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# Comments on the classical gauge field with sources

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Two results for the classical SU(2) gauge field interacting with a static external source are given: (1) The smooth solutions associated with a smooth non-Abelian source are non-Abelian. (2) The minimum-energy field configuration for a smooth source is not smooth. It exhibits a pathological property we refer to as high-wave-number screening.

## I. INTRODUCTION

We present two results for the classical SU(2) gauge field in interaction with a static external source[1]

$$j^\mu(t,\vec{x}) = \delta^{\mu 0}\rho(\vec{x}). \tag{1.1}$$

The first says that the smooth solutions associated with a smooth non-Abelian source are non-Abelian and the second that the minimum-energy field configuration for a smooth source is not smooth.

A precise statement of the results requires some definitions. A *good function*[2] is one which is everywhere differentiable any number of times, and all of its derivatives are

$$O(|x|^{-N}) \text{ as } |x| \to \infty \text{ for all } N. \tag{1.2}$$

A *good source* will be one for which each of the functions $\rho^A(\vec{x})$ is good. We will say that two good sources $\rho$ and $\rho'$ are *gauge equivalent* if

$$\rho' = g^{-1}\rho g, \tag{1.3}$$

with $g$ a *smooth* SU(2) gauge transformation. A source is said to be in *Abelian form* if

$$\rho^A = \delta^{A3}\rho^3. \tag{1.4}$$

A good source is an *Abelian source* if it is gauge equivalent to a good source in Abelian form. Otherwise, it is a *non-Abelian source.* A gauge field will be said to be an *Abelian* or *non-Abelian field* according to whether its holonomy group[3] is Abelian or non-Abelian. With this tedium behind us, we return to the introductory discussion.

Many authors[4] have discussed the solutions to the field equations with static sources. However, little attention has been given to the geometry[5] of the source. In most cases, it has been assumed (either explicitly or implicitly) that the source is Abelian. But what can be said about the gauge field solution if it is known that the source is non-Abelian? We find that the smooth solutions (whether static or not) must be non-Abelian. In this way, the gauge field inherits a geometrical property of its source. Abelian sources admit both Abelian and non-Abelian solutions (Sikivie and Weiss, Ref. 4).

To discuss the second result, we begin by considering the simple Abelian solution for an Abelian source. When the source is in Abelian form, this is a Coulomb solution. Mandula[6] has shown that for sufficiently strong sources this solution is unstable. So what is the form of the field at the global minimum of the energy? A body[4] of recent work can be interpreted as contributions to the understanding of this question. We will give a general result which shows that the problem is pathological.

The system is the energy functional

$$H = \int d^3x \tfrac{1}{2}(E_i^A E_i^A + B_i^A B_i^A), \tag{1.5}$$

with constraint

$$\partial_i E_i^A + \epsilon_{B\,C}^{\;\;A} A_i^B E_i^C = \rho^A. \tag{1.6}$$

Consider it defined on the space of smooth $E$ and $A$ and assume only that $\rho$ is good. Our result is that $H$ does not have a global minimum on the space of smooth $E$ and $A$ constrained by (1.6). Indeed on this space, which will be referred to as $S$, $H$ can be made smaller than any positive number. We refer to this effect as high-wave-number screening. It implies that there is no force between good sources at the classical level.

In earlier discussions[7] of this problem, the proofs were given for Abelian sources. Our work is based on the same idea, but the treatment is simpler, more precise, and valid for all good sources. It shows that the earlier results represent an important general insight into the problem of the gauge field with static external sources. A discussion of the relationship of our contribution to the work in Ref. 7 appears in Sec. III.

The result does not apply to singular (e.g., point) sources.[8] However, we can see that it will be very difficult to treat the point-charge problem as the limit of the problem with a smooth source.

It is also worth noting that the second result makes the first seem somewhat irrelevant. The restriction to smooth solutions may be ill-motivated. This is discouraging because a systematic treatment of fields with complicated singularities will be very difficult.

Finally, we may ask how the second result will be expressed in a quantum-mechanical treatment of the field with a classical source. Section IV contains the speculation that the vacuum energy will be independent of the source in order $\hbar^0$. A very simple model is discussed.

