\mu\nu}$ represents the magnetic field and $H_{\mu\nu}$ stands for the induced field

$$H_{\mu\nu} = B_{\mu\nu} - 4\pi^2 M_{\mu\nu}. \tag{3.8}$$

The magnetization $M_{\mu\nu}$ is in first approximation obtained by neglecting all the interdipole interactions and taking into account only the interaction with the external field $B_{\mu\nu}$. This then leads to the dilute-gas relation for the permeability, i.e., $\mu = 1+\eta$. An improvement is achieved by invoking the famous Onsager argument that the local field does not equal the field at infinity and can be approximated by the field in a vacuum cavity. The Callan-Dashen-Gross (CDG) result for the latter is

$$B^{\text{local}}_{\mu\nu} = \frac{2}{1+\mu} B, \tag{3.9}$$

and then they obtain the analog of the Onsager formula which now reads

$$\frac{(\mu-1)(\mu+1)}{\mu} = 2\eta. \tag{3.10}$$

Based on this result they proceed to argue that a first-order phase transition occurs when a strong external electric field is introduced into the medium.

Now, concerning the above expression (3.10) for the permeability, one would like to know what kind of approximation this is to the full statistical-mechanics problem associated with the original instanton-antiinstanton partition function (3.2). This question can be rather easily understood in the framework of the collective field approach which we present in what follows.

Inasmuch as we are not going to consider quarks in the present paper, let us proceed using the notationally simpler Abelian dipoles (3.6). The interaction potential can be explicitly written in a more compact form

$$V(i,j) = 8\pi^2\rho^2\bar{\rho}^2 \sum_{a,b=1}^{3} \hat{n}_a(i) v_{ab}(x_i - \bar{x}_j)\hat{m}_b(j) \tag{3.11}$$

with

$$v_{ab}(x-y) = \bar{\eta}_{a\alpha\mu}\frac{(\delta_{\mu\nu} - 4\hat{r}_\mu\hat{r}_\nu)}{r^4}\eta_{b\alpha\nu}. \tag{3.12}$$

We proceed as explained in detail in Sec. II by introducing the density fields

$$\vec{S}_{(+)}(x) = \sum_i \hat{n}_i \delta^{(4)}(x - x_i), \qquad \vec{S}_{(-)}(x) = \sum_j \hat{m}_j \delta^{(4)}(x - \bar{x}_j). \tag{3.13}$$

The total interaction energy then becomes

$$S_{\text{int}} = \frac{8\pi^2}{g^2}\rho^2\bar{\rho}^2 \int dx \int dy \sum_{a,b} S^a_{(+)}(x) v_{ab}(x-y) S^b_{(-)}(y) \tag{3.14}$$

and the field-theoretic representation of the instanton-antiinstanton partition function equals

$$Z = \int \prod_{\epsilon=\pm} [d\vec{S}_\epsilon d\vec{\phi}_\epsilon] \exp\left\{-S_{\text{int}} - i\int dx S\cdot\phi + \int dx\left[z\int d\hat{n}\, e^{i\hat{n}\cdot\phi_{(+)}} + \bar{z}\int d\hat{m}\, e^{i\hat{m}\cdot\phi_{(-)}}\right]\right\}. \tag{3.15}$$

Based on this representation we can now address the problem of computing the permeability. Since here we are dealing with magnetic dipoles there are no charges to test the system but the permeability (and in fact the dielectric constant also) can be defined by alternatively introducing external dipoles. A similar calculation, as the one done in Sec. II, shows that in the quadratic approximation one obtains the four-dimensional analog of the Clausius-Mosotti formula, i.e.,

$$\frac{\mu-1}{\mu+1}=\frac{\eta}{2}. \tag{3.16}$$

The problem of diverging μ is present here also. In connection with this result we now have to confront the fact that the partition function (3.2) involves only dipolar interactions, and that the system is then not stable. What is needed to stabilize the system is a presence of short-range repulsive forces.

