Equation of motion in classical electrodynamics. II

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With the aid of the principle of equivalence, we extend our development of an equation of motion for a radiating charged particle to include movement in both a gravitational and an electromagnetic field. The rate of electromagnetic radiation is given by the covariant generalization of the Larmor formula for flat space-time. The proposed equation of motion exhibits none of the difficulties that are associated with the covariant extension of the Lorentz-Dirac equation. We apply our results to the simple examples of a charged particle falling in a gravitational field and a charged particle held fixed in a quasiuniform gravitational field.

I. INTRODUCTION

In a previous publication[1] we proposed an equation of motion for a radiating charged particle in an external electromagnetic field. This equation exhibited none of the problems characteristic of the Lorentz-Dirac (LD) equation,[2,3] and it yielded trajectories having the essential physical features of the usual perturbation solutions to the LD equation. However, the treatment was limited to motion with respect to an inertial system. In the present paper we would like to extend our discussion to the motion of a charged particle in a background gravitational field. To do this, we will use the principle of equivalence in the manner advocated by Weinberg.[4] In this way, we obtain a comparatively simple equation that describes the motion of a radiating charged particle in a gravitational and an electromagnetic field. Other efforts[5,6] in this direction have been covariant generalizations of Dirac's approach,[2] with the result that the equations of motion arrived at have inherently manifested difficulties similar to those of the Lorentz-Dirac equation in flat space-time.

II. MOTION AND RADIATION IN AN INERTIAL SYSTEM

In this section, it is desirable that we first express the results of paper I in a way that can readily be generalized to arbitrary coordinate systems. To do this conveniently we adopt the notation and units (rational Heaviside-Lorentz with the velocity of light $c=1$) employed in Ref. 4. The equation of motion, Eq. (I3) (the prefix I refers to the equations of paper I), can then be written as

$$m\dot{v}^{\alpha}=eF^{\alpha}{}_{\beta}v^{\beta}-\sigma_T v^{\alpha}F^{\beta}{}_{\gamma}F_{\beta\delta}v^{\gamma}v^{\delta}+\sigma_T F^{\alpha}{}_{\beta}F^{\beta}{}_{\gamma}v^{\gamma}\,, \tag{1}$$

where the dot over a variable denotes differentiation with respect to the proper time of the particle, and the repeated Greek indices indicate a summation over the four space-time coordinate labels 1, 2, 3, 0. The Thomson cross section of the charge, which depends on its classical radius r_o, has been designated σ_T $(=8\pi r_o^2/3)$. The externally applied electromagnetic field tensor $F_{\alpha\beta}$ can be expressed (for inertial systems) in terms of the component electric and magnetic fields in a fashion similar to Eq. (I4) as

$$F_{\alpha\beta}=\eta_{\alpha j}\eta_{\beta k}\epsilon_{jkl}B_l+(\eta_{\alpha 0}\eta_{\beta k}-\eta_{\alpha k}\eta_{\beta 0})E_k\,. \tag{2}$$

Here $\eta_{\alpha\beta}$ is the Minkowski metric tensor (signature +2) for the inertial coordinate system, and ϵ_{jkl} is the total antisymmetric tensor with $\epsilon_{123}=1$. The repeated Latin indices in Eq. (2) run over the space labels 1, 2, and 3. In accordance with the metric relation

$$d\tau^2=-\eta_{\alpha\beta}d\xi^{\alpha}d\xi^{\beta}\,, \tag{3}$$

the four-velocity satisfies the equation

$$1=-\eta_{\alpha\beta}v^{\alpha}v^{\beta}\,. \tag{4}$$

With these notational preliminaries out of the way, we continue by writing the equation of motion, which is quadratic in the applied field tensor, as a function of the energy-momentum density tensor of this field

$$T^{\alpha\beta}=F^{\alpha}{}_{\gamma}F^{\beta\gamma}-\tfrac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}\,. \tag{5}$$