## II. NON-ABELIAN SOURCES AND NON-ABELIAN FIELDS

It will be shown that non-Abelian sources require non-Abelian fields. But first an example of a non-Abelian source will be given. Consider

$$\rho^A(\vec{x}) = x^A \sigma(|\vec{x}|), \tag{2.1}$$

with $\sigma$ some good function satisfying

$$\sigma(y) = 0 \text{ for } y < R_0 . \tag{2.2}$$

A proof by contradiction shows that (2.1) is non-Abelian. Assume that (2.1) is Abelian. Then there exists a smooth $g$ such that

$$\rho'^A = \delta^{A3}|\vec{x}|\sigma = (g^{-1}\rho g)^A . \tag{2.3}$$

Now consider a fixed $|\vec{x}| = R > R_0$ for which $\sigma(R) \neq 0$. Restricted to this sphere, $\rho$ and $\rho'$ each induce maps $S^2 \to S^2$. They are in different homotopy classes. The restriction of $g$ to the sphere $g|_R$ is a map $S^2 \to S^3$. Since $\pi_2(S^3)$ is trivial, we can find a homotopy which connects 1 and $g|_R$. This will provide a homotopy between $\rho$ and $\rho'$. But that is impossible; so (2.1) is a non-Abelian source. An attempt to force the issue will net a $g$ with a string singularity. An example of this is the string singularity of the SU(2) magnetic monopole in the gauge where the scalar field is

$$\phi^A = \delta^{A3}\phi^3 . \tag{2.4}$$

Recall that the *holonomy group* at $x$ of a gauge field $A$ is all elements of the gauge group obtained by varying the expression

$$P \exp\left(\int dx^\mu A_\mu\right)$$

over all closed paths beginning and ending at $x$. It is easy to show[9] that the holonomy groups at different points are isomorphic. Further,[9] if the smooth field $A$ has an Abelian holonomy group, then a smooth gauge transformation will take $A$ to an Abelian form

$$A'^A_\mu = \delta^{A3} A'^3_\mu . \tag{2.5}$$

Now suppose that $A$ is a smooth solution to the field equations

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + \epsilon_{BC}{}^A A^B_\mu A^C_\nu , \tag{2.6}$$

$$\partial_\mu F^{A\mu\nu} + \epsilon_{BC}{}^A A^B_\mu F^{C\mu\nu} = j^{A\nu} , \tag{2.7}$$

that the current is given by (1.1), and that $\rho$ is non-Abelian and good. Then $A$ must be non-Abelian. To prove this, we assume the opposite, i.e., that $A$ is Abelian. Then gauge transform $A$ to an Abelian form and the field equations become

$$\delta^{A3}\partial_\mu F^{3\mu\nu} = \delta^{\nu 0}(g^{-1}\rho g)^A . \tag{2.8}$$

This implies that $\rho$ is Abelian and is a contradiction with the assumptions.

Thus, a non-Abelian source requires a non-Abelian field and does not admit an Abelian solution. These results require the smoothness assumptions. They are not true if singular fields are allowed.

## III. MINIMIZING THE ENERGY

The problem of finding the absolute minimum of the energy

$$H = \int d^3x \tfrac{1}{2}(E^A_i E^A_i + B^A_i B^A_i) , \tag{3.1}$$

subject to the constraint

$$\partial_i E^A_i + \epsilon_{BC}{}^A A^B_i E^C_i = \rho^A , \tag{3.2}$$

with $\rho$ static and good, will be discussed in this section. It is known[10] that a minimum-energy configuration with reasonable properties is a static solution to the field equations. (The time evolution is a gauge transformation.) We will show that the absolute minimum configuration does not have reasonable properties. In particular, it is not smooth where $\rho$ is nonzero. This generalizes earlier proofs[7] which used Abelian sources.

It is convenient to work in the

$$A_0 = 0 \tag{3.3}$$

gauge where the field equations provide a sensible initial-value problem. If the constraint (3.2) is satisfied at the initial time, then it is preserved in the time evolution as is the value of the energy (3.1). Thus, it will be sufficient for our purposes to study the energies of certain initial configurations. It is only the constraint (3.2) which prevents the problem from being trivial. Even so, it is not difficult.