Now one may wonder whether there are such forces in the instanton-antiinstanton case. Since one is dealing with configurations which are not exact solutions, the analysis of the short-distance behavior is conceptually and technically rather difficult and has not been done up to now. One way of approaching this problem should be through the formalism of constraints developed in Ref. 19. On the other hand, one can give a rather simple argument for necessity of such forces. Namely, when an instanton and the antiinstanton are close to each other, the configuration looks like the classical vacuum and one would double count. Consequently, it is certainly more appropriate to forbid them from approaching each other by, for example, a repulsive hard-core potential

$$u(r)=\begin{cases}\infty, & r<R\\ 0, & r>R.\end{cases} \tag{3.17}$$

Here R should be thought of as double the radius of the instanton.[20]

Let us now describe how one is to incorporate this shoft-range force. In order to clarify the statements made at the end of Sec. II, consider the three-dimensional case first. The total potential now contains two parts $V_{\rm tot}=V(1,2)+u(1,2)$ and, furthermore, one knows that collective fields are useful for long-range forces only. Consequently, the hard-core part requires an explicit treatment and the partition function is now written in the form

$$Z(z,\beta)=\int[d\vec{S}\,d\vec{\phi}]\exp\left(-\frac{\beta}{2}\vec{S}\cdot V\cdot\vec{S}-i\vec{S}\cdot\vec{\phi}\right)\left\{\sum\frac{z^N}{N!}\int dx_1\cdots dx_N\prod_i\left(\int d\hat{n}\,e^{i\hat{n}\cdot\vec{\phi}(x_i)}\right)\exp\left[-\beta\sum_{i<1}u(x_i,x_j)\right]\right\}. \tag{3.18}$$

Denoting

$$\int d\hat{n}\,e^{i\hat{n}\cdot\vec{\phi}(x)}=e^{\phi(x)}, \tag{3.19}$$

we see that the expression in the curly brackets represents the total partition function of the hard-core problem in an external field $\phi(x)$:

$$Z[z,\phi(\cdot)]=\sum_{N=0}^{\infty}\frac{z^N}{N!}\int dx_1\cdots dx_N\exp\left[-\beta\sum_{i<j}u(x_i,x_j)+\sum\phi(x_i)\right]. \tag{3.20}$$

This is formally written in terms of exact hard-core correlation functions as

$$Z[z,\phi(\cdot)]=\exp[\beta V p_{\rm hc}(z)]\exp[\Delta(\phi)], \tag{3.21}$$

where $p_{\rm hc}$ is the exact pressure and

$$\Delta(\phi)=\sum_{m=1}^{\infty}\frac{1}{m!}\int dx_1\cdots dx_m\mu(x_1,\ldots,x_m)\sum_{i=1}^{m}\phi(x_i). \tag{3.22}$$

Here the μ's are the so-called modified Ursell distribution functions and are defined as follows.[21] First, one has the ordinary correlation functions

$$\rho(1,\ldots,m)=\frac{1}{Z_{\rm hc}}\sum_N\frac{z^N}{(N-m)!}\times\int d(m+1)d(N)\times\exp\left[-\beta\sum_{i<j}u(i,j)\right], \tag{3.23}$$

which in terms of functional derivatives over $z(x)=\exp[\phi(x)]$ read

$$\rho(1,\ldots,m)=\prod_i z(i)\frac{\delta^m Z(\phi)}{\delta z(1)\cdots\delta z(m)}\bigg|_{\phi=0}. \tag{3.24}$$

The connected parts are the Ursell functions and

are given by

$$u(n,\ldots,m)=\prod_i z(i)\frac{\delta^m \ln Z(\phi)}{\delta z(1)\cdots\delta z(m)}\bigg|_{\phi=0}. \quad (3.25)$$

Finally, the modified Ursell functions are appropriate combinations of

$$\mu(x)=u(x),$$
$$\mu(x_1,x_2)=u(x_1,x_2)+\delta(x_1-x_2)u(x_1), \quad (3.26)$$

and they satisfy

$$\mu(1,\ldots,m)=\frac{\delta^m nZ(\varphi)}{\delta\varphi(x_1)\cdots\delta\varphi(x_m)}\bigg|_{\phi=0}, \quad (3.27)$$

which is equivalent to (4.21).

The total action in (4.18) now equals

$$\tfrac{1}{2}\beta\vec{S}\cdot V\cdot\vec{S}+i\vec{S}\cdot\vec{\phi}+\Delta[\phi], \quad (3.28)$$

and comparing with the earlier expression (2.39) we see that the presence of short-range forces induces an infinite set of vertices given by $\Delta[\phi]$.