Equation (1) then becomes

$$m\dot{v}^{\alpha}=eF^{\alpha}{}_{\beta}v^{\beta}-\sigma_T v^{\alpha}T_{\beta\gamma}v^{\beta}v^{\gamma}-\sigma_T T^{\alpha}{}_{\gamma}v^{\gamma}\,. \tag{6}$$

To show why this form of the equation is interesting, we specialize to the case of the motion of a charge referred to its instantaneous rest frame, that is, one in which the particle has the four-velocity components $v^k=0$ and $v^0=1$. We see that in such a proper frame of reference, Eq. (6) is represented by

$$m\dot{v}_j=eE_j+\sigma_T\epsilon_{jkl}E_kB_l \tag{7a}$$

and

$$\dot{\gamma}=0. \tag{7b}$$

As we have pointed out in paper I, Eq. (7a) lends itself to the interpretation that the particle acquires momentum due to the external electric field and the Poynting momentum which it intercepts in its own rest frame. Equation (6) can then be viewed as the generalization of Eq. (7) when the motion is referred to an arbitrary inertial system.

Given the particle moving according to Eq. (6), it will radiate energy irreversibly depending on the value of its four-acceleration, $\dot{v}^\alpha$. The scalar, which equals the time rate of this energy emission, has been studied in some detail by Rohrlich[7] and Schild.[8] It can be written as

$$R=\frac{e^2}{6\pi}\eta_{\alpha\beta}\dot{v}^\alpha\dot{v}^\beta. \tag{8}$$

We observe that this expression is an extension of the well-known Larmor formula to arbitrary inertial systems. Its derivation[8] depends only on Maxwell's equations, and it is evidently independent of the particular form of the particle equation of motion. In addition, it plays the role of a local criterion for the existence of a radiation flux that propagates away from the accelerating particle.

III. MOTION IN A GRAVITATIONAL FIELD

Our discussion up to this point has been limited to that of motion referred to the inertial coordinate systems of special relativity. To introduce the effect of gravitation, we envision such a system as a "local inertial system." Specifically, we use the alternative version of the principle of equivalence, that is, the principle of general covariance. This idea has been presented in a succinct way by Weinberg[9]: "Write the appropriate special relativistic equations that hold in the absence of gravitation, replace $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and replace all derivatives with covariant derivatives." Accordingly, Eq. (6) assumes the form

$$m\left(\frac{dv^\lambda}{d\tau}+\Gamma^\lambda_{\mu\nu}v^\mu v^\nu\right)=eF^\lambda{}_\mu v^\mu-\sigma_T v^\lambda T_{\mu\nu}v^\mu v^\nu - \sigma_T T^\lambda{}_\mu v^\mu . \tag{9}$$

Here the symbol $\Gamma^\lambda_{\mu\nu}$ is the affine connection related to the metric tensor $g_{\mu\nu}$, and $T^{\mu\nu}$ is the energy-momentum tensor density of the external electromagnetic field $F^{\mu\nu}$, that is,

$$T^{\mu\nu}=F^\mu{}_\lambda F^{\nu\lambda}-\tfrac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{10}$$

Equation (9) is then the classical equation of motion for a radiating point particle of rest mass m and charge e (with the Thomson cross section σ_T) which moves in a background gravitational field and an applied electromagnetic field. It is to be compared with the equations of Refs. 5 and 6. These, as previously stated, are based on a direct extension of Dirac's treatment and contain not only derivatives of the acceleration, but also an integral over the entire past history of the particle.