Consider the space $S$ of smooth $E^A_i$ and $A^A_i$ satisfying (3.2). Suppose that at some point of $S$, $H$ has an absolute minimum $H_m$. Since (3.2) implies that $E^A_i$ is not zero on $S$, we conclude from (3.1) that

$$0 < H_m . \tag{3.4}$$

We will give an example of a path (parametrized by $\alpha$) *in the space* $S$ with

$$\lim_{\alpha\to\infty} H(\alpha) = 0 . \tag{3.5}$$

This contradicts (3.4), and we conclude that the energy functional constrained by (3.2) does not have an absolute minimum on the space of smooth fields.

A path in $S$ satisfying (3.5) will now be given. $A_0$ is zero at all times. At the initial time, we take

$$A_i = g^{-1}\partial_i g \tag{3.6}$$

and

$$E_i = g^{-1}E_i' g . \tag{3.7}$$

To satisfy the constraint, we require that

$$\partial_i E_i' = g\rho g^{-1} = \rho' . \tag{3.8}$$

If we take

$$E_i'^A = -\partial_i V^A , \tag{3.9}$$

then (3.8) becomes

$$-\nabla^2 V^A = \rho'^A , \tag{3.10}$$

and

$$V^A = -\frac{1}{\nabla^2}\rho'^A . \tag{3.11}$$

The energy of this configuration is

$$H = \tfrac{1}{2}\int \rho'^A\left(\frac{-1}{\nabla^2}\right)\rho'^A \tag{3.12}$$

$$= \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,|\tilde\rho'^A(k)|^2\frac{1}{k^2} . \tag{3.13}$$

Since a smooth $g$ will be used, $\rho'$ is good and so is $\tilde\rho'$. Thus, the convergence of (3.13) is assured.

For $g$ take

$$g^{-1} = e^{\alpha T_1}e^{\beta T_2} , \tag{3.14}$$

with $\alpha$ and $\beta$ some yet to be specified functions of $\vec{x}$. This gives

$$\rho'^A = R^{AB}\rho^B , \tag{3.15}$$

with

$$R = \begin{bmatrix} \cos\beta & \sin\beta\sin\alpha & -\sin\beta\cos\alpha \\ 0 & \cos\alpha & +\sin\alpha \\ \sin\beta & -\cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix} . \tag{3.16}$$

If we take

$$\alpha = \vec\alpha\cdot\vec{x} \text{ and } \beta = \vec\beta\cdot\vec{x} , \tag{3.17}$$

then each component of $\tilde\rho'$ is a sum of terms each of which is of the form

$$\tilde\sigma(\vec{k}+\vec{K}) , \tag{3.18}$$

with $\tilde\sigma$ some good function independent of $\vec\alpha$ and $\vec\beta$ and

$$\vec{K} = \pm\vec\alpha,\ \pm\vec\beta,\ \pm(\vec\alpha+\vec\beta),\ \text{or } \pm(\vec\alpha-\vec\beta) . \tag{3.19}$$

Then (3.13) will be a sum of terms each of which is of the form

$$\int d^3k\,\tilde\sigma(\vec{k}+\vec{K})\tilde\sigma'(\vec{k}+\vec{K}')\frac{1}{k^2} . \tag{3.20}$$

Now specify $\vec\alpha$ and $\vec\beta$ further by

$$\alpha_i = \alpha\delta_{i1} \text{ and } \beta_i = \alpha\delta_{i2} . \tag{3.21}$$

Each of the possible $\vec{K}$'s then goes to infinity in a different direction as $\alpha\to\infty$.

If we shift by $-K$ in (3.20), it becomes

$$\int d^3k\,\tilde\sigma(\vec{k})\tilde\sigma'(\vec{k}+\vec{K}'-\vec{K})\frac{1}{(\vec{k}-\vec{K})^2} . \tag{3.22}$$

Since $\tilde\sigma'$ is a good function, it has a finite maximum modulus somewhere and the modulus of (3.22) is bounded above by

$$|\tilde\sigma'|_{\max}\int d^3k\,|\tilde\sigma(k)|\frac{1}{(\vec{k}-\vec{K})^2} , \tag{3.23}$$

which goes to zero as $\alpha$ goes to infinity. Thus, in (3.6), (3.7), (3.9), (3.11), (3.14), (3.17), and (3.21) we have the desired path.