Before proceeding to demonstrate the relevance of these short-range contributions let us mention an important point. Since the total potential now contains a hard-core part obviously the long-range part is not uniquely defined inside the core. Namely, in (3.18) we can write

$$V(1,2)=\begin{cases} p^2v(1,2), & |x_1-x_2|>R \\ w(1,2), & |x_1-x_2|<R. \end{cases} \quad (3.29)$$

Now on the exact level nothing depends on this arbitrary potential $w(1,2)$. However, in any approximate calculation things do depend on $w(1,2)$. For example, the quadratic part of the action (3.28) now reads

$$A_{(2)}=\frac{\beta}{2}\vec{S}\cdot(p^2v+w)\cdot\vec{S}+\frac{3}{2\rho}\vec{S}^2, \quad (3.30)$$

where v is the cutoff dipolar potential and we have explicitly written out the dependence on w. In deriving (3.30) we have integrated ϕ out and used the fact that

$$\phi(x)=\ln\int d\hat{n}\, e^{i\hat{n}\cdot\phi}\approx -\tfrac{1}{6}\phi^2(x), \quad (3.31)$$

with $\mu(x)=\rho$ being the density. The $\vec{S}$-field propagator in momentum space is now obtained as the inverse of the 3×3 matrix

$$M_{ab}(w)=\beta p^2 v_{ab}(k)+\beta\tilde{w}_{ab}(k)+\frac{3}{\rho}\delta_{ab}. \quad (3.32)$$

Now the dielectric constant is obtained, as explained in Sec. II, from the small momentum behavior of the propagator. Denoting

$$\lim_{k\to0}\tilde{w}_{ab}(k)=\frac{4\pi}{3}p^2w_0\delta_{ab}, \quad (3.33)$$

and using the fact that at small momenta the Fourier transform of the cutoff dipolar potential equals (2.28), we have for the propagator in this limit the form

$$\tilde{G}^{-1}{}_{ab}(k\to0)=\frac{4\pi}{3}\beta p^2[3k_ak_b-(1-w_0)\delta_{ab}]+\frac{3}{\rho}\delta_{ab}. \quad (3.34)$$

The corresponding dielectric constant now equals

$$\frac{\epsilon-1}{\epsilon+2}=\frac{y}{1+w_0y}, \quad (3.35)$$

and we see the explicit dependence on $w(1,2)$.

Now the question is how should one fix this arbitrariness. First, obviously there is a specific choice for w_0 so that (3.35) reproduces the Onsager formula for the dielectric constant given by (2.51). However, there is no special reason for this choice and the correct procedure which leads to a much better approximation to ϵ is to treat $w(x)$ as a variational function.

Namely, one can turn this arbitrariness of $w(1,2)$ into an advantage and variationally determine the best possible choice. What one has to optimize is obviously the total pressure of the system. In the present approximation it is obtained by performing the $\vec{S}$-field Gaussian integration and then

$$\exp(Vp)=\exp(Vp_{\rm hc})(\det M(w))^{-1}. \quad (3.36)$$

Consequently, w is to be determined from the variational equation

$$\frac{\delta p(w)}{\delta w(1,2)}=\frac{\delta\,\mathrm{Tr}\ln M(w)}{\delta w(1,2)}=0. \quad (3.37)$$

In view of the fact that the S-field Green's function equals

$$G=M^{-1}=\left(\beta V+\frac{3}{\rho}1\right)^{-1}, \quad (3.38)$$

the above equation is nothing but the requirement that

$$G_{ab}(x;y)=0 \quad \text{for} \quad |x-y|<R. \quad (3.39)$$

One can understand these two equations (3.38) and (3.39) as a system of equations for $G(x,y)$ and $V(1,2)$. The unknowns are $V(1,2)$ inside the core, i.e., for $|x-y|<R$ and $G_{ab}(x,y)$ outside the core, i.e., for $|x-y|>R$.

Concerning the above equations and their solution, we mention that they are analogs in the present language to the well-known Percus-Yevick (PY) approximate integral equation for classical fluids. To summarize, the PY integral equation is defined as follows.[21] One considers the two-point correlation function

$$g(1,2)=\frac{1}{\rho^2}\rho(1,2)\equiv 1+h(1,2), \tag{3.40}$$

and also the Ornstein-Zernire direct correlation function $c(1,2)$ which is defined through the integral equation

$$h(1,2)=c(1,2)+\rho\int c(1,3)h(3,2)d(3). \tag{3.41}$$

Now the Percus-Yevick approximation demands that

$$h(1,2)=-1 \quad \text{for } |x_1-x_2|<R, \tag{3.42}$$

and

$$c(1,2)=-\beta p^2 v(1,2) \quad \text{for } |x_1-x_2|>R. \tag{3.43}$$

The first requirement is in fact exact since $g(1,2)=0$ for $r<R$ when the hard core is present while the second equation (3.43) is the approximation.