Employing the principle of equivalence once again, we arrive at the covariant generalization of Eq. (8) for the electromagnetic power radiated. Thus, we have

$$R=\frac{e^2}{6\pi}g_{\mu\nu}\left(\frac{dv^\mu}{d\tau}+\Gamma^\mu_{\rho\sigma}v^\rho v^\sigma\right)\left(\frac{dv^\nu}{d\tau}+\Gamma^\nu_{\xi\eta}v^\xi v^\eta\right). \tag{11}$$

This relationship corresponds to the Larmor formula for curved space recently investigated by Villarroel.[10]

IV. FALLING CHARGED PARTICLE

The problem of a charged particle falling in a gravitational field has been discussed frequently in the past,[11,12] and it is, consequently, appropriate to consider it briefly. When there is no applied electromagnetic field, the right-hand side of Eq. (9) equals zero and the charged particle follows a geodesic path

$$\frac{dv^\lambda}{d\tau}+\Gamma^\lambda_{\mu\nu}v^\mu v^\nu=0 \tag{12}$$

in the background gravitational field. Under this condition then, our equation of motion predicts that a charged point particle follows the same trajectory as an uncharged one. Corresponding to this type of motion, the rate of electromagnetic radiation, Eq. (11), vanishes. Both of these results were to be expected since, in agreement with the principle of equivalence, the point particle finds itself free, under the action of no force, in the local inertial frame of reference.

V. QUASIUNIFORM GRAVITATIONAL FIELD

Our development up to now has led us to equations that are valid in an arbitrary curved space-time. It is, therefore, desirable, by way of illustration, that we study their consequences for a specific model of a gravitational field. To simplify the analysis we shall employ the approximation to Schwarzschild's metric which Whittaker[13] has called "the quasi-uniform gravitational field." Such a space-time region is described by the metric relation

$$d\tau^2=-g_{\mu\nu}dx^\mu dx^\nu \tag{13}$$

equal to

$$d\tau^2=-\frac{1}{1+2gx}dx^2-dy^2-dz^2+(1+2gx)dt^2, \tag{14}$$

where g is the accelerating strength of the gravitational field. Recalling that the affine connection is defined by

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} + \frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right), \tag{15}$$

we find that it has the following nonvanishing components:

$$\Gamma^{1}_{11} = \frac{-g}{1+2gx}, \tag{16a}$$

$$\Gamma^{1}_{00} = g(1+2gx), \tag{16b}$$

and

$$\Gamma^{0}_{10} = \Gamma^{0}_{01} = \frac{g}{1+2gx}. \tag{16c}$$

If we assume that the geodesic trajectory starts from rest at $x^{\mu}=0$, we obtain the path for a falling particle

$$x = -\tfrac{1}{2} g\tau^{2}, \tag{17a}$$

$$y = 0, \tag{17b}$$

$$z = 0, \tag{17c}$$

and

$$t = \frac{1}{g}\arctan(g\tau). \tag{17d}$$

It is to be noted that $x^{\mu} \equiv (x, y, z, t)$ are the coordinates in the gravitational frame of reference (G system), and τ is the proper time of the falling particle. If we set up a local coordinate system centered on the falling particle, which we label $\xi^{\alpha} \equiv (\xi^1, \xi^2, \xi^3, \xi^0)$ and call the I system, we can derive the transformation equations[14] from the G system to the I system. These are

$$\xi^1 = \frac{1}{g}[(1+2gx)^{1/2}\cosh(gt) - 1], \tag{18a}$$

$$\xi^2 = y, \tag{18b}$$

$$\xi^3 = z, \tag{18c}$$

and

$$\xi^0 = \frac{1}{g}(1+2gx)^{1/2}\sinh(gt). \tag{18d}$$

Applying the tensor transformation equation for the metric tensor

$$g^{\alpha\beta}(I) = \frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}} g^{\mu\nu}(G), \tag{19}$$

one can easily verify that indeed the metric of the falling system is Minkowskian, and, as a consequence,

$$d\tau^2 = -(d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 + (d\xi^0)^2. \tag{20}$$