To summarize: If the energy had an absolute minimum on $S$, the field configuration would be static and would have a positive energy less than that of any time-dependent configuration. However, we have just shown that there are time-dependent configurations *in* $S$ with arbitrarily small energy. Thus, the energy does not have an absolute minimum on the space $S$. It is clear from the discussion that in attempting to minimize the energy, one is driven toward fields with arbitrarily rapid spatial oscillations. We refer to this as high-wave-number screening.

This basic idea has been discussed by several authors.[7] The limitations of those treatments can be characterized by saying either that the sources were assumed Abelian or that for arbitrary sources the path would not lie in $S$.

## IV. QUANTUM MECHANICS

What does the result of Sec. III tell us about the quantum mechanics of the gauge field interacting with a classical static source? A careful treatment of this problem may be difficult. One strategy would be to begin with a Hamiltonian formulation.[11] We will just discuss one speculation and present a highly simplified model which shares one feature with the real problem and which tends to support our conjecture.

In simple field theories, the vacuum energy can be expanded in powers of $\hbar$. When the field is coupled to a classical source, the vacuum energy often depends on the source in order $\hbar^0$. This is certainly the case for an ordinary scalar field and for an Abelian gauge field. It is a reflection of the fact that the minimum-energy configuration of the classical field theory depends upon the source. However, this is not always the case. A free massive scalar field can be coupled to a source with a term which is quadratic in both:

$$\mathfrak{L}_s = -\rho^2\phi^2 . \tag{4.1}$$

The classical minimum has zero field and zero energy independent of $\rho$. The vacuum energy is independent of $\rho$ in order $\hbar^0$.

The situation with which we are confronted has an additional feature. Although the minimum value of the classical energy is zero independent of the source, the field configuration at the minimum is something horrible. Nevertheless, we speculate that the vacuum energy of the quantum gauge field interacting with a classical, static, good source is independent of the source in order $\hbar^0$.

For a relatively simple example,[12] consider the field theory with Lagrangian

$$L = \int d^3x[\tfrac{1}{2}\dot{\phi}^A\dot{\phi}^A - \lambda(\phi^A\phi^A - \sigma^2)^2] - \int d^3x \int d^3y\,\gamma\phi^A\left(\frac{-1}{\nabla^2}\right)\phi^A . \tag{4.2}$$

The constants $\lambda$, $\gamma$, and $\sigma^2$ are positive and the field has two real components. Think of $\gamma$ as a space-time-independent source. The classical energy is bounded below by zero and can be zero only if both

$$\int d^3x(\phi^A\phi^A - \sigma^2)^2 \tag{4.3}$$

and

$$\int d^3x \int d^3y\, \phi^A\left(\frac{-1}{\nabla^2}\right)\phi^A \tag{4.4}$$

are zero. With the choice

$$\phi^1 = \sigma\cos\alpha x_1 \text{ and } \phi^2 = \sigma\sin\alpha x_1 , \tag{4.5}$$

(4.3) is zero and (4.4) becomes

$$\frac{\sigma^2}{4\pi}\int d^3x \int d^3y \cos\alpha x_1 \frac{1}{[(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2]^{1/2}}\cos\alpha y_1 + (\cos\to\sin) \tag{4.6}$$

which vanishes as $\alpha\to\infty$. Thus, the classical energy can go to zero independent of the "source" $\gamma$, and in so doing it runs out of the space of smooth fields. This is similar to the gauge field.

A still simpler model is obtained by dropping the interaction and reducing the field to one component

$$L = \int d^3x\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\gamma\int d^3x\int d^3y\,\phi\left(\frac{-1}{\nabla^2}\right)\phi . \tag{4.7}$$

The classical minimum of the energy is zero independent of $\gamma$. But besides the zero field, fields of the form

$$\phi = \phi_0\cos\alpha x_1 , \tag{4.8}$$

with $\alpha$ going to infinity also give zero energy. This model is so simple that we can see that this possibility is the result of symmetry.