Now the integral equation can be written as

$$\hat{g}=1+\frac{1}{1-\rho\hat{c}}\hat{c} \tag{3.44}$$

and in this form we see the equivalence to our equations. Specifically $G_{ab}(x,y)$ is the analog of $g(1,2)$, and $V(1,2)$ stands for $c(1,2)$.

An advantage of the Percus-Yevick system is that it can be solved exactly for the present case of three-dimensional dipolar interactions. This was done explicitly by Wertheim,[22] who showed that the solution can in fact be given in terms of the solution to the pure hard-core problem, in which case

$$c(1,2)=0 \quad \text{for } |x_1-x_2|>R \tag{3.45}$$

replaces (3.43). Now the analytic solution to the pure hard-core PY integral equation was achieved by the same author earlier and is given in terms of a finite power series.

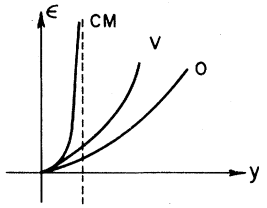

FIG. 1. Comparison of various approximate expressions for the dielectric constant: CM, V, and O stand for the Clausius-Mosotti, variational, and the Onsager curve, respectively.

The dielectric constant obtained in this approximation lies in between the Clausius-Mossotti and Onsager curves (see Fig. 1) and compares quite well with numerical Monte Carlo computations. For more details and a review of the subject the reader is referred to Ref. 23.

Having understood how one has to incorporate short-range forces and also their physical implications, let us now summarize the analogous treatment of the instanton-antiinstanton ensemble. First, for the canonical partition function we have

$$Z(N_+,N_-)=\int [d\vec{S}_+ d\vec{S}_-]J[S]\exp\left(-\frac{8\pi^2}{g^2}\bar{\rho}^2\rho^2\vec{S}_+\cdot v\cdot S_-\right). \tag{3.46}$$

Here, the Jacobian which is obtained in transition from the $N=N_++N_-$ body integrations to the functional integral is given by

$$J[S]=\int\prod_i^{N_+}d(i)\prod_j^{N_-}d(j)\prod_x\delta\left(S_+(x)-\sum_i\hat{n}_i\delta(x-x_i)\right)\delta\left(S_-(x)-\sum\hat{m}_j\delta(x-\bar{x}_j)\right)\exp\left[-\beta\sum_{i,j}u(x_i,\bar{x}_j)\right]. \tag{3.47}$$

Introducing two scalar fields $(\vec{\phi}_+,\vec{\phi}_-)$ and the notation

$$e^{\phi_\pm(x)}=\int d\hat{n}\,e^{i\hat{n}\cdot\vec{\phi}_\pm(x)}, \tag{3.48}$$

we have for the Jacobian the expression

$$J[S]=\int[d\vec{\phi}_+ d\vec{\phi}_-]\exp\left(-i\int\vec{\phi}\cdot\vec{S}\right)Z_{\rm hc}[\phi_+,\phi_-], \tag{3.49}$$

where

$$Z_{\rm hc}[\phi_+,\phi_-]=\int\prod_i(dx_i)\prod_j(dx_j)\exp\left[-\beta\sum_{i<j}u(x_i-\bar{x}_j)+\sum_i\phi_+(x_i)+\sum_j\phi_-(\bar{x}_j)\right]. \tag{3.50}$$

Consequently, the Jacobian is obtained as a Fourier transform of the pure, hard-core partition function with external potentials $\phi_+(x)$ and $\varphi_-(x)$ present.