VI. CHARGED PARTICLE STATIONARY IN A GRAVITATIONAL FIELD

In the initial portion of this section we shall treat the case of a charged particle, which is held fixed by an electric field in a quasiuniform gravitational field, from the viewpoint of the falling coordinate system I. Since the spatial coordinates of the particle with respect to the G system are $x^k = 0$, the coordinates of the path of the particle (with respect to the I frame) are

$$\xi^1 = \frac{1}{g}[\cosh(gt) - 1], \tag{21a}$$

$$\xi^2 = 0, \tag{21b}$$

$$\xi^3 = 0, \tag{21c}$$

and

$$\xi^0 = \frac{1}{g}\sinh(gt). \tag{21d}$$

It is clear that the motion is of the hyperbolic type[1] with t (the coordinate time of the particle in the G system) taking the part of the particle proper time. The four-velocity of this charge is then

$$\zeta^{\alpha} = \frac{d\xi^{\alpha}}{dt} = (\sinh gt, 0, 0, \cosh gt). \tag{22}$$

The application of Eq. (8) now gives the radiation rate

$$R = \frac{e^2 g^2}{6\pi} = R(I). \tag{23}$$

Since for the inertial frame of reference the components of the affine connection all vanish, the general equation of motion, Eq. (9), reduces to Eq. (6). Under this circumstance, the electric field $E_k = (E, 0, 0)$, which is causing the four-acceleration of the charge, is readily shown, via the equation

$$m\frac{d\zeta^{\alpha}}{dt} = eF^{\alpha}{}_{\beta}\zeta^{\beta} - \sigma_T\zeta^{\alpha}T_{\beta\gamma}\zeta^{\beta}\zeta^{\gamma} - \sigma_T T^{\alpha}{}_{\beta}\zeta^{\beta}, \tag{24}$$

to satisfy the anticipated relationship

$$eE = mg. \tag{25}$$

In arriving at this result we have used the following nonvanishing components of the field tensor and of the energy-momentum density tensor:

$$F^{1}{}_{0} = E, \tag{26a}$$

$$T_{11} = -E^2/2, \tag{26b}$$

and

$$T_{00} = E^2/2. \tag{26c}$$

An alternative approach to the problem of a stationary charge in the quasiuniform field is to adopt the viewpoint of the G system of coordinates. The velocity is then constant and given by

$$v^{\mu} = (0, 0, 0, 1) . \tag{27}$$

According to the tensor transformation rule for the electromagnetic field tensor, the electric field (which is now seen supporting the charge against the action of gravity) is unchanged and equal to E. When we now apply Eq. (9) and take into account the components of the affine connection, Eq. (16), we deduce the fact that the electric field again fulfills the condition Eq. (25). In addition, the radiation rate, as specified by Eq. (11), yields the value given in Eq. (23).

In short, when referred to either the falling (I) or the stationary (G) coordinate systems, the charge fixed at $x^k = 0$ behaves in agreement with the general equation of motion, Eq. (9), and radiates in agreement with the generalization of the Larmor relation, Eq. (11). Though, in practical terms, such a charge only radiates a very small amount of power (i.e., an electron supported at the earth's surface would emit about 5×10^{-52} W), the special example we have analyzed is important because of the concepts which it brings out. It may seem strange that a charge fixed under the action of an electric and a gravitational field should radiate, but the idea is in agreement with the principle of equivalence.

VII. DISCUSSION

Our development of a general equation of motion for a radiating charged particle shows some interesting overall features which we would like to emphasize. These, as we have stated, follow necessarily because of the way in which we have introduced the principle of equivalence. Thus, within the classical domain of electrodynamics, a point particle deviates from its "natural motion," the geodesic path, only when acted on by an external electromagnetic field. In addition, only then does it radiate. Consistent with this idea, a charge supported by an electric field is not following a geodesic trajectory and, therefore, it radiates electromagnetic energy. Since the theory which we have presented represents a covariant generalization of that given in Ref. 1, it also has the advantage of not requiring the concept of the self-force acting on the charged particle. The trajectory of the point charge is completely determined by the background gravitational field and the applied electromagnetic field.

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