To show this, a change of variables will be made. Write

$$\phi(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{x}}\tilde{\phi}(t,\vec{k}) . \tag{4.9}$$

Equation (4.7) becomes

$$L = \int\frac{d^3k}{(2\pi)^3}\tfrac{1}{2}\dot{\tilde{\phi}}(t,\vec{k})\dot{\tilde{\phi}}(t,-\vec{k}) - \frac{\gamma}{2}\int\frac{d^3k}{(2\pi)^3}\tilde{\phi}(t,\vec{k})\frac{1}{k^2}\tilde{\phi}(t,-\vec{k}) . \tag{4.10}$$

Now let

$$\tilde{\phi}(t,\vec{k}) = \left(\frac{1}{k^3}\right)\tilde{\beta}\left(t,\frac{\vec{k}}{k^2}\right), \tag{4.11}$$

which gives

$$L = \tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^6}\left[\dot{\tilde{\beta}}\left(t,\frac{\vec{k}}{k^2}\right)\dot{\tilde{\beta}}\left(t,-\frac{\vec{k}}{k^2}\right) - \gamma\tilde{\beta}\left(t,\frac{\vec{k}}{k^2}\right)\frac{1}{k^2}\tilde{\beta}\left(t,-\frac{\vec{k}}{k^2}\right)\right]. \tag{4.12}$$

Changing integration variables to

$$\vec{p} = \vec{k}/k^2 \tag{4.13}$$

gives (up to an overall constant)

$$L = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[\dot{\tilde{\beta}}(t,\vec{p})\dot{\tilde{\beta}}(t,-\vec{p}) - \gamma\tilde{\beta}(t,\vec{p})p^2\tilde{\beta}(t,-\vec{p})\right] \tag{4.14}$$

$$= \frac{1}{2}\int d^3x(\dot{\beta}^2 - \gamma\partial_i\beta\partial_i\beta)\,. \tag{4.15}$$

Thus, a choice such as (4.8) with $\alpha \to \infty$ corresponds to taking $\beta$ constant. $L$ has a field translation symmetry and the classical energy is zero for any constant field. Further, it is clear that the vacuum energy is independent of $\gamma$ in order $\hbar^0$.

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[1]We will jump between matrix and component notation according to convenience. Generators of the SU(2) algebra satisfying $[T_A, T_B] = \epsilon_{AB}{}^C T_C$ will be used. We have $\rho(\vec{x}) = \rho^A(\vec{x})T_A$, $A_\mu = A_\mu^A T_A$, etc.

[2]Here we follow M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1970). The assumption that the source is good can be weakened considerably. However, the discussion would become more technical and no further physical insight would result.

[3]This is defined after Eq. (2.4) in Sec. II.

[4]J. Mandula, Phys. Rev. D 14, 3497 (1976); Phys. Lett. 67B, 175 (1977); P. Sikivie and N. Weiss, Phys. Rev. D 18, 3809 (1978); 20, 487 (1979); Phys. Rev. Lett. 40, 1411 (1978); M. Magg, Phys. Lett. 78B, 481 (1978); 77B, 199 (1978); K. Cahill, Phys. Rev. Lett. 41, 599 (1978); R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D 20, 474 (1979); Y. Leroyer and A. Raychaudhuri, Phys. Rev. D 20, 2086 (1979); University of Oxford Report No. 10/79 (unpublished); R. Jackiw and P. Rossi, *ibid.* (to be published); P. Pirilä and P. Presnajder, Nucl. Phys. B142, 229 (1978); A. Gillespie, University of Durham report (unpublished); J. Harnad, S. Shnider, and Luc Vinet, J. Math. Phys. 20, 931 (1979); University of Montreal Report No. CRM 795 (unpublished); J. Kiskis, Phys. Rev. D (to be published).

[5]Although there are many geometrical attributes of a source, we will be referring only to the Abelian or non-Abelian property when we use the word geometry.

[6]J. Mandula, see Ref. 4.

[7]P. Sikivie and N. Weiss, Ref. 4; P. Pirilä and P. Presnajder, Ref. 4; A. Gillespie, Ref. 4.

[8]For some discussion of the point source, see P. Sikivie and N. Weiss, Phys. Rev. D 20, 487 (1979).

[9]C. Bernard, N. Christ, A. Guth, and E. Weinberg, Phys. Rev. D 16, 2967 (1977).

[10]P. Sikivie and N. Weiss, Phys. Rev. D 20, 487 (1979); J. Kiskis, *ibid.* (to be published).

[11]J. Kiskis, Ref. 10.

[12]This is motivated by the Hamiltonian of Ref. 11.