Next, for the grand partition function we obtain the following representation:

$$Z(z,\bar{z}) = \exp(\beta V p_{\rm hc}) \int \prod_{\epsilon=\pm} dS_\epsilon \, d\phi_\epsilon \exp\left\{ -\frac{8\pi^2}{g^2}\rho^2\bar{\rho}^2 \int \vec{S}_+ \cdot v \cdot \vec{S}_- - i \int \vec{S}\cdot\vec{\phi} + \sum_{M=1}^{\infty} \frac{1}{M!} \sum_{(\epsilon)} \int (dx)\mu(x_1,\epsilon_1,\ldots,x_M,\epsilon_M) \prod_i \phi_{\epsilon_i}(x_i) \right\}. \tag{3.51}$$

Here $p_{\rm hc}$ and $\mu(x_1,\ldots,x_M)$ represent the pressure and the correlation functions characterizing the pure hard-core system of two types of particles. As mentioned before, only different types interact.

It is this improved field-theoretic representation which takes into account both the long- and the short-range forces that are appropriate for investigating the properties of the instanton medium. For example, making the quadratic approximation for the action in (3.51) we can easily reproduce the CDG formula for the permeability. This computation proceeds in precisely the same fashion as the three-dimensional case.

What one is more interested in is deriving the analog of the variational formula. As in the three-dimensional case, we have to solve in this way the analog of the Percus-Yevick integral equation. However, in attempting to solve this equation we have found that in four dimensions it does not allow for a closed analytic solution. What happens is that the series expansion in this case does not terminate and, in fact, this is true for any even number of dimensions. Consequently, in order to obtain an improved variation expression for the permeability one would need to use approximate numerical methods.

IV. LIQUID-GAS PHASE TRANSITION OF THE INSTANTON MEDIUM

In the vacuum, instantons and antiinstantons are densely packed and one can think of this medium as a liquid. In recent works[4,5] the important observation was made that in the presence of strong electric fields the instanton-antiinstanton medium suffers a first-order phase transition to a more dilute phase. From this observation there emerges then a baglike picture of hadrons in QCD.

The argument of Callan, Dashen, and Gross consists of employing their formula (3.10) for the permeability as a function of the density and deriving an approximate dependence of the density on the external electric field E. In this way they derive in some sense a self-consistent relation between the induced field D and E. Plotting the curve $D(E)$ one finds that for a critical value of the electric field, a first-order phase transition takes place.

It is our purpose to present in this section a more direct derivation of this phase transition. We consider directly the instanton-antiinstanton partition function and since we have incorporated both short-range and long-range forces there is obviously a possibility for a first-order liquid-gas phase transition.

It is well known that the liquid-gas phase transition arises due to a competition between repulsive short-range forces and attractive long-range forces. The standard mean-field theory of such phase transitions is given by the Van der Waals equation.

Let us first summarize how the Van der Waals theory is derived for usual potentials. We consider the grand partition function corresponding to a system with the long-range potential $v(1,2)$ and a repulsive core $u(1,2)$. In the random-field representation it reads

$$Z(\beta,z) = z_{\rm hc}(\beta,z) \int [d\phi] \exp\left\{ \frac{1}{2\beta}\phi v^{-1}\phi + \sum_m \frac{1}{m!} d(1)\cdots d(m)\mu(1,\ldots,m)\phi(1)\cdots\phi(m) \right\}. \tag{4.1}$$

The pressure and the density are given by

$$p = \frac{1}{\beta V}\ln Z, \quad \rho = \frac{1}{V} z \frac{\partial \ln Z}{\partial z}, \tag{4.2}$$

where V stands for the volume. Using the second equation, one can eliminate the fugacity z in favor of the density and obtain the equation of state $p = p(\rho,\beta)$.

The Van der Waals equation of state after Maxwell's construction has the well-known form given in Fig. 2. For densities larger than

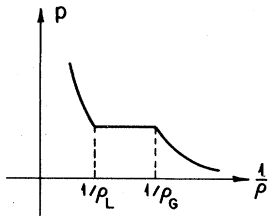

FIG. 2. Equation of state exhibiting the first-order liquid-gas phase transition.

ρ_L one is in the liquid phase and for $\rho<\rho_G$ the system is in the gaseous phase.

Now, in the framework of the functional integral representation (4.1) the Van der Waals behavior arises already at the tree-approximation level. In this approximation one considers ϕ to be constant, i.e., space independent, and then using the identity

$$Z_{\rm hc}(z,\beta)\exp\left\{\sum_m \frac{1}{m!}\int \mu(1,\ldots,m)\phi^m\right\}=Z_{\rm hc}(ze^{\phi},\beta), \tag{4.3}$$

the total pressure becomes

$$\beta Vp=\beta Vp_{\rm hc}(ze^{\phi},\beta)+\frac{\phi^2}{2\beta}\int dx\,dy\,v^{-1}(x,y). \tag{4.4}$$

The appropriate value of ϕ is the classical stationary point of the action and is determined from

$$\frac{\partial p}{\partial\phi}=0. \tag{4.5}$$

Denoting $\phi\to\phi-\mu$, $\mu=\ln z$, and also

$$v_0=\frac{1}{V}\int d(1)\,d(2)v(1,2), \tag{4.6}$$

this equation becomes

$$\frac{\phi-\mu}{(-\beta v_0)}=\rho_{\rm hc}(e^{\phi}). \tag{4.7}$$

Now to solve this equation for ϕ one would really first need to know the exact hard-core pressure, but for understanding the qualitative behavior it is sufficient to know that it has the approximate form depicted in Fig. 3. From this we see that for attractive potentials ($v_0<0$) and when we are below a certain critical temperature, depending on the value of the chemical potential μ, we have one or three solutions for φ. Namely, for smaller values of μ only a single solution for ϕ exists, while for larger μ's two additional stationary points appear. The correct one is chosen by demanding that the pressure be maximal. Consequently, in increasing the value of μ ϕ changes

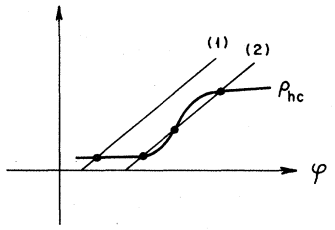

FIG. 3. Graphical representation of Eq. (4.7) showing the multiplicity of solutions as a function of μ: (1) for $\mu<\mu_c$ we have a single solution and (2) for $\mu>\mu_c$ three distinct solutions are possible. The right one follows from an area rule.

continuously on them at some μ and a discontinuous jump occurs to the upper solution. Since the density of the system can be shown from (4.2) to equal

$$\rho=\rho_{\rm hc}(e^{\phi}), \tag{4.8}$$

we see that this implies a sudden jump in the density and a gas to a liquid-phase transition. For a more detailed discussion the reader should consult Ref. 24.

Now, in the present case of dipolar interactions, the situation is not as simple as above and there is no phase transition in the tree approximation. Owing to orientational integrations the analog of v_0 now vanishes, i.e.,

$$v_0=\int dx\int d\hat{n}_1 d\hat{n}_2 v(|x|;\hat{n}_1,\hat{n}_2)=0. \tag{4.9}$$

However, as we shall show, the Van der Waals phase transition, nevertheless, arises in the instanton medium and now comes from higher quantum corrections in our field-theoretic language.

To obtain immediately the equation of state let us work in the canonical ensemble with $n_{\pm}=N_{\pm}/V$ denoting the instanton and the antiinstanton densities. Keeping the first two terms in the expansion for $Z_{\rm hc}[\phi]$ given in (3.49) we have

$$Z_{\rm hc}[\phi]=Z_{\rm hc}[\phi=0]\exp\left\{\sum_{\epsilon} n_{\epsilon}\int \phi_{\epsilon}(x)+\tfrac{1}{2}\sum_{\epsilon_1\epsilon_2}\int dx_1 dx_2 h_{\epsilon_1\epsilon_2}(x_1-x_2)\phi_{\epsilon_1}(x_1)\phi_{\epsilon_2}(x_2)\right\}, \tag{4.10}$$

where h represents the hard-core two-point function. Consequently the action of the canonical partition function (3.46) can be approximated as

$$A\approx\frac{8\pi^2}{g^2}\rho^2\bar{\rho}^2\vec{S}_+\cdot v\cdot\vec{S}_- +i\vec{S}\cdot\vec{\phi}-\tfrac{1}{6}\int(n_+\vec{\phi}_+^{\,2}+n_-\vec{\phi}_-^{\,2})$$

$$+\frac{1}{2}\times\frac{1}{6^2}\int h_{\epsilon_1\epsilon_2}(x_1-x_2)\vec{\phi}_{\epsilon_1}^{\,2}(x_1)\vec{\phi}_{\epsilon_2}^{\,2}(x_2), \tag{4.11}$$

where we have used the definition (3.48) to approximate

$$\phi_{\pm}\approx-\tfrac{1}{6}\vec{\phi}_{\pm}^{\,2}(x). \tag{4.12}$$

Since in the action (4.11), ϕ_{ϵ} appears as a quartic term we now integrate out $\vec{S}_{\epsilon}$ and obtain the following propagator for the ϕ field:

$$\langle\phi_+^a(x_1)\phi_-^b(x_2)\rangle_0=v_{ab}(x_1-x_2), \tag{4.13}$$

which is nothing but the instanton-antiinstanton interaction (3.12).

The perturbation expansion based on the above propagator is a high-temperature expansion and that is the reason we did not include the $n_\epsilon \phi_\epsilon^2$ terms in the propagator but treat them as vertices. The high-temperature expansion is very useful since in coalition with the Padé methods it can lead to extremely good approximations for the equation of state.

Let us now evaluate only the first term in this expansion, which as will be seen already exhibits the Van der Waals behavior. It is given by the two graphs shown in Fig. 4. These two graphs come, respectively, from the last two terms in (4.11) and taken together give the following correction to the Helmholtz free energy:

$$F_1 = -\beta\rho^2\bar{\rho}^2 n_+ n_- (\tfrac{1}{6})^2 \int dx_1 dx_2 g_{+-}(x_1 - x_2) \times \sum_{ab} [v_{ab}(x_1 - x_2)]^2 . \quad (4.14)$$

Here,

$$g_{+-}(x_1 - x_2) = 1 + h_{+-}(x_1 - x_2) . \quad (4.15)$$

Using the properties of the 't Hooft tensors which enter the definition of v_{ab} one easily simplifies and

$$\sum_{ab} v^{ab}(x_1 - x_2)^2 = 12 \frac{1}{|x_1 - x_2|^8} . \quad (4.16)$$

Consequently, in this approximation the total free energy reads

$$F = F_{hc} + F_1 = F_{hc} - \tfrac{1}{3}\beta\rho^2\bar{\rho}^2 n_+ n_- V \int d^4x \frac{1}{r^8} g_{+-}(r) . \quad (4.17)$$

Here the second term effectively resembles a contribution of an attractive $1/r^8$ potential and therefore we have obtained the Van der Waals behavior.

The approximate result (4.17) exhibiting the liquid-gas phase transition is obviously of qualitative nature only. However, we mention that one can obtain a much more accurate numerical answer for the equation of state by simply evaluating the next term in the large temperature series and employing the Padé methods.

Let us now summarize the implications of this result on the behavior of the instanton-antiinstanton medium. First of all, the vacuum is described by the grand-canonical partition function with fixed fugacities $(z, \bar{z})$, and the unperturbed system is in the liquid state. It is plausible that this instanton liquid state represents the intermediate phase in transition from weak to strong coupling.[25,26]

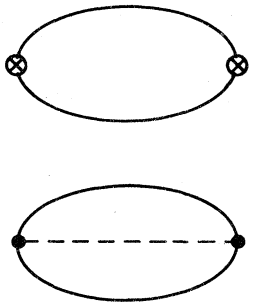

FIG. 4. Feynman graphs which give the first correction to the Helmholtz free energy.

The above demonstrated Van der Waals behavior means that changing by hand the fugacities one can induce a first-order phase transition to a gaseous state. In Ref. 4 it was argued that a constant external electric field can cause such a transition. Ultimately, what one would like to do is to introduce external quark sources and see how they induce a space-dependent variation of the fugacity and consequently a gaseous phase in their vicinity.

The present formulation offers an effective computational framework for discussion of this problem and the associated baglike hadron states at least for the case of heavy quarks. As the first step, one is to couple point quarks to the Yang-Mills field which can be done, for example, by following the work of Ref. 27. Then one needs to deduce how this quark degrees of freedom couple to the collective fields and write down the effective scalar-field theory in presence of quark sources. In the framework of this effective field theory one should proceed using soliton-type methods and, after an analytic continuation, look for real time baglike classical solutions. A WKB quantization procedure would then produce the bound-state spectrum.

The execution of the above steps is not a trivial task and obviously a considerably amount of hard work is needed to achieve a complete understanding of the instanton-generated spectrum. However, we believe that the necessary ingredient for such a picture, namely, the existence of a phase boundary, is certainly generated by instantons in this semiclassical approach.